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V

by

Pierre Robert

1. Introduction

This paper is the fifth of a series published under the same title and numbered I, II, ... The reader is assumed to be familiar with the definitions, notations and results of Parts I to IV.

The problem studied by C. A. Swanson and M. Schulzer in [32] and [33] is that of the existence and the approximation of the solutions of a class of equations in Banach spaces.

In this Part we generalize Theorems 4 and 5 of [33] to arbitrary V-algebras and V-spaces. The hypotheses of [33] are slightly weakened.

2. Equations in V-algebras

In this Section, X is a V-algebra.

We consider two points, $x, y \in X$ which have the following finite or infinite expansions:

$$x = x_0+x_1+x_2+ \cdots,$$
$$y = y_0+y_1+y_2+ \cdots,$$

and we assume that x_0 admits a pseudo-inverse x_0^{-1} such that

$$\text{(V.1)} \qquad |x-x_0| < |x_0^{-1}|^{-1}.$$

It follows from Theorem II-6.6 that:

THEOREM 2.1. The element x admits a pseudo-inverse x^{-1} and the equation $xw = y$ admits a pseudo-solution $z = x^{-1}y$ (i.e. $xz = y$).

The problem is to make use of the known expansions of x and y to obtain approximations to z and x^{-1}, as defined in the above theorem. The sequences $\{z_n\}$ and $\{u_n\}$ defined by

$$\text{(V.2)} \qquad z_0 = x_0^{-1} y_0,\ z_n = x_0^{-1}\Big(\sum_{i=0}^{n} y_i - \sum_{i=1}^{n} x_i z_{n-i}\Big),$$

$$\text{(V.3)} \qquad u_0 = x_0^{-1},\ u_n = x_0^{-1}\Big(e - \sum_{i=1}^{n} x_i u_{n-i}\Big),$$

will be shown to approximate z and x^{-1}, respectively, provided the rates of convergence of the series $\sum x_n$ and $\sum y_n$ satisfy certain conditions.

More precisely, we shall consider two sets of assumptions on the rates of convergence of the series $\sum x_n$ and $\sum y_n$ and, under these assumptions, we shall obtain upper bounds for the values of $|z-z_n|$ and $|x^{-1}-u_n|$.

In the first case we assume that

$$\text{(V.4a)} \quad |x_n| \leqq \rho^{-n}|x_0^{-1}|^{-1} \quad \text{for } n \geqq 1,$$

$$\text{(V.4b)} \quad |y_n| \leqq \rho^{-n}|y_0| \quad \text{for } n \geqq 1.$$

In the second case, our assumptions are that

$$\text{(V.5a)} \quad |x_0| \geqq |x_1| \geqq |x_2| \geqq \ldots,$$

$$\text{(V.5b)} \quad |x_n|\cdot|x_m| \leqq |x_{m+n}|\cdot|x_0^{-1}|^{-1} \text{ for all } n,\ m \geqq 1 \text{ such that } |x_{m+n}| \neq 0,$$

$$\text{(V.5c)} \quad |y_0| \geqq |y_n| \text{ for all } n \geqq 1,$$

$$\text{(V.5d)} \quad |y_n| \leqq |x_{n-1}|\cdot|x_0^{-1}|\cdot \operatorname{Max}\{|x_0|, |y_0|\} \text{ for all } n \geqq 1.$$

The interest of the second case lies in its applicability in V-algebras which admit distinguished bases with many elements having the same norm (e.g. the V-algebra $\mathscr{K}_0$ of III-5). In such cases, the norms of the terms in the expansions of x or y will not necessarily decrease as rapidly as required by (V.4), and to sum up the terms having the same norms may be inconvenient or difficult.

THEOREM 2.2. (i) If (V.4a) and (V.4b) hold, then the sequence $\{z_n\}$ defined by (V.2) converges to z and

$$\text{(V.6)} \quad |z-z_n| \leqq \rho^{-n}|x_0^{-1}||y_0| \text{ for all } n = 0, 1, 2, \ldots.$$

(ii) If (V.5a), (V.5b), (V.5c) and (V.5d) hold, then

$$\text{(V.7)} \quad |z-z_n| \leqq |x_n||x_0^{-1}|^2 \operatorname{Max}\{|x_0|, |y_0|\} \text{ for all } n \text{ such that } |x_n| \neq 0;$$

if for all integers n, $|x_n| \neq 0$, then $\{z_n\}$ converges to z.

Before proving the theorem, we note that if $y_0 \equiv e$ and $y_n \equiv \theta$ for all $n \geqq 1$, then (V.4b), (V.5c) and (V.5d) are satisfied and, hence, the following corollary is deduced from Theorem 2.2:

COROLLARY 2.3. (i) If (V.4a) holds, then the sequence $\{u_n\}$ defined by (V.3) converges to x^{-1} and

$$|x^{-1}-u_n| \leqq \rho^{-n}|x_0^{-1}| \text{ for all } n = 0, 1, 2, \ldots.$$

(ii) If (V.5a) and (V.5b) hold, then

$$|x^{-1}-u_n| \leqq |x_n||x_0^{-1}|^2 \operatorname{Max}\{|x_0|, 1\} \text{ for all } n \text{ such that } |x_n| \neq 0;$$

if for all integers n, $|x_n| \neq 0$, then $\{u_n\}$ converges to x^{-1}.

PROOF OF THEOREM 2.2. One verifies directly that

$$z = x_0^{-1}[y-(x-x_0)z] = x_0^{-1}[\sum_{i\geqq 0} y_i - (\sum_{i\geqq 1} x_i)z].$$

Thus,

$$|z-z_n| = |x_0^{-1}(\sum_{i\geqq n+1} y_i) - x_0^{-1}(\sum_{i\geqq n+1} x_i)z - x_0^{-1}(\sum_{i=1}^{n} x_i(z-z_{n-i}))|,$$

and

$$\text{(V.8)} \quad \begin{aligned} &|z-z_n| \leqq \operatorname{Max}\{\alpha_n, \beta_n, \gamma_n\}, \text{ where} \\ &\alpha_n = |x_0^{-1}|\cdot|\sum_{i\geqq n+1} y_i|, \\ &\beta_n = |x_0^{-1}|\cdot|\sum_{i\geqq n+1} x_i|\cdot|z|, \\ &\gamma_n = |x_0^{-1}|\cdot|\sum_{i=1}^{n} x_i(z-z_{n-i})|. \end{aligned}$$

Both (V.4b) and (V.5c) imply $|y| \leqq |y_0|$; hence, from Theorem II-6.6(ii) and the relation $1 \leqq |x_0||x_0^{-1}|$:

$$\text{(V.9)} \quad \begin{aligned} &|z| \leqq |x^{-1}||y| \leqq |x_0^{-1}||y_0|, \\ &|z-z_0| \leqq |z-x_0^{-1}y_0| \leqq |x_0^{-1}|\cdot|y_0| \leqq |x_0||x_0^{-1}|^2 \operatorname{Max}\{|x_0|, |y_0|\}. \end{aligned}$$

This shows that both (V.6) and (V.7) are satisfied for $n = 0$. We complete the proof by induction, and for each set of assumptions separately.

(i) In the first case, suppose that (V.6) is satisfied for $n = 0, 1, 2, \ldots, m-1$.

From (V.4b) and Theorem I-3.2 (ii):

$$\alpha_m \leqq |x_0^{-1}||y_{m+1}| \leqq \rho^{-m-1}|x_0^{-1}||y_0|;$$

from (V.4a), Theorem I-3.2 (ii) and (V.9):

$$\beta_m \leqq |x_0^{-1}||x_{m+1}||z| \leqq \rho^{-m-1}|x_0^{-1}||y_0|;$$

from (V.2a) and the induction hypothesis:

$$\gamma_m \leqq |x_0^{-1}| \operatorname*{Max}_{1\leqq i\leqq m} \{|x_i||z-z_{m-i}|\} \leqq \rho^{-m}|x_0^{-1}||y_0|.$$

It follows from (V.8) and these three inequalities that (V.6) holds for $n = m$ and hence for all n.

The convergence of ρ^{-n} to 0 implies that $\lim_{n\to\infty}|z-z_n| = 0$ and, consequently, $\{z_n\}$ converges to z.

(ii) In the second case, we note that (V.5a) implies that when $|x_m| \neq 0$, then $|x_n| \neq 0$ for $n = 0, 1, \ldots, m-1$. Suppose that (V.7) holds for $n = 0, 1, \ldots, m-1$. Then an argument similar to that conducted in the first case shows that (V.7) holds also for $n = m$.

If for each integer n, $|x_n| \neq 0$, the convergence of the series $\sum x_n$ implies, as in the first case, the convergence of $\{z_n\}$ to z.

Application. Let Z be an arbitrary V-space. Let $\{A_n : n = 0, 1, 2, \ldots\}$ be a sequence of linear operators in the V-algebra $\mathscr{I}(Z)$.

Assume that, in the norm of $\mathscr{I}(Z)$,

$$A = A_0+A_1+A_2+\ldots.$$

Assume also that A_0 is pseudo-regular, with pseudo-inverse A_0^{-1}, and that

$$\text{(V.10)} \qquad |A-A_0| < |A_0^{-1}|^{-1},$$

$$\text{(V.11)} \qquad |A_n| \leqq \rho^{-n}|A_0^{-1}|^{-1}.$$

Under these assumptions the equation

$$\text{(V.12)} \qquad Az = w, \quad z, w \in Z$$

has a solution z for each $w \in Z$.

Indeed, from Corollary 2.3, A has a pseudo-inverse A^{-1}, so that $z = A^{-1}w$ is a solution of (V.12).

Furthermore, it follows from (V.3) that A^{-1} is a limit of the sequence $\{B_n\}$:

$$B_0 = A_0^{-1}, \; B_n = A_0^{-1}(I - \sum_{i=1}^{n} A_i B_{n-i})$$

and, by the linearity of the operators A_0^{-1}, z is a limit of the sequence $\{z_n\}$:

$$z_0 = A_0^{-1}w, \; w_n = B_n w = A_0^{-1}(w - \sum_{i=1}^{n} A_i w_{n-i}).$$

(Compare this result with the results of Section 4 below.)

3. The equation $Ax = y$

In this section, X and Y are V-spaces, $A \in \mathcal{O}(X, Y)$.

DEFINITION 3.1. Let $y \in Y$ and $D \subset X$.

(i) The equation $Ax = y$ is said to have the *pseudo-solution* z in D if $z \in D$ and $Az = y$.

(ii) The equation $Ax = y$ is said to have a *unique pseudo-solution* in D if it has at least one pseudo-solution z in D and if $z' = z$ for all pseudo-solutions in D.

We consider the linear operator $A_0 \in \mathscr{I}(X, Y)$ and assume that A_0 has a bounded pseudo-inverse A_0^{-1} on its range $A_0(X)$. The operator A_0^{-1} is linear.

THEOREM 3.2. Let $y_0 \in A_0(X)$ and $u = A_0^{-1}y_0$. If there exists a ball $D = S'(u, r)$, $r > 0$, such that $K = |A_0^{-1}|_{A_0(D)}$ and such that the following conditions (V.13) and (V.14) are satisfied:

$$|A - A_0|_D < K^{-1}; \tag{V.13}$$

$$(A - A_0)x \in S'(\theta, rK^{-1}) \subset A_0(X) \quad \text{for all } x \in D; \tag{V.14}$$

then, for all $y \in A_0(D)$ the equation $Ax = y$ has a unique pseudo-solution z in D. Furthermore, the sequence $\{z_n\}$ defined by

$$z_0 = A_0^{-1}y, \; z_n = A_0^{-1}y - A_0^{-1}(A - A_0)z_{n-1}, \tag{V.15}$$

converges to z.

PROOF: Let $y' = y - y_0$. Since $y \in A_0(D)$, $A_0^{-1}y \in D$ and

$$|A_0^{-1}y'| = |A_0^{-1}y - u| \leqq r. \tag{V.16}$$

Since, from (V.14), $(A - A_0)x \in A_0(X)$ for all $x \in D$, the equation

$$Ax = y = y_0 + y' \tag{V.17}$$

is equivalent, for $x \in D$, to the equation

$$Lx = x, \tag{V.18}$$

where

$$Lx = u + A_0^{-1}y' - A_0^{-1}(A - A_0)x, \quad x \in D.$$

From (V.14)

$$|A_0^{-1}(A - A_0)x| \leqq K \cdot rK^{-1} = r \quad \text{for all } x \in D.$$

From this inequality and (V.16), it follows that

$$|Lx - u| = |A_0^{-1}y' - A_0^{-1}(A - A_0)x| \leqq r \text{ for all } x \in D.$$

Thus, L maps D into itself.

From (V.13), we have, for all $x_1, x_2 \in D$:

$$\begin{aligned}|Lx_1-Lx_2| &= |A_0^{-1}(A-A_0)x_1-A_0^{-1}(A-A_0)x_2| \\ &\leqq K\cdot|A-A_0|_D|x_1-x_2| < |x_1-x_2|.\end{aligned}$$

Since 0 is the only accumulation point of the norm range of a V-space, it follows that

$$|L|_D \leqq \rho^{-1} < 1.$$

The contraction mapping principle ([19], Vol. I, p. 43) can be applied to L on the closed sphere D, to conclude that the equation (V.17) and, hence, the equation (V.18) have a unique pseudo-solution z in D.

The contraction mapping principle also asserts that the sequence $\{z_n\}$ defined by

$$z_0 = A_0^{-1}y, \quad z_n = Lz_{n-1}$$

converges to the pseudo-solution z.

Since $K = |A_0^{-1}|_{A_0(D)} \leqq |A_0^{-1}|_{A_0(X)}$ and $|A-A_0|_D \leqq |A-A_0|_X$, we see that the theorem holds if, in (V.13), $|A-A_0|_D$ is replaced by $|A-A_0|_X$ and/or if, in one or both of (V.13) and (V.14), K is replaced by $|A^{-1}|_{A_0(X)}$.

This theorem extends Theorem 4 of [33] (Th. 7.1 of [32]) to arbitrary V-spaces.

APPLICATION. For some integer $k \geqq 1$, let $X = Y = \mathscr{P}_k$, where $\mathscr{P}_k$ is defined in Section 4 of Part III.

We consider an operator $F \in \mathscr{O}(\mathscr{P}_k)$ such that

(V.19) $$0 < |I-F| < 1.$$

(Examples of such operators are $F = F_n$ where $F_nx = x+x^n$, $n = 2, 3, \ldots$, or $F_nx = x(1+\varphi_n)$, $n = 1, 2, \ldots$).

Let $\mathscr{L}$ be the operator defined in Section 6 of Part IV; namely:

$$(\mathscr{L}x)(\lambda) = \int_0^\infty \frac{1}{\lambda} e^{-t/\lambda} x(t)\, dt.$$

Consider the equation

(V.20) $$y+\mathscr{L}Fx = \alpha x,$$

(i.e.: $y(\lambda)+\int_0^\infty(1/\lambda)e^{-t/\lambda}F(x(t))dt = \alpha x(\lambda)$), where $y \in \mathscr{P}_k$ and α is a real number.

We shall apply Theorem 3.2 to prove that (V.20) has a unique pseudo-solution in $\mathscr{P}_k$ when

(V.21) $$\alpha \neq n! \quad \text{for each integer } n \geqq k.$$

Define, for $x \in \mathscr{P}_k$:

$$A x = \alpha x - \mathscr{L} F x = (\alpha I - \mathscr{L} F) x$$
$$A_0 x = \alpha x - \mathscr{L} x = (\alpha I - \mathscr{L}) x$$

The equation (V.20) is equivalent to the equation

(V.22) $$A x = y$$

It follows from the results of Section 6, Part IV, that if $\alpha \neq n!$ for each integer $n \geqq k$, $A_0 = \alpha I - \mathscr{L}$ is pseudo-regular and that its pseudo-inverse A_0^{-1} is defined on all of $\mathscr{P}_k$, with $|A_0^{-1}| = |A_0| = 1$.

To apply Theorem 3.2, select $y_0 = u = 0$ and $r = \rho^{-k}$. Then, $D = \mathscr{P}_k$.

Since $A - A_0 = \mathscr{L}(I - F)$ and $|\mathscr{L}| = 1$, we have from (V.19)

$$|A - A_0| \leqq |\mathscr{L}| |I - F| < 1 = |A_0^{-1}|^{-1},$$

and hence, (V.13) is satisfied.

Clearly (V.14) is also satisfied since $(A - A_0)$ maps $\mathscr{P}_k$ into itself and since $|A_0^{-1}| = 1$.

The conclusion is that (V.22) and (V.20) have a unique pseudo-solution z in $\mathscr{P}_k$ when (V.21) holds. Furthermore, z is a limit of the sequence $\{z_n\}$ defined by

$$z_0 = A_0^{-1} y, \quad z_n = A_0^{-1} y - A_0^{-1} \mathscr{L}(I - F) z_{n-1}.$$

Other examples of applications of Theorem 3.2 will be found in [32].

4. The equation $Ax = y$ involving expansions of A and y

As in the previous Section, X and Y are V-spaces, $A \in \mathcal{O}(X, Y)$ and we consider the equation $Ax = y$. However, we now suppose that A and y are known from their finite or infinite expansions

$$A = A_0 + A_1 + A_2 + \dots$$
$$y = y_0 + y_1 + y_2 + \dots$$

We assume that $A_n \in \mathcal{O}(X, Y)$ for $n = 1, 2, \dots$; that $A_0 \in \mathscr{I}(X, Y)$ and that $y_0 \in A_0(X)$. We also assume that A_0 has a pseudo-inverse A_0^{-1} on its range $A_0(X)$. Let $u = A_0^{-1} y_0$.

Suppose that there exists a ball $D = S'(u, r)$, $r > 0$, such that $K = |A_0^{-1}|_D$ and such that the following conditions (V.23)—(V.26) are satisfied:

(V.23) $$|A - A_0|_D < K^{-1};$$

(V.24) $|A_n x| \leqq rK^{-1} \min \{1, |A_n|_D\}$ for all $n \geqq 1$ and all $x \in D$;

(V.25) $$|y_n| \leqq rK^{-1} \text{ for all } n \geqq 1;$$

(V.26) In Y, the ball $S'(\theta, rK^{-1})$ is contained in $A_0(X)$.

THEOREM 4.1. Under the conditions above, the equation $Ax = y$ has a unique pseudo-solution z in D.

PROOF: The convergence on D of the series $\sum_{n \geqq 0} A_n$ implies that $\lim_{n\to\infty} |A_n|_D = 0$. Therefore, from (V.24), $\lim_{n\to\infty}|A_n x| = 0$ for all $x \in D$ and, hence, the series $\sum_{n \geqq 1} A_n x$ is convergent on D. Then, it follows, also from (V.24) that

(V.27) $$|(A-A_0)x| = |\sum_{n\geqq 1} A_n x| \leqq rK^{-1} \text{ for all } x \in D.$$

A consequence of (V.25) and (V.26) is that $y_n \in A_0(X)$ for all $n \geqq 1$ and that
$$|A_0^{-1} y_n| \leqq K \cdot rK^{-1} = r.$$

From the linearity of A_0 we conclude that $y \in A_0(X)$ and the last inequality gives
$$|A_0^{-1}y - u| = |A_0^{-1}(y-y_0)| = |\sum_{n\geqq 1} A_0^{-1} y_n| \leqq r.$$
Hence:

(V.28) $$y \in A_0(D).$$

The relations (V.23), (V.24), (V.27) and (V.28) establish the applicability of Theorem 3.2. Thus, $Ax = y$ has a unique pseudo-solution z in D.

As in Theorem 2.2, we now seek an approximation to the pseudo-solution z. We consider the sequence $\{z_n\}$ defined by

(V.29) $$z_0 = A_0^{-1} y_0, \; z_n = A_0^{-1}(\sum_{i=0}^{n} y_i - \sum_{i=1}^{n} A_i z_{n-i}).$$

The existence of this sequence is guaranteed by the following lemma.

LEMMA 4.2. Let $u_n = \sum_{i=0}^{n} y_i - \sum_{i=1}^{n} A_i z_{n-i}$, $n = 1, 2, \ldots$. The domain of A_0^{-1} contains all u_n, $n = 1, 2, \ldots$ and $z_n \in D$ for all $n = 0, 1, 2, \ldots$.

PROOF: Clearly $z_0 \in D$. Suppose that $z_i \in D$ for $i = 0, 1, 2, \ldots$, $n-1$. Then from (V.24)
$$|A_i z_{n-i}| \leqq rK^{-1} \text{ for } i = 1, 2, \ldots, n,$$
and (V.26) implies that $A_i z_{n-i} \in A_0(X)$ for $i = 1, 2, \ldots, n$.

It was just shown that $y_i \in A_0(X)$ for all $i \geqq 0$. Hence, $u_i \in A_0(X)$ for $i = n$. This induction shows that $u_i \in A_0(X)$ for all $i \geqq 1$, provided $z_i \in D$ for all $i \geqq 0$.

By induction, $z_i \in D$ for all $i \geqq 0$, since by (V.25) and the above inequality:

$$|A_0^{-1}u_n - A_0^{-1}y_0| \leqq K|\sum_{i=1}^{n}(y_i - A_i z_{n-i})| \leqq K \cdot rK^{-1} = r.$$

In Theorem 4.3 we show that z_n is an approximation to z and we give an upper bound for $|z-z_n|$. As in Theorem 2.2, the degree of this approximation depends on the rates of convergence of the series $\sum_{n\geqq 0} A_n$ and $\sum_{n\geqq 0} y_n$. In that respect, we make two distinct sets of additional assumptions on A_n and y_n. First:

(V.30a) $$|A_n|_D \leqq \rho^{-n}K^{-1} \text{ for } n \geqq 1,$$

(V.30b) $$|y_n| \leqq r\rho^{-n+1}K^{-1} \text{ for } n \geqq 1;$$

secondly:

(V.31a) $$|A_0|_D \geqq |A_1|_D \geqq |A_2|_D \geqq \dots,$$

(V.31b) $$|A_m|_D |A_n|_D \leqq K^{-1}|A_{m+n}|_D \text{ for all } m \geqq 1 \text{ and } n \geqq 1 \text{ such that } |A_{m+n}| \neq 0,$$

(V.31c) $$|y_0| \geqq |y_n| \text{ for all } n \geqq 1,$$

(V.31d) $$|y_n| \leqq rK^{-1} \min\{1, |A_{n-1}|_D\} \text{ for all } n \geqq 1.$$

Assumptions (V.30b) and (V.31d) imply (V.25).

THEOREM 4.3. (i) (V.30a) and (V.30b) hold, then the sequence $\{z_n\}$ defined by (V.29) converges to z and

(V.32) $$|z-z_n| \leqq r\rho^{-n} \operatorname{Max}\{1, K^{-1}\} \text{ for } n = 0, 1, 2, \dots.$$

(ii) If (V.31a), (V.31b), (V.31c) and (V.31d) hold, then

(V.33) $$|z-z_n| \leqq r|A_n|_D \operatorname{Max}\{1, K^{-1}\} \text{ for all } n \text{ such that } |A_n|_D \neq 0;$$

if for each integer $n \geqq 0$, $|A_n|_D \neq 0$, then the sequence $\{z_n\}$ converges to z.

PROOF: From (V.27), it follows that $(A-A_0)z \in A_0(X)$. Hence $[y-(A-A_0)z]$ belongs to the domain of A_0^{-1}. It may be verified directly that

(V.34) $$z = A_0^{-1}[y-(A-A_0)z] = A_0^{-1}(\sum_{n\geqq 0} y_n - \sum_{n\geqq 1} A_n z).$$

From the definition of $\{z_n\}$ and the linearity of A_0^{-1}, we have, for $n = 1, 2, 3, \ldots$

$$|z-z_n| = |A_0^{-1}(\sum_{i\geqq n+1} y_i)-A_0^{-1}(\sum_{i\geqq n+1} A_i z)-A_0^{-1}(\sum_{i=1}^{n} (A_i z-A_i z_{n-i}))|.$$

Hence,

$$|z-z_n| \leqq \mathrm{Max}\ \{\alpha_n, \beta_n, \gamma_n\} \text{ for } n = 1, 2, \ldots,$$

where

$$\alpha_n = |A_0^{-1}|_D \cdot |\sum_{i\geqq n+1} y_i|,$$

$$\beta_n = |A_0^{-1}|_D \cdot |\sum_{i\geqq n+1} A_i z|,$$

and, since $z \in D$ and $z_i \in D$ for all $i \geqq 0$ (Lemma 4.2),

$$\gamma_n = |A_0^{-1}|_D \cdot \mathrm{Max}_{1\leqq i\leqq n}\ \{|A_i|_D \cdot |z-z_{n-i}|\}.$$

Since $1 \leqq |A_0|_D \cdot |A_0^{-1}|_D$,

$$|z-z_0| \leqq r \leqq r\ \mathrm{Max}\ \{1, K^{-1}\},$$

$$|z-z_0| \leqq r \leqq r|A_0|_D\ \mathrm{Max}\ \{1, K^{-1}\}.$$

So, (V.32) and (V.33) are both satisfied for $n = 0$.

The rest of the proof is conducted, for each set of assumptions (V.30) and (V.31), by induction, and exactly as in Theorem 2.2. We omit this later part of the proof.

It is easily verified that Theorems 4.1, 4.3 and Lemma 4.2 hold if in the hypotheses (V.23), (V.24), (V.30a), (V.31a), (V.31b), (V.31d) and the estimate (V.33), we change all norms on $D(|A_n|_D)$ to norms on $X(|A_n|_X)$.

If we assume that $X = Y$ and that all operators A_n are linear, and that the above change to norms on X is made, the results of Theorem 4.1 and 4.3 are refinements of those of the Application of Theorem 2.2.

Applications can be found in [32].

(Oblatum 18–3–66)