

# COMPOSITIO MATHEMATICA

KULDIP KUMAR

## **On the geometric means of an integral function**

*Compositio Mathematica*, tome 19, n° 4 (1968), p. 271-277

[http://www.numdam.org/item?id=CM\\_1968\\_\\_19\\_4\\_271\\_0](http://www.numdam.org/item?id=CM_1968__19_4_271_0)

© Foundation Compositio Mathematica, 1968, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

# On the geometric means of an integral function

by

Kuldip Kumar

## 1.

Let  $f(z)$  be an integral function of order  $\rho$  and lower order  $\lambda$  and

$$(1.1) \quad \lim_{r \rightarrow \infty} \frac{\sup \log n(r)}{\inf \log r} = \frac{\rho_1}{\lambda_1},$$

$n(r)$  being the number of zeros of  $f(z)$  in  $|z| \leq r$ . Let  $G(r)$  and  $g_\delta(r)$  denote the geometric means of  $|f(z)|$ , defined as

$$(1.2) \quad G(r) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \right\}$$

and

$$(1.3) \quad g_\delta(r) = \exp \left\{ \frac{(\delta+1)}{2\pi r^{\delta+1}} \int_0^r \int_0^{2\pi} \log |f(xe^{i\theta})| x^\delta dx d\theta \right\}$$

In this paper we have obtained some of the properties of  $G(r)$  and  $g_\delta(r)$ . Let

$$(1.4) \quad N(r) = \int_0^r \frac{n(x)}{x} dx$$

and

$$(1.5) \quad \lim_{r \rightarrow \infty} \frac{\sup n(r)}{\inf r^\rho} = \frac{c}{d}.$$

Using Jensen's formula in (1.2), we have

$$(1.6) \quad \log G(r) = \log |f(0)| + \int_0^r \frac{n(x)}{x} dx.$$

From (1.1), we have, for any  $\varepsilon > 0$  and  $r > r_0 = r_0(\varepsilon)$

$$r^{\lambda_1 - \varepsilon} < n(r) < r^{\rho_1 + \varepsilon}.$$

Using this in (1.6), we have, for almost all values of  $r > r_0$

$$r^{\lambda_1-1-\varepsilon}G(r) < G'(r) < r^{\rho_1-1+\varepsilon}G(r).$$

Again from (1.5), we have for any  $\varepsilon > 0$  and  $r > r_0$

$$(d-\varepsilon)r^\rho < n(r) < (c+\varepsilon)r^\rho.$$

Substituting for  $n(r)$  from (1.6), we have, for almost all  $r > r_0$

$$(d-\varepsilon)r^{\rho-1}G(r) < G'(r) < (c+\varepsilon)r^{\rho-1}G(r).$$

## 2.

We shall now obtain some of the properties of  $g_\delta(r)$ . We may write (1.3) as

$$\log g_\delta(r) = \frac{(\delta+1)}{r^{\delta+1}} \int_0^r \log G(x)x^\delta dx.$$

Now, using (1.4) and (1.6), we get

$$(2.1) \quad \log g_\delta(r) = 0(1) + \frac{(\delta+1)}{r^{\delta+1}} \int_{r_0}^r N(x)x^\delta dx.$$

**THEOREM 1.** *Let  $f(z)$  be an integral function of order  $\rho$  ( $0 < \rho < \infty$ ) and let  $f(0) \neq 0$ . Further,*

(i) *if*

$$\liminf_{r \rightarrow \infty} \frac{N(r)}{r^\rho} = \beta,$$

*then*

$$\liminf_{r \rightarrow \infty} \frac{\log g_\delta(r)}{r^\rho} \geq \frac{\beta(\delta+1)}{(\rho+\delta+1)};$$

(ii) *if*

$$\limsup_{r \rightarrow \infty} \frac{N(r)}{r^\rho} = \alpha,$$

*then*

$$\limsup_{r \rightarrow \infty} \frac{\log g_\delta(r)}{r^\rho} \leq \frac{\alpha(\delta+1)}{(\rho+\delta+1)}.$$

**PROOF.** (i) Since

$$\liminf_{r \rightarrow \infty} \frac{N(r)}{r^\rho} = \beta,$$

therefore, for any  $\varepsilon > 0$  and  $r > r_1 = r_1(\varepsilon)$ , we have

$$N(r) > (\beta - \varepsilon)r^\rho,$$

and so from (2.1)

$$\log g_\delta(r) > o(1) + \frac{(\beta - \varepsilon)(\delta + 1)}{(\rho + \delta + 1)} (r^{\rho + \delta + 1} - r_0^{\rho + \delta + 1})r^{-\delta - 1}.$$

Taking limit on both the sides leads to

$$\liminf_{r \rightarrow \infty} \frac{\log g_\delta(r)}{r^\rho} \geq \frac{\beta(\delta + 1)}{(\rho + \delta + 1)}.$$

(ii) If

$$\limsup_{r \rightarrow \infty} \frac{N(r)}{r^\rho} = \alpha,$$

we have, for any  $\varepsilon > 0$  and  $r > r_1 = r_1(\varepsilon)$ ,  $N(r) < (\alpha + \varepsilon)r^\rho$ . Substituting this in (2.1), integrating and proceeding to limits, the result follows.

**THEOREM 2.** *Let  $f(z)$  be an integral function of finite non-integral order  $\rho$ , and let*

$$\lim_{r \rightarrow \infty} \frac{\log g_\delta(r)}{r^\rho} = \nu \quad \text{and} \quad \lim_{r \rightarrow \infty} \frac{\log G(r)}{r^\rho} = \mu,$$

then

$$\nu(\rho + \delta + 1) = \mu(\delta + 1).$$

**PROOF.** Since

$$\lim_{r \rightarrow \infty} \frac{\log g_\delta(r)}{r^\rho} = \nu,$$

therefore,

$$r^\rho(\nu - \varepsilon) < \log g_\delta(r) < r^\rho(\nu + \varepsilon), \quad \text{for } r > r_0(\varepsilon).$$

Hence,

$$\begin{aligned} \frac{(\delta + 1)}{r^{\delta + 1}} \int_{(1-\eta)r}^r \log G(x)x^\delta dx &= \frac{(\delta + 1)}{r^{\delta + 1}} \int_{r_0}^r \log G(x)x^\delta dx \\ &\quad - \frac{(\delta + 1)}{r^{\delta + 1}} \int_{r_0}^{(1-\eta)r} \log G(x)x^\delta dx, \quad \left(0 < \eta < \frac{1}{\delta + 1}\right) \\ &= o(1) + \log g_\delta(r) - (1-\eta)^{\delta + 1} \log g_\delta\{(1-\eta)r\} \\ &> o(1) + \nu\{(\rho + \delta + 1)\eta - \dots\}r^\rho - \varepsilon\{2 - (\rho + \delta + 1)\eta + \dots\}r^\rho. \end{aligned}$$

But,

$$\begin{aligned} \frac{(\delta+1)}{r^{\delta+1}} \int_{(1-\eta)r}^r \log G(x)x^\delta dx &\leq \frac{(\delta+1) \log G(r)}{r^{\delta+1}} \int_{(1-\eta)r}^r x^\delta dx \\ &< \frac{(\delta+1)\eta \log G(r)}{1-(\delta+1)\eta}, \quad \text{for } \rightarrow (\delta+1)\eta < 1. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\log G(r)}{r^\rho} &> o(1) + \frac{\nu\{(\rho+\delta+1)\eta - \dots\}\{1-(\delta+1)\eta\}}{(\delta+1)\eta} \\ &\quad - \frac{\varepsilon\{2-(\rho+\delta+1)\eta + \dots\}\{1-(\delta+1)\eta\}}{(\delta+1)\eta}. \end{aligned}$$

Since  $\eta$  is arbitrary, we get

$$(2.2) \quad \liminf_{r \rightarrow \infty} \frac{\log G(r)}{r^\rho} \geq \frac{\nu(\rho+\delta+1)}{(\delta+1)}.$$

Further,

$$\begin{aligned} \frac{(\delta+1)}{r^{\delta+1}} \int_r^{(1+\eta)r} \log G(x)x^\delta dx &= \frac{(\delta+1)}{r^{\delta+1}} \int_{r_0}^{(1+\eta)r} \log G(x)x^\delta dx \\ &\quad - \frac{(\delta+1)}{r^{\delta+1}} \int_{r_0}^r \log G(x)x^\delta dx < o(1) + \nu\{(\rho+\delta+1)\eta + \dots\} \\ &\quad \times r^\rho + \varepsilon\{2+(\rho+\delta+1)\eta + \dots\}r^\rho, \quad \text{for } \eta < 1, \end{aligned}$$

but,

$$\frac{(\delta+1)}{r^{\delta+1}} \int_r^{(1+\eta)r} \log G(x)x^\delta dx > (\delta+1)\eta \log G(r),$$

hence,

$$\frac{\log G(r)}{r^\rho} < o(1) + \frac{\nu\{(\rho+\delta+1)\eta + \dots\}}{(\delta+1)\eta} + \frac{\varepsilon\{2+(\rho+\delta+1)\eta + \dots\}}{(\delta+1)\eta}.$$

Therefore we get

$$(2.3) \quad \limsup_{r \rightarrow \infty} \frac{\log G(r)}{r^\rho} \leq \frac{\nu(\rho+\delta+1)}{(\delta+1)}.$$

Combining (2.2) and (2.3), we get the result.

### 3.

Combining (1.2) and (1.3), we obtain

$$\frac{g_\delta(r)}{G(r)} = \exp \left\{ \frac{-1}{r^{\delta+1}} \int_0^r x^{\delta+1} \frac{d}{dx} (\log G(x)) dx \right\}.$$

Using (1.6) in this, we get

$$(3.1) \quad \left\{ \frac{g_\delta(r)}{G(r)} \right\}^{1/N(r)} = \exp \left\{ \frac{-1}{r^{\delta+1}N(r)} \int_0^r n(x)x^\delta dx \right\}.$$

Let us set

$$\limsup_{r \rightarrow \infty} \left[ \frac{\{g_\delta(r)\}^{1/N(r)}}{G(r)} \right] = \frac{P}{p}.$$

We now prove the following:

**THEOREM 3.** *If  $f(z)$  is an integral function of order  $\rho$  ( $0 < \rho < \infty$ ) and  $f(0) \neq 0$ , such that  $n(r) \sim \phi(r)r^{\rho_1}$ , where  $\phi(r)$  is a positive continuous and indefinitely increasing function of  $r$  and  $\phi(cr) \sim \phi(r)$  as  $r \rightarrow \infty$  for every constant  $c > 0$ , then*

$$P = p = \exp \left\{ \frac{-\rho_1}{\rho_1 + \delta + 1} \right\}.$$

**PROOF.** Since  $n(r) \sim \phi(r)r^{\rho_1}$  we have for any  $\varepsilon > 0$  and  $r \geq r_0(\varepsilon)$

$$(3.2) \quad (1 - \varepsilon)\phi(r)r^{\rho_1} < n(r) < (1 + \varepsilon)\phi(r)r^{\rho_1}$$

or

$$(1 - \varepsilon) \int_{r_0}^r \phi(x)x^{\rho_1 + \delta} dx < \int_{r_0}^r n(x)x^\delta dx < (1 + \varepsilon) \int_{r_0}^r \phi(x)x^{\rho_1 + \delta} dx.$$

Now, by Lemma V([1], p. 54),

$$\int_{r_0}^r \phi(u)u^{\alpha-1} du \sim \frac{\phi(r)r^\alpha}{\alpha},$$

for every positive  $\alpha$ , and so we get

$$(3.3) \quad \frac{(1 - \varepsilon)}{\rho_1 + \delta + 1} \phi(r)r^{\rho_1 + \delta + 1} + o(1) < \int_0^r n(x)x^\delta dx < \frac{(1 + \varepsilon)}{\rho_1 + \delta + 1} \phi(r)r^{\rho_1 + \delta + 1} + o(1).$$

Again, from (3.2), we have

$$(1 - \varepsilon) \int_{r_0}^r \phi(x)x^{\rho_1 - 1} dx < \int_{r_0}^r \frac{n(x)}{x} dx < (1 + \varepsilon) \int_{r_1}^r \phi(x)x^{\rho_1 - 1} dx,$$

giving,

$$(3.4) \quad \frac{(1 - \varepsilon)}{\rho_1} \phi(r)r^{\rho_1} + o(1) < N(r) < \frac{(1 + \varepsilon)}{\rho_1} \phi(r)r^{\rho_1} + o(1).$$

Combining (3.3) and (3.4) leads to

$$\begin{aligned} \frac{-\rho_1}{(\rho_1 + \delta + 1)} \left[ \frac{(1 - \varepsilon)\phi(r)r^{\rho_1} + o(1)}{(1 + \varepsilon)\phi(r)r^{\rho_1}} \right] &> \frac{-1}{r^{\delta+1}N(r)} \int_0^r n(x)x^\delta dx \\ &> \frac{-\rho_1}{(\rho_1 + \delta + 1)} \left[ \frac{(1 + \varepsilon)\phi(r)r^{\rho_1} + o(1)}{(1 - \varepsilon)\phi(r)r^{\rho_1}} \right]. \end{aligned}$$

Taking exponentials and proceeding to limits, we have, since  $\varepsilon$  is arbitrary and  $n(r) \sim \phi(r)r^{\rho_1}$ ,

$$\lim_{r \rightarrow \infty} \exp \left\{ \frac{-1}{r^{\delta+1}N(r)} \int_0^r n(x)x^\delta dx \right\} = \exp \left\{ \frac{-\rho_1}{\rho_1 + \delta + 1} \right\}.$$

**THEOREM 4.** *If  $f(z)$  has at least one zero and  $f(0) \neq 0$ , then*

- (i)  $e^{-1} \leq p \leq P \leq 1$ ;
- (ii)  $P \geq \exp \left( \frac{-\lambda_1}{\delta + 1} \right)$ .

**PROOF.** (i) Integrating by parts the integral in (3.1), we get

$$\left\{ \frac{g_\delta(r)}{G(r)} \right\}^{1/N(r)} = \exp \left\{ -1 + \frac{(\delta + 1)}{r^{\delta+1}N(r)} \int_0^r N(x)x^\delta dx \right\}.$$

Since  $N(r)$  is non-decreasing function of  $r$ , we get

$$p \geq e^{-1} \quad \text{and} \quad P \leq 1.$$

Since  $n(r)$  is non-decreasing function of  $r$ , (3.1) gives

$$\left\{ \frac{g_\delta(r)}{G(r)} \right\}^{1/N(r)} > \exp \left\{ \frac{-n(r)}{(\delta + 1)N(r)} \right\}.$$

But we know ([2], p. 17) that

$$\liminf_{r \rightarrow \infty} \frac{n(r)}{N(r)} \leq \lambda_1,$$

and so

$$\limsup_{r \rightarrow \infty} \left[ \left\{ \frac{g_\delta(r)}{G(r)} \right\}^{1/N(r)} \right] \geq \exp \left( \frac{-\lambda_1}{\delta + 1} \right).$$

I wish to express my sincere thanks to Dr. S. K. Bose for his guidance in the preparation of this paper.

## REFERENCES

G. H. HARDY and W. W. ROGOSINSKI

- [1] 'Note on Fourier Series (III)', *Quarterly Journal of Mathematics*, 16, (1945), pp. 49—58.

R. P. BOAS, JR.

- [2] 'Entire Functions', New York, 1954.

(Oblatum 9-3-1966)

Department of Mathematics  
and Astronomy,  
Lucknow University,  
Lucknow (India)