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# On the geometric means of an integral function

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# On the geometric means of an integral function

by

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## 1.

Let $f(z)$ be an integral function of order $\rho$ and lower order $\lambda$ and

$$(1.1) \qquad \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log n(r)}{\log r} = \frac{\rho_1}{\lambda_1},$$

$n(r)$ being the number of zeros of $f(z)$ in $|z| \leqq r$. Let $G(r)$ and $g_\delta(r)$ denote the geometric means of $|f(z)|$, defined as

$$(1.2) \qquad G(r) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})| d\theta\right\}$$

and

$$(1.3) \qquad g_\delta(r) = \exp\left\{\frac{(\delta+1)}{2\pi r^{\delta+1}}\int_0^r\int_0^{2\pi} \log|f(xe^{i\theta})|x^\delta \, dx \, d\theta\right\}$$

In this paper we have obtained some of the properties of $G(r)$ and $g_\delta(r)$. Let

$$(1.4) \qquad N(r) = \int_0^r \frac{n(x)}{x}\, dx$$

and

$$(1.5) \qquad \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{n(r)}{r^\rho} = \frac{c}{d}.$$

Using Jensen's formula in (1.2), we have

$$(1.6) \qquad \log G(r) = \log |f(0)| + \int_0^r \frac{n(x)}{x}\, dx.$$

From (1.1), we have, for any $\varepsilon > 0$ and $r > r_0 = r_0(\varepsilon)$

$$r^{\lambda_1-\varepsilon} < n(r) < r^{\rho_1+\varepsilon}.$$

Using this in (1.6), we have, for almost all values of $r > r_0$

$$r^{\lambda_1-1-\varepsilon}G(r) < G'(r) < r^{\rho_1-1+\varepsilon}G(r).$$

Again from (1.5), we have for any $\varepsilon > 0$ and $r > r_0$

$$(d-\varepsilon)r^{\rho} < n(r) < (c+\varepsilon)r^{\rho}.$$

Substituting for $n(r)$ from (1.6), we have, for almost all $r > r_0$

$$(d-\varepsilon)r^{\rho-1}G(r) < G'(r) < (c+\varepsilon)r^{\rho-1}G(r).$$

## 2.

We shall now obtain some of the properties of $g_\delta(r)$. We may write (1.3) as

$$\log g_\delta(r) = \frac{(\delta+1)}{r^{\delta+1}}\int_0^r \log G(x)x^\delta\, dx.$$

Now, using (1.4) and (1.6), we get

$$\log g_\delta(r) = 0(1)+\frac{(\delta+1)}{r^{\delta+1}}\int_{r_0}^r N(x)x^\delta\, dx. \tag{2.1}$$

THEOREM 1. *Let $f(z)$ be an integral function of order $\rho(0 < \rho < \infty)$ and let $f(0) \neq 0$. Further,*

(i) *if*

$$\liminf_{r\to\infty}\frac{N(r)}{r^\rho} = \beta,$$

*then*

$$\liminf_{r\to\infty}\frac{\log g_\delta(r)}{r^\rho} \geqq \frac{\beta(\delta+1)}{(\rho+\delta+1)};$$

(ii) *if*

$$\limsup_{r\to\infty}\frac{N(r)}{r^\rho} = \alpha,$$

*then*

$$\limsup_{r\to\infty}\frac{\log g_\delta(r)}{r^\rho} \leqq \frac{\alpha(\delta+1)}{(\rho+\delta+1)}.$$

PROOF. (i) Since

$$\liminf_{r\to\infty}\frac{N(r)}{r^\rho} = \beta,$$

therefore, for any $\varepsilon > 0$ and $r > r_1 = r_1(\varepsilon)$, we have

$$N(r) > (\beta-\varepsilon)r^{\rho},$$

and so from (2.1)

$$\log g_{\delta}(r) > 0(1)+\frac{(\beta-\varepsilon)(\delta+1)}{(\rho+\delta+1)}(r^{\rho+\delta+1}-r_0^{\rho+\delta+1})r^{-\delta-1}.$$

Taking limit on both the sides leads to

$$\liminf_{r\to\infty}\frac{\log g_{\delta}(r)}{r^{\rho}} \geqq \frac{\beta(\delta+1)}{(\rho+\delta+1)}.$$

(ii) If

$$\limsup_{r\to\infty}\frac{N(r)}{r^{\rho}}=\alpha,$$

we have, for any $\varepsilon > 0$ and $r > r_1 = r_1(\varepsilon)$, $N(r) < (\alpha+\varepsilon)r^{\rho}$. Substituting this in (2.1), integrating and proceeding to limits, the result follows.

**Theorem 2.** *Let $f(z)$ be an integral function of finite non-integral order $\rho$, and let*

$$\lim_{r\to\infty}\frac{\log g_{\delta}(r)}{r^{\rho}}=\nu \quad \textit{and} \quad \lim_{r\to\infty}\frac{\log G(r)}{r^{\rho}}=\mu,$$

*then*

$$\nu(\rho+\delta+1)=\mu(\delta+1).$$

**Proof.** Since

$$\lim_{r\to\infty}\frac{\log g_{\delta}(r)}{r^{\rho}}=\nu,$$

therefore,

$$r^{\rho}(\nu-\varepsilon) < \log g_{\delta}(r) < r^{\rho}(\nu+\varepsilon), \quad \text{for} \quad r > r_0(\varepsilon).$$

Hence,

$$\begin{aligned}\frac{(\delta+1)}{r^{\delta+1}}\int_{(1-\eta)r}^{r}\log G(x)x^{\delta}\,dx &= \frac{(\delta+1)}{r^{\delta+1}}\int_{r_0}^{r}\log G(x)x^{\delta}\,dx\\ &\quad -\frac{(\delta+1)}{r^{\delta+1}}\int_{r_0}^{(1-\eta)r}\log G(x)x^{\delta}\,dx, \left(0<\eta<\frac{1}{\delta+1}\right)\\ &= o(1)+\log g_{\delta}(r)-(1-\eta)^{\delta+1}\log g_{\delta}\{(1-\eta)r\}\\ &> o(1)+\nu\{(\rho+\delta+1)\eta-\cdots\}r^{\rho}-\varepsilon\{2-(\rho+\delta+1)\eta+\cdots\}r^{\rho}.\end{aligned}$$

But,

$$\frac{(\delta+1)}{r^{\delta+1}}\int_{(1-\eta)r}^{r}\log G(x)x^{\delta}\,dx \leqq \frac{(\delta+1)\log G(r)}{r^{\delta+1}}\int_{(1-\eta)r}^{r}x^{\delta}\,dx$$
$$< \frac{(\delta+1)\eta\log G(r)}{1-(\delta+1)\eta}, \quad \text{for}\rightarrow(\delta+1)\eta < 1.$$

Hence,

$$\frac{\log G(r)}{r^{\rho}} > o(1)+\frac{\nu\{(\rho+\delta+1)\eta-\cdots\}\{1-(\delta+1)\eta\}}{(\delta+1)\eta}$$
$$-\frac{\varepsilon\{2-(\rho+\delta+1)\eta+\cdots\}\{1-(\delta+1)\eta\}}{(\delta+1)\eta.}.$$

Since $\eta$ is arbitrary, we get

$$\liminf_{r\to\infty}\frac{\log G(r)}{r^{\rho}} \geqq \frac{\nu(\rho+\delta+1)}{(\delta+1)}. \tag{2.2}$$

Further,

$$\frac{(\delta+1)}{r^{\delta+1}}\int_{r}^{(1+\eta)r}\log G(x)x^{\delta}\,dx = \frac{(\delta+1)}{r^{\delta+1}}\int_{r_0}^{(1+\eta)r}\log G(x)x^{\delta}\,dx$$
$$-\frac{(\delta+1)}{r^{\delta+1}}\int_{r_0}^{r}\log G(x)x^{\delta}\,dx < o(1)+\nu\{(\rho+\delta+1)\eta+\cdots\}$$
$$\times r^{\rho}+\varepsilon\{2+(\rho+\delta+1)\eta+\cdots\}r^{\rho}, \quad \text{for} \quad \eta < 1,$$

but,

$$\frac{(\delta+1)}{r^{\delta+1}}\int_{r}^{(1+\eta)r}\log G(x)x^{\delta}\,dx > (\delta+1)\eta\log G(r),$$

hence,

$$\frac{\log G(r)}{r^{\rho}} < o(1)+\frac{\nu\{(\rho+\delta+1)\eta+\cdots\}}{(\delta+1)\eta}+\frac{\varepsilon\{2+(\rho+\delta+1)\eta+\cdots\}}{(\delta+1)\eta}.$$

Therefore we get

$$\limsup_{r\to\infty}\frac{\log G(r)}{r^{\rho}} \leqq \frac{\nu(\rho+\delta+1)}{(\delta+1)}. \tag{2.3}$$

Combining (2.2) and (2.3), we get the result.

## 3.

Combining (1.2) and (1.3), we obtain

$$\frac{g_{\delta}(r)}{G(r)} = \exp\left\{\frac{-1}{r^{\delta+1}}\int_{0}^{r}x^{\delta+1}\frac{d}{dx}\left(\log G(x)\right)dx\right\}.$$

Using (1.6) in this, we get

$$\left\{\frac{g_\delta(r)}{G(r)}\right\}^{1/N(r)} = \exp\left\{\frac{-1}{r^{\delta+1}N(r)}\int_0^r n(x)x^\delta\,dx\right\}. \tag{3.1}$$

Let us set

$$\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\left[\left\{\frac{g_\delta(r)}{G(r)}\right\}^{1/N(r)}\right] = \begin{matrix}P\\ p\end{matrix}.$$

We now prove the following:

THEOREM 3. *If $f(z)$ is an integral function of order $\rho(0 < \rho < \infty)$ and $f(0) \neq 0$, such that $n(r) \sim \phi(r)r^{\rho_1}$, where $\phi(r)$ is a positive continuous and indefinitely increasing function of $r$ and $\phi(cr) \sim \phi(r)$ as $r \to \infty$ for every constant $c > 0$, then*

$$P = p = \exp\left\{\frac{-\rho_1}{\rho_1+\delta+1}\right\}.$$

PROOF. Since $n(r) \sim \phi(r)r^{\rho_1}$ we have for any $\varepsilon > 0$ and $r \geqq r_0(\varepsilon)$

$$(1-\varepsilon)\phi(r)r^{\rho_1} < n(r) < (1+\varepsilon)\phi(r)r^{\rho_1} \tag{3.2}$$

or

$$(1-\varepsilon)\int_{r_0}^r \phi(x)x^{\rho_1+\delta}\,dx < \int_{r_0}^r n(x)x^\delta\,dx < (1+\varepsilon)\int_{r_0}^r \phi(x)x^{\rho_1+\delta}\,dx.$$

Now, by Lemma V([1], p. 54),

$$\int_{r_0}^r \phi(u)u^{\alpha-1}\,du \sim \frac{\phi(r)r^\alpha}{\alpha},$$

for every positive $\alpha$, and so we get

$$\frac{(1-\varepsilon)}{\rho_1+\delta+1}\phi(r)r^{\rho_1+\delta+1}+0(1) < \int_0^r n(x)x^\delta\,dx < \frac{(1+\varepsilon)}{\rho_1+\delta+1}\phi(r)r^{\rho_1+\delta+1}+0(1). \tag{3.3}$$

Again, from (3.2), we have

$$(1-\varepsilon)\int_{r_0}^r \phi(x)x^{\rho_1-1}\,dx < \int_{r_0}^r \frac{n(x)}{x}\,dx < (1+\varepsilon)\int_{r_1}^r \phi(x)x^{\rho_1-1}\,dx,$$

giving,

$$\frac{(1-\varepsilon)}{\rho_1}\phi(r)r^{\rho_1}+0(1) < N(r) < \frac{(1+\varepsilon)}{\rho_1}\phi(r)r^{\rho_1}+0(1). \tag{3.4}$$

Combining (3.3) and (3.4) leads to

$$\frac{-\rho_1}{(\rho_1+\delta+1)}\left[\frac{(1-\varepsilon)\phi(r)r^{\rho_1}+o(1)}{(1+\varepsilon)\phi(r)r^{\rho_1}}\right] > \frac{-1}{r^{\delta+1}N(r)}\int_0^r n(x)x^\delta\,dx$$

$$> \frac{-\rho_1}{(\rho_1+\delta+1)}\left[\frac{(1+\varepsilon)\phi(r)r^{\rho_1}+o(1)}{(1-\varepsilon)\phi(r)r^{\rho_1}}\right].$$

Taking exponentials and proceeding to limits, we have, since $\varepsilon$ is arbitrary and $n(r)\sim\phi(r)r^{\rho_1}$,

$$\lim_{r\to\infty}\exp\left\{\frac{-1}{r^{\delta+1}N(r)}\int_0^r n(x)x^\delta\,dx\right\} = \exp\left\{\frac{-\rho_1}{\rho_1+\delta+1}\right\}.$$

**Theorem 4.** *If $f(z)$ has at least one zero and $f(0)\neq 0$, then*

(i) $$e^{-1}\leqq p\leqq P\leqq 1;$$

(ii) $$P\geqq \exp\left(\frac{-\lambda_1}{\delta+1}\right).$$

**Proof.** (i) Integrating by parts the integral in (3.1), we get

$$\left\{\frac{g_\delta(r)}{G(r)}\right\}^{1/N(r)} = \exp\left\{-1+\frac{(\delta+1)}{r^{\delta+1}N(r)}\int_0^r N(x)x^\delta\,dx\right\}.$$

Since $N(r)$ is non-decreasing function of $r$, we get

$$p\geqq e^{-1} \quad\text{and}\quad P\leqq 1.$$

Since $n(r)$ is non-decreasing function of $r$, (3.1) gives

$$\left\{\frac{g_\delta(r)}{G(r)}\right\}^{1/N(r)} > \exp\left\{\frac{-n(r)}{(\delta+1)N(r)}\right\}.$$

But we know ([2], p. 17) that

$$\liminf_{r\to\infty}\frac{n(r)}{N(r)}\leqq\lambda_1,$$

and so

$$\limsup_{r\to\infty}\left[\left\{\frac{g_\delta(r)}{G(r)}\right\}^{1/N(r)}\right]\geqq\exp\left(\frac{-\lambda_1}{\delta+1}\right).$$

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