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On the geometric means of an integral function

by

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1.

Let $f(z)$ be an integral function of order ρ and lower order λ and

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log n(r)}{\log r} = \frac{\rho_1}{\lambda_1}, \tag{1.1}$$

$n(r)$ being the number of zeros of $f(z)$ in $|z| \leqq r$. Let $G(r)$ and $g_\delta(r)$ denote the geometric means of $|f(z)|$, defined as

$$G(r) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})| d\theta\right\} \tag{1.2}$$

and

$$g_\delta(r) = \exp\left\{\frac{(\delta+1)}{2\pi r^{\delta+1}}\int_0^r\int_0^{2\pi} \log|f(xe^{i\theta})| x^\delta \, dx \, d\theta\right\} \tag{1.3}$$

In this paper we have obtained some of the properties of $G(r)$ and $g_\delta(r)$. Let

$$N(r) = \int_0^r \frac{n(x)}{x} dx \tag{1.4}$$

and

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{n(r)}{r^\rho} = \frac{c}{d}. \tag{1.5}$$

Using Jensen's formula in (1.2), we have

$$\log G(r) = \log |f(0)| + \int_0^r \frac{n(x)}{x} dx. \tag{1.6}$$

From (1.1), we have, for any $\varepsilon > 0$ and $r > r_0 = r_0(\varepsilon)$

$$r^{\lambda_1-\varepsilon} < n(r) < r^{\rho_1+\varepsilon}.$$

Using this in (1.6), we have, for almost all values of $r > r_0$

$$r^{\lambda_1-1-\varepsilon}G(r) < G'(r) < r^{\rho_1-1+\varepsilon}G(r).$$

Again from (1.5), we have for any $\varepsilon > 0$ and $r > r_0$

$$(d-\varepsilon)r^\rho < n(r) < (c+\varepsilon)r^\rho.$$

Substituting for $n(r)$ from (1.6), we have, for almost all $r > r_0$

$$(d-\varepsilon)r^{\rho-1}G(r) < G'(r) < (c+\varepsilon)r^{\rho-1}G(r).$$

2.

We shall now obtain some of the properties of $g_\delta(r)$. We may write (1.3) as

$$\log g_\delta(r) = \frac{(\delta+1)}{r^{\delta+1}}\int_0^r \log G(x)x^\delta\, dx.$$

Now, using (1.4) and (1.6), we get

$$\log g_\delta(r) = 0(1)+ \frac{(\delta+1)}{r^{\delta+1}}\int_{r_0}^r N(x)x^\delta\, dx. \tag{2.1}$$

THEOREM 1. *Let $f(z)$ be an integral function of order $\rho(0 < \rho < \infty)$ and let $f(0) \neq 0$. Further,*

(i) *if*

$$\liminf_{r\to\infty} \frac{N(r)}{r^\rho} = \beta,$$

then

$$\liminf_{r\to\infty} \frac{\log g_\delta(r)}{r^\rho} \geqq \frac{\beta(\delta+1)}{(\rho+\delta+1)};$$

(ii) *if*

$$\limsup_{r\to\infty} \frac{N(r)}{r^\rho} = \alpha,$$

then

$$\limsup_{r\to\infty} \frac{\log g_\delta(r)}{r^\rho} \leqq \frac{\alpha(\delta+1)}{(\rho+\delta+1)}.$$

PROOF. (i) Since

$$\liminf_{r\to\infty} \frac{N(r)}{r^\rho} = \beta,$$

therefore, for any $\varepsilon > 0$ and $r > r_1 = r_1(\varepsilon)$, we have

$$N(r) > (\beta-\varepsilon)r^\rho,$$

and so from (2.1)

$$\log g_\delta(r) > 0(1)+ \frac{(\beta-\varepsilon)(\delta+1)}{(\rho+\delta+1)} (r^{\rho+\delta+1}-r_0^{\rho+\delta+1})r^{-\delta-1}.$$

Taking limit on both the sides leads to

$$\liminf_{r\to\infty} \frac{\log g_\delta(r)}{r^\rho} \geqq \frac{\beta(\delta+1)}{(\rho+\delta+1)}.$$

(ii) If

$$\limsup_{r\to\infty} \frac{N(r)}{r^\rho} = \alpha,$$

we have, for any $\varepsilon > 0$ and $r > r_1 = r_1(\varepsilon)$, $N(r) < (\alpha+\varepsilon)r^\rho$. Substituting this in (2.1), integrating and proceeding to limits, the result follows.

THEOREM 2. *Let $f(z)$ be an integral function of finite non-integral order ρ, and let*

$$\lim_{r\to\infty} \frac{\log g_\delta(r)}{r^\rho} = \nu \quad \textit{and} \quad \lim_{r\to\infty} \frac{\log G(r)}{r^\rho} = \mu,$$

then

$$\nu(\rho+\delta+1) = \mu(\delta+1).$$

PROOF. Since

$$\lim_{r\to\infty} \frac{\log g_\delta(r)}{r^\rho} = \nu,$$

therefore,

$$r^\rho(\nu-\varepsilon) < \log g_\delta(r) < r^\rho(\nu+\varepsilon), \quad \text{for} \quad r > r_0(\varepsilon).$$

Hence,

$$\begin{aligned}\frac{(\delta+1)}{r^{\delta+1}}\int_{(1-\eta)r}^{r} \log G(x)x^\delta\,dx &= \frac{(\delta+1)}{r^{\delta+1}}\int_{r_0}^{r} \log G(x)x^\delta\,dx \\ &\quad - \frac{(\delta+1)}{r^{\delta+1}}\int_{r_0}^{(1-\eta)r} \log G(x)x^\delta\,dx, \left(0 < \eta < \frac{1}{\delta+1}\right) \\ &= o(1)+ \log g_\delta(r)-(1-\eta)^{\delta+1} \log g_\delta\{(1-\eta)r\} \\ &> o(1)+\nu\{(\rho+\delta+1)\eta- \cdots\}r^\rho-\varepsilon\{2-(\rho+\delta+1)\eta+ \cdots\}r^\rho.\end{aligned}$$

But,

$$\frac{(\delta+1)}{r^{\delta+1}}\int_{(1-\eta)r}^{r} \log G(x)x^{\delta}\,dx \leqq \frac{(\delta+1)\log G(r)}{r^{\delta+1}}\int_{(1-\eta)r}^{r} x^{\delta}\,dx$$

$$< \frac{(\delta+1)\eta\log G(r)}{1-(\delta+1)\eta}, \quad \text{for}\to(\delta+1)\eta < 1.$$

Hence,

$$\frac{\log G(r)}{r^{\rho}} > o(1)+\frac{\nu\{(\rho+\delta+1)\eta-\cdots\}\{1-(\delta+1)\eta\}}{(\delta+1)\eta}$$

$$-\frac{\varepsilon\{2-(\rho+\delta+1)\eta+\cdots\}\{1-(\delta+1)\eta\}}{(\delta+1)\eta.}.$$

Since η is arbitrary, we get

$$\liminf_{r\to\infty}\frac{\log G(r)}{r^{\rho}} \geqq \frac{\nu(\rho+\delta+1)}{(\delta+1)}. \tag{2.2}$$

Further,

$$\frac{(\delta+1)}{r^{\delta+1}}\int_{r}^{(1+\eta)r}\log G(x)x^{\delta}\,dx = \frac{(\delta+1)}{r^{\delta+1}}\int_{r_0}^{(1+\eta)r}\log G(x)x^{\delta}\,dx$$

$$-\frac{(\delta+1)}{r^{\delta+1}}\int_{r_0}^{r}\log G(x)x^{\delta}\,dx < o(1)+\nu\{(\rho+\delta+1)\eta+\cdots\}$$

$$\times r^{\rho}+\varepsilon\{2+(\rho+\delta+1)\eta+\cdots\}r^{\rho}, \quad \text{for} \quad \eta<1,$$

but,

$$\frac{(\delta+1)}{r^{\delta+1}}\int_{r}^{(1+\eta)r}\log G(x)x^{\delta}\,dx > (\delta+1)\eta\log G(r),$$

hence,

$$\frac{\log G(r)}{r^{\rho}} < o(1)+\frac{\nu\{(\rho+\delta+1)\eta+\cdots\}}{(\delta+1)\eta}+\frac{\varepsilon\{2+(\rho+\delta+1)\eta+\cdots\}}{(\delta+1)\eta}.$$

Therefore we get

$$\limsup_{r\to\infty}\frac{\log G(r)}{r^{\rho}} \leqq \frac{\nu(\rho+\delta+1)}{(\delta+1)}. \tag{2.3}$$

Combining (2.2) and (2.3), we get the result.

3.

Combining (1.2) and (1.3), we obtain

$$\frac{g_{\delta}(r)}{G(r)} = \exp\left\{\frac{-1}{r^{\delta+1}}\int_{0}^{r} x^{\delta+1}\frac{d}{dx}\left(\log G(x)\right)dx\right\}.$$

Using (1.6) in this, we get

$$\left\{\frac{g_\delta(r)}{G(r)}\right\}^{1/N(r)} = \exp\left\{\frac{-1}{r^{\delta+1}N(r)}\int_0^r n(x)x^\delta\,dx\right\}. \tag{3.1}$$

Let us set

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left[\left\{\frac{g_\delta(r)}{G(r)}\right\}^{1/N(r)}\right] = \begin{matrix}P\\ p\end{matrix}.$$

We now prove the following:

THEOREM 3. *If $f(z)$ is an integral function of order $\rho(0 < \rho < \infty)$ and $f(0) \neq 0$, such that $n(r) \sim \phi(r)r^{\rho_1}$, where $\phi(r)$ is a positive continuous and indefinitely increasing function of r and $\phi(cr) \sim \phi(r)$ as $r \to \infty$ for every constant $c > 0$, then*

$$P = p = \exp\left\{\frac{-\rho_1}{\rho_1+\delta+1}\right\}.$$

PROOF. Since $n(r) \sim \phi(r)r^{\rho_1}$ we have for any $\varepsilon > 0$ and $r \geqq r_0(\varepsilon)$

$$(1-\varepsilon)\phi(r)r^{\rho_1} < n(r) < (1+\varepsilon)\phi(r)r^{\rho_1} \tag{3.2}$$

or

$$(1-\varepsilon)\int_{r_0}^r \phi(x)x^{\rho_1+\delta}\,dx < \int_{r_0}^r n(x)x^\delta\,dx < (1+\varepsilon)\int_{r_0}^r \phi(x)x^{\rho_1+\delta}\,dx.$$

Now, by Lemma V([1], p. 54),

$$\int_{r_0}^r \phi(u)u^{\alpha-1}\,du \sim \frac{\phi(r)r^\alpha}{\alpha},$$

for every positive α, and so we get

$$\begin{aligned}\frac{(1-\varepsilon)}{\rho_1+\delta+1}\phi(r)r^{\rho_1+\delta+1}+0(1) &< \int_0^r n(x)x^\delta\,dx \\ &< \frac{(1+\varepsilon)}{\rho_1+\delta+1}\phi(r)r^{\rho_1+\delta+1}+0(1).\end{aligned} \tag{3.3}$$

Again, from (3.2), we have

$$(1-\varepsilon)\int_{r_0}^r \phi(x)x^{\rho_1-1}\,dx < \int_{r_0}^r \frac{n(x)}{x}\,dx < (1+\varepsilon)\int_{r_1}^r \phi(x)x^{\rho_1-1}\,dx,$$

giving,

$$\frac{(1-\varepsilon)}{\rho_1}\phi(r)r^{\rho_1}+0(1) < N(r) < \frac{(1+\varepsilon)}{\rho_1}\phi(r)r^{\rho_1}+0(1). \tag{3.4}$$

Combining (3.3) and (3.4) leads to

$$\frac{-\rho_1}{(\rho_1+\delta+1)}\left[\frac{(1-\varepsilon)\phi(r)r^{\rho_1}+o(1)}{(1+\varepsilon)\phi(r)r^{\rho_1}}\right] > \frac{-1}{r^{\delta+1}N(r)}\int_0^r n(x)x^\delta\,dx$$
$$> \frac{-\rho_1}{(\rho_1+\delta+1)}\left[\frac{(1+\varepsilon)\phi(r)r^{\rho_1}+o(1)}{(1-\varepsilon)\phi(r)r^{\rho_1}}\right].$$

Taking exponentials and proceeding to limits, we have, since ε is arbitrary and $n(r) \sim \phi(r)r^{\rho_1}$,

$$\lim_{r\to\infty}\exp\left\{\frac{-1}{r^{\delta+1}N(r)}\int_0^r n(x)x^\delta\,dx\right\} = \exp\left\{\frac{-\rho_1}{\rho_1+\delta+1}\right\}.$$

THEOREM 4. *If $f(z)$ has at least one zero and $f(0)\neq 0$, then*

(i) $$e^{-1}\leqq p \leqq P \leqq 1;$$

(ii) $$P \geqq \exp\left(\frac{-\lambda_1}{\delta+1}\right).$$

PROOF. (i) Integrating by parts the integral in (3.1), we get

$$\left\{\frac{g_\delta(r)}{G(r)}\right\}^{1/N(r)} = \exp\left\{-1+\frac{(\delta+1)}{r^{\delta+1}N(r)}\int_0^r N(x)x^\delta\,dx\right\}.$$

Since $N(r)$ is non-decreasing function of r, we get

$$p \geqq e^{-1} \quad\text{and}\quad P\leqq 1.$$

Since $n(r)$ is non-decreasing function of r, (3.1) gives

$$\left\{\frac{g_\delta(r)}{G(r)}\right\}^{1/N(r)} > \exp\left\{\frac{-n(r)}{(\delta+1)N(r)}\right\}.$$

But we know ([2], p. 17) that

$$\liminf_{r\to\infty}\frac{n(r)}{N(r)} \leqq \lambda_1,$$

and so

$$\limsup_{r\to\infty}\left[\left\{\frac{g_\delta(r)}{G(r)}\right\}^{1/N(r)}\right] \geqq \exp\left(\frac{-\lambda_1}{\delta+1}\right).$$

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