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On some non-Archimedean normed linear spaces IV

by

Pierre Robert

Introduction

This paper is the fourth of a series published under the same title and numbered I, II, ... The reader is assumed to be familiar with the definitions, notations and results of Parts I, II and III.

This Part is devoted to the study of bounded linear and non-linear operators on V-spaces.

We recall two conventions established in previous Parts:

(i) In a V-space the relation $x = y$ means that $|x-y| = 0$; strict identity between x and y is indicated by symbol "$x \equiv y$".

(II) Whenever two or more V-spaces will be considered simultaneously, it will be assumed that the real number ρ which serves to establish the norms is the same for all of these spaces (See Definition II-1.1).

1. Definitions and notations

Unless otherwise specified, X and Y will denote two V-spaces over the same field of scalars F; Z will denote a closed subset of A. An operator from Z to Y is a single valued mapping defined on all of Z with range in Y.

DEFINITION 1.1. An operator A from Z to Y is said to be *linear on* Z if

$$A(\alpha u+\beta v) = \alpha Au+\beta Av$$

for all α, $\beta \in F$ and all $u, v \in Z$ such that $\alpha u+\beta v \in Z$.

DEFINITION 1.2. Let A be an operator from Z to Y.

(i) The *norm of* A *on* Z, denoted by $|A|_Z$ is defined by

$$\text{(IV.1)} \quad |A|_Z = \inf\{M \geqq 0 : |Au-Av| \leqq M|u-v| \text{ for all } u, v \in Z\}.$$

(ii) If $Z = X$, *the norm of* A *on* X is denoted by $|A|$, i.e. $|A|_X = |A|$.

(iii) A is said to be *bounded on* Z if $|A|_Z < \infty$.

It follows that $|A|_Z = 0$ if and only if for some fixed $y \in Y$ and all $u \in Z$, $|Au-y| = 0$.

If Z is a linear subspace of X and A is linear on Z, (IV.1) is equivalent to

$$\text{(IV.2)} \quad |A|_Z = \inf \{M \geqq 0 : |Au| \leqq M|u| \text{ for all } u \in Z\}.$$

In X, the balls $S(\theta, r)$, $S'(\theta, r)$ are closed subspaces of X and the quotient spaces $X/S(\theta, r)$, $X/S'(\theta, r)$ are discrete V-spaces (See Theorem I-2.2). Consequently, the norm on X of an operator, even a linear operator, cannot be determined by consideration of its values on these balls only (unless, of course, $X = S(\theta, r)$ or $X = S'(\theta, r)$). This is in striking contrast to the case of a linear operator on a Banach space ([7], [36]) where

$$||A|| = \inf \{M \geqq 0 : ||Ax|| \leqq M||x|| \text{ for all } x \in S'(\theta, r)\}.$$

In a V-space, if $Z \supset Z'$, then $|A|_Z \geqq |A|_{Z'}$.

$\mathcal{O}(Z, Y)$ will denote the set of all bounded operators from Z to Y. $\mathcal{I}(Z, Y)$ will denote the set of all bounded linear operators from Z to Y.

If $Z = X = Y$, we shall use the notations $\mathcal{O}(X)$ and $\mathcal{I}(X)$ in place of $\mathcal{O}(X, X)$ and $\mathcal{I}(X, X)$.

The product AB of two elements A, B of $\mathcal{O}(X)$ is defined by $(AB)x \equiv A(Bx)$ for all $x \in X$. It is simple to verify that

$$\text{(IV.3)} \qquad |AB| \leqq |A| \cdot |B|.$$

In general, the product of non-linear operators does not satisfy conditions (II.9) and (II.10) of Definition II-6.1. and hence $\mathcal{O}(X)$ is not an algebra. These conditions are satisfied for linear operators and hence $\mathcal{I}(X)$ is a non-commutative subalgebra of the space $\mathcal{O}(X)$.

0 will denote the zero-operator in $\mathcal{O}(Z, Y) : 0u \equiv \theta \in Y$ for all $u \in Z$. I will denote the identity operator in $\mathcal{O}(X)$, i.e. $Ix \equiv x$ for all $x \in X$.

2. The spaces $\mathcal{O}(Z, Y)$ and $\mathcal{O}(X)$ of bounded operators

The spaces $\mathcal{O}(Z, Y)$ and $\mathcal{O}(X)$ are linear spaces over the field of scalars F. Clearly the elements of $\mathcal{O}(Z, Y)$ or of $\mathcal{O}(X)$ are continuous mappings on their domains of definition, Z or X.

The norm on Z, defined by (IV.1) has the following properties:

(i) In accordance with our convention, both the norms on X and on Y are expressed in terms of the same real number ρ (Definition II-1.1). It follows that the norm of an operator in $\mathcal{O}(Z, Y)$ has a norm equal to zero or to ρ^{-n} for some integer n.

(ii) $|\alpha A|_Z = |A|_Z$ for all $A \in \mathcal{O}(Z, Y)$ and all $\alpha \in F$, $\alpha \neq 0$.

(iii) $|A+B|_Z \leqq \text{Max}\,\{|A|_Z, |B|_Z\}$ for all $A, B \in \mathcal{O}(Z, Y)$.

Indeed, for all $u, v \in Z$:

$$\begin{aligned} |Au+Bu-Av-Bv| &\leqq \text{Max}\,\{|Au-Av|, |Bu-Bv|\} \\ &\leqq (\text{Max}\,\{|A|_Z, |B|_Z\})|u-v|. \end{aligned}$$

(iv) $|A+B|_Z = \text{Max}\,\{|A|_Z, |B|_Z\}$ whenever $|A|_Z \neq |B|_Z$.

To prove this, suppose without loss of generality that $|A|_Z > |B|_Z$. Then, for every $\varepsilon > 0$ such that

$$0 < \varepsilon < (|A|_Z - |B|_Z),$$

there exist $u = u(\varepsilon)$ and $v = v(\varepsilon)$, in Z, such that

$$\begin{aligned} |Au-Av| > (|A|_Z - \varepsilon)|u-v| &> |B|_Z \cdot |u-v| \\ &\geqq |Bu-Bv|. \end{aligned}$$

Thus,

$$|Au+Bu-Av-Bv| = |Au-Av|.$$

It follows that for every $\varepsilon > 0$,

$$\begin{gathered} |A+B|_Z > |A|_Z - \varepsilon, \\ |A+B|_Z = |A|_Z = \text{Max}\,\{|A|_Z, |B|_Z\}. \end{gathered}$$

These results lead to the following theorem on the structure of $\mathcal{O}(Z, Y)$, (and of $\mathcal{O}(X)$ when $Z = X = Y$).

Theorem 2.1. The space $\mathcal{O}(Z, Y)$, under the norm on Z defined by (IV.1), is a V-space.

Proof: It follows from (i)—(iv) above that $\mathcal{O}(Z, Y)$ satisfies all the defining properties of V-spaces, except possibly for completeness.

To prove the completeness of $\mathcal{O}(Z, Y)$, consider a Cauchy sequence $\{A_n\}$ in $\mathcal{O}(Z, Y)$. Since (Th. I-3.1)

$$\lim_{n\to\infty} |A_{n+1} - A_n|_Z = 0,$$

for any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all u, $v \in Z$ and all $n > N(\varepsilon)$,

$$|(A_{n+1}u - A_{n+1}v) - (A_n u - A_n v)| < \varepsilon |u-v|.$$

Let us select an arbitrary point x_0 of Z. For each $x \in Z$, the sequence $\{A_n x - A_n x_0\}$ is a Cauchy sequence in Y; since Y is complete, this sequence has a limit. Let A be an operator from Z to Y defined by

$$Ax = \lim_{n\to\infty} (A_n x - A_n x_0), \quad x \in Z.$$

We shall show that A is a limit of $\{A_n\}$. For $u, v \in Z$, define

$$\begin{aligned} y_{n,p}(u, v) \equiv {} & [(A_{n+p}u - A_{n+p}x_0) - (A_n u - A_n x_0)] \\ & - [(A_{n+p}v - A_{n+p}x_0) - (A_n v - A_n x_0)]. \end{aligned}$$

For any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all $u, v \in Z$, all $n > N(\varepsilon)$ and all $p > 0$,

$$\begin{aligned} |y_{n,p}(u, v)| &= |(A_{n+p}u - A_n u) - (A_{n+p}v - A_n v)| \\ &\leqq |A_{n+p} - A_n|_Z \cdot |u - v| < \varepsilon |u - v|. \end{aligned}$$

With n fixed, we have, for all $u, v \in Z$:

$$\lim_{p\to\infty} y_{n,p}(u, v) \equiv (Au - A_n u) - (Av - A_n v).$$

Since $\lim_{p\to\infty} |y_{n,p}(u, v)| = |\lim_{p\to\infty} y_{n,p}(u, v)|$, we have for $n > N(\varepsilon)$ and all $u, v \in Z$:

$$|(Au - A_n u) - (Av - A_n v)| < \varepsilon |u - v|.$$

Hence

$$\lim_{n\to\infty} |A - A_n|_Z = 0.$$

This shows that A is a limit of the sequence $\{A_n\}$. As a limit of a Cauchy sequence A is bounded on Z and

$$|A|_Z = \lim_{n\to\infty} |A_n|_Z.$$

We note that the operator A defined above depends on the selected point x_0. Clearly two different selections of x_0 will in general generate two distinct limits for the sequence $\{A_n\}$; the norm of the difference between two such limits is obviously zero.

3. The spaces $\mathscr{I}(Z, Y)$ and $\mathscr{I}(X)$ of bounded linear operators

$\mathscr{I}(Z, Y)$ is the set of bounded linear operators from Z to Y. To avoid meaningless or trivial statements we shall assume that Z properly lends itself to linearity arguments. This is achieved by requiring that Z be *linearly non-trivial*, defined as follows: A subset Z of a linear space is said to be linearly non-trivial if and only if there exist $u, v \in Z$ and $\alpha, \beta \in F$ such that

$$z = \alpha u + \beta v \in Z, \quad z \neq u, \quad z \neq v \quad \text{and} \quad |z| \neq 0.$$

Obviously, any non-trivial subspace of a V-space is linearly non-trivial.

THEOREM 3.1. The space $\mathscr{I}(Z, Y)$ is a V-space. The space $\mathscr{I}(X)$ is a V-algebra.

PROOF: Using continuity of the operators involved, it is easy to verify that $\mathscr{I}(Z, Y)$ is a closed linear subspace of $\mathscr{O}(Z, Y)$. In $\mathscr{I}(X)$, the product of two linear operators is a linear operator. Then, the theorem is a corollary of Theorem 2.1.

The following theorems are analogous to theorems valid in topological normed linear spaces over the real or complex fields with their usual valuations. The proofs are similar to those of the corresponding theorems in [36], pp. 18, 85—86, and are omitted. We shall use the following definition:

DEFINITION 3.2. Let $A \in \mathscr{O}(Z, Y)$. An operator A^{-1} from $A(Z)$ to Z is called a *pseudo-inverse of A on $A(Z)$* if $A^{-1}(Az) = z$ for all $z \in Z$.

THEOREM 3.3. If Z is a subspace of X, then $A \in \mathscr{I}(Z, Y)$ is continuous either at every point of Z or at no point of Z.

THEOREM 3.4. Let Z be a subspace of X and $A \in \mathscr{I}(Z, Y)$.

(i) A pseudo-inverse of A on $A(Z)$, when it exists, is linear on $A(Z)$.

(ii) A admits a bounded pseudo-inverse on $A(Z)$ if and only if there exists a constant $m > 0$ such that $m|z| \leqq |Az|$ for all $z \in Z$.

A linear operator from a Banach space to another is bounded if and only if it is continuous ([36], p. 85). In V-spaces boundedness implies continuity but the converse is not true. (See Example 1, below). A. F. Monna ([24], Part III, p. 1136) has proved that linear operators from a V-space to its field of scalars F, considered as a V-space over itself, are continuous if and only if they are bounded. The following theorem generalizes this result; the proof is modelled after that of Monna.

THEOREM 3.5. Let $A \in \mathscr{I}(Z, Y)$ and suppose that $A(Z)$ is a discrete and bounded subspace of Y. Then, A is bounded if and only if it is continuous.

PROOF: Boundedness implies continuity.

To prove the converse, suppose that A is continuous. Since $A(Z)$ is a discrete subspace, there exists $\varepsilon > 0$ such that

$$\text{(IV.4)} \qquad y \in A(Z) \text{ and } |y| < \varepsilon \text{ imply } |y| = 0.$$

Since A is continuous, there exists $\delta(\varepsilon)$ such that

$$z \in Z \text{ and } |z| < \delta(\varepsilon) \text{ imply } |Az| < \varepsilon,$$

and therefore, by (IV.4)

$$z \in Z \text{ and } |z| < \delta(\varepsilon) \text{ imply } |Az| = 0.$$

Since $A(Z)$ is bounded, there exists $M > 0$ such that $|y| \leqq M$ for all $y \in A(Z)$.

For all $z \in Z$ such that $|z| \geqq \delta(\varepsilon)$, we have

$$|Az| \leqq M = \frac{M}{\delta(\varepsilon)} \cdot \delta(\varepsilon) \leqq \frac{M}{\delta(\varepsilon)} |z|.$$

Hence, $|A|_Z \leqq (M/\delta(\varepsilon))$, and A is bounded.

We conclude this section with two examples. The first is an example of a continuous unbounded linear operator from a V-space to itself; the second shows that the Uniform Boundedness Principle ([36], p. 204; [7], p. 66) does not hold in V-spaces, i.e. a family of bounded linear operators on a V-space which is point-wise bounded is not necessarily uniformly bounded.

EXAMPLE 1. Let X be a V-space over the real numbers, with a countable distinguished basis $H = \{h_0, h_1, h_2, \ldots\}$ such that, for some integer k,

$$|h_n| = \rho^{-k-n}, \quad n = 0, 1, 2, \ldots.$$

(The space $\mathscr{P}_k$ of III-4 is such a space.)

Every non-trivial element of X has an expansion in terms of H:

$$\text{(IV.5)} \qquad x = \sum_{n=N}^{\infty} (\alpha_n h_{2n} + \beta_n h_{2n+1}), \quad \alpha_n, \beta_n \in R,$$

where $N \geqq 0$ and $|\alpha_N| + |\beta_N| \neq 0$.

Let A be an operator from X to itself defined by

$$Ax = \begin{cases} \theta & \text{if } |x| = 0, \\ \sum_{n=N}^{\infty} (\alpha_n + \beta_n) h_n & \text{if } x \text{ is given by (IV.5).} \end{cases}$$

Clearly A is linear. A is unbounded since

$$|Ah_{2n}| = \rho^n |h_{2n}| \text{ for all } n = 0, 1, 2, \ldots.$$

Yet, given any integer $n \geqq 0$,

$$A(S(\theta, \rho^{-k-2n})) \subset S(\theta, \rho^{-k-n}).$$

This shows that A is continuous at θ and, from Theorem 3.3, that A is continuous on all of X.

EXAMPLE 2. Let X be as in Example 1. Every non-trivial element x of X has an expansion in terms of H:

$$x = \sum_{n=N}^{\infty} \alpha_n h_n, \quad \alpha_N \neq 0, \quad N \geqq 0. \tag{IV.6}$$

For each non-negative integer p, let A_p be an operator from X to itself defined by

$$A_p x = \begin{cases} \theta & \text{if } |x| = 0, \\ \sum_{n=N}^{\infty} \alpha_n h_{n-p} & \text{if } x \text{ is given by (IV.6), } N \geqq p, \\ \sum_{n=N}^{p-1} \alpha_n h_0 + \sum_{n=p}^{\infty} \alpha_n h_{n-p} & \text{if } x \text{ is given by (IV.6), } N < p; \end{cases}$$

i.e.: the image of h_n is h_0 if $n \leqq p$ and is h_{n-p} if $n \geqq p$.

The linearity of A_p is easily verified. We have

$$|A_p x| = \rho^{-k-N+p} = \rho^p |x| \text{ if } N \geqq p \text{ in (IV.6)},$$
$$|A_p x| = \rho^{-k} \leqq \rho^p |x| \qquad \text{if } N < p \text{ in (IV.6)}.$$

Hence: $|A_p| = \rho^p$, $p = 0, 1, 2, \ldots$.

This shows that the family of linear operators $\{A_p\}$ is a family of bounded linear operators which is not uniformly bounded since $\lim_{p\to\infty} \rho^p = \infty$.

Yet, the family $\{A_p\}$ is point-wise bounded since $|A_p x| \leqq \rho^{-k}$ for all $x \in X$. We have shown that the Uniform Boundedness Principle does not hold in V-spaces. ([36], p. 204.)

4. Characterization of bounded linear operators

In this section X and Y are V-spaces, Z is a linear subspace of X which is not necessarily closed, H is a distinguished basis of Z.

With each element $h \in H$, let there be associated an element $Ah \in Y$ such that for some $M \geqq 0$,

$$\begin{aligned} |Ah| &\leqq M|h| \text{ for all } h \in H, \\ |Ah_0| &= M|h_0| \text{ for some } h_0 \in H. \end{aligned} \tag{IV.7}$$

Each $z \in Z$ is a sum of an expansion in terms of H:

$$z = \alpha_1 h_1 + \alpha_2 h_2 + \ldots, \ \alpha_i \in F, \ \alpha_i \neq 0, \ |h_i| \geqq |h_{i+1}|. \tag{IV.8}$$

If $\{h_1, h_2, \ldots\}$ is infinite, then $\lim_{n\to\infty}|h_n| = 0$ and, by (IV.7), $\lim_{n\to\infty}|Ah_n| = 0$.

We extend the definition of A to all of Z, by setting

$$Az = \alpha_1 Ah_1 + \alpha_2 Ah_2 + \ldots, \; z \text{ given by (IV.8).} \tag{IV.9}$$

Since Y is complete, this series converges and Az is defined, up to addition of trivial elements.

Clearly, A is a linear operator from Z to Y. It is also bounded since, by Lemma I-5.5,

$$|Az| \leqq \sup_n \{|Ah_n|\} \leqq M \cdot \sup_n \{|h_n|\} = M|h_1| = M|z|.$$

In view of (IV.7), $|A|_Z = M$.

We have constructed an element of $\mathscr{I}(Z, Y)$. It is important to note that the values of A on H were completely arbitrary, except for conditions (IV.7).

Now, suppose that B is a continuous linear operator and that

$$Bh - Ah \in [\theta] \text{ for all } h \in H.$$

Then

$$Bx - Ax \in [\theta] \text{ for all } x \in (H);$$

(recall that (H) is the set of all finite linear combinations of elements of H).

Since (H) is dense in Z and $B - A$ is continuous, we must have

$$Bz - Az \in [\theta] \text{ for all } z \in Z.$$

This result can be stated as follows:

THEOREM 4.1. Let X and Y be V-spaces and let Z be a linear subspace (not necessarily closed) of X.

(i) An element A of $\mathscr{I}(Z, Y)$ is determined, up to addition of trivial elements, by its values on a distinguished basis H of Z, and

$$|A|_Z = \inf\{M \geqq 0 : |Ah| \leqq M|h| \text{ for all } h \in H\}.$$

(ii) If a single valued mapping A is arbitrarily defined on H except for the requirement that

$$\sup_{h\in H} \frac{|Ah|}{|h|} \tag{IV.10}$$

be finite, then A can be extended by linearity to all of Z and $|A|_Z$ is equal to (IV.10). Furthermore, if B is a continuous linear operator from Z to Y and $|Bh - Ah| = 0$ for all $h \in H$, then $B \in \mathscr{I}(Z, Y)$ and $|B - A|_Z = 0$.

APPLICATION 1. The important feature of the assertion of part (ii) of the above theorem is that, provided (IV.10) is finite, the values of A on the elements of H are arbitrary.

The same is not true in an infinite dimensional Hilbert space X in which $H = \{h_1, h_2, h_3, \ldots\}$ would represent a countable complete orthonormal basis. As two examples, consider A and B defined on H by

$$Ah_n = h_1 \quad \text{for all } n, \tag{IV.11}$$

$$Bh_n = h_{2^i-1} \text{ for } 2^{i-1} \leqq n \leqq 2^i - 1. \tag{IV.12}$$

Then

$$\sup_{h \in H} \frac{||Ah||}{||h||} = \sup_{h \in H} \frac{||Bh||}{||h||} = 1 < \infty.$$

However, neither A nor B can be extended by linearity to all of X: they are not defined at the point $S = \sum_{n=1}^{\infty} h_n/n$.

If we suppose now that X is a V-space with distinguished basis $H = \{h_1, h_2, \ldots\}$ and $\lim_{n\to\infty} |h_n| = 0$, then the mapping A of (IV.11) is not acceptable under Theorem 4.1 since $\sup_{h\in H} |Ah|/|h| = \infty$; the mapping B can be extended into an element of $\mathscr{I}(X)$ since $\sup_{h\in H} |Bh|/|h| = 1$.

APPLICATION 2. Theorem 4.1 finds an application in a paper of H. F. Davis [4]. We present the problem of [4] in our own terminology. The notation is that of Section 4, Part III.

Let $X = Y = \mathscr{R}$. Let Z be the open subspace of $\mathscr{R}$ for which the set $\Phi_0 \cup E$ (see (III.16)) is a Hamel basis.

Let A' be a single valued mapping defined on $\Phi_0 \cup E$ by

$$\begin{aligned} A'\varphi_n &\equiv \beta(n)\varphi_n, \quad n = 0, 1, 2, \ldots, \\ A'e_\alpha &\equiv f_\alpha, \qquad -\infty < \alpha < \infty, \quad \alpha \neq 0, \end{aligned}$$

where $\beta(n)$ is a scalar and f_α is an element of $\mathscr{R}$.

Since every element of Z is a unique finite linear combination of the elements of $\Phi_0 \cup E$, A' can be uniquely extended by linearity to all of Z. Let B denote this extension.

The main theorem of Davis [4] asserts that a necessary and sufficient condition for B to be continuous on Z is that, in the Φ-asymptotic norm on $\mathscr{R}$:

$$f_\alpha = \sum_{n=0}^{\infty} \frac{\alpha^n \beta(n)}{n!} \varphi_n, \text{ for all } \alpha \neq 0. \tag{IV.13}$$

The result is obtained from Theorem 4.1 through the following argument:

Φ_0 is a distinguished basis of $Z \subset \mathscr{P}_0$ (see Section III-4). The mapping A' is defined on Φ_0 in such a way that $\sup_{n\geq 0}|A'\varphi_n|/|\varphi_n| = 1$. Thus, by Theorem 4.1 (ii), there exists a "unique" linear operator A from Z to $\mathscr{R}$ which agrees with A' on Φ_0; this extension A is such that $|A| = 1$ and since

$$e_\alpha = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \varphi_n,$$

we must have

$$Ae_\alpha = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} A'\varphi_n = \sum_{n=0}^{\infty} \frac{\alpha^n \beta(n)}{n!} \varphi_n.$$

Hence, the above linear operator B is continuous on Z if and only if $|Bz-Az| = 0$ for all $z \in Z$; i.e. B is continuous if and only if $f_\alpha = Be_\alpha = Ae_\alpha$, so that (IV.13) is satisfied.

The reader will notice that the definition of continuity which we use and the definition of "asymptotic continuity" given by Davis ([4], p. 91) are different. Keeping to our own terminology, Davis calls an operator A "asymptotically continuous" if $|x| < \rho^n$ implies $|Ax| < \rho^n$. This is equivalent to saying that A is continuous if and only if $|A|_Z \leqq 1$. However, in the example above $|A|_Z = 1$ and the two definitions of continuity lead to identical results.

This definition of "asymptotic continuity" is too restrictive. An operator which is asymptotically continuous is also continuous in the topological sense but the converse is not true. The desirability of removing such restrictions is comparable to the desirability of accepting asymptotic expansions which are not of the Poincaré type (see Section III-3, e).

Application 3. Define the linear operator L from the space $\mathscr{K}_0$ to the space $\mathscr{K}'_0$ of Section III-5 by:

$$(Lx)(z, w) = \int_0^\infty \int_0^\infty e^{-zu-wv} x(u, v) du\, dv.$$

Since

$$Lx_{mn} = m!n!y_{m+1,n+1}, \quad m, n \geqq 0,$$

and

$$|Lx_{mn}| = \rho^{-2}|x_{mn}|, \text{ for all } m, n \geqq 0,$$

L can be extended (by Theorem 4.1) to all of $\mathscr{K}_0$ and $|L| = \rho^{-2}$.

If, in the asymptotic norm on $\mathscr{K}_0$, the function x admits the expansion

$$x = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \alpha_{n,j} x_{n-j,j},$$

then, in the asymptotic norm on $\mathscr{K}'_0$,

$$Lx = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \alpha_{n,j}(n-j)!\,j!\,y_{n-j+1,j+1}.$$

This result provides means to obtain asymptotic expansions of Laplace transforms in two variables. See [6]. For example, since $J_0(\sqrt{uv})$ has the asymptotic expansion:

$$J_0(\sqrt{uv}) = \sum_{n=0}^{\infty} \frac{u_n v_n}{2^{2n}(n!)^2},$$

$$LJ_0(\sqrt{uv}) = \frac{1}{zw} \sum_{n=0}^{\infty} \frac{1}{(4zw)^n} = \frac{4}{4zw-1}.$$

(Compare with [6], p. 100.)

5. Inverses and spectra in $\mathscr{I}(X)$

The V-space $\mathscr{I}(X)$ is a V-algebra.

In accordance with Definition II-6.2, a pseudo-identity in $\mathscr{I}(X)$ is a linear operator I' such that $|I'-I| = 0$.

The definition of a pseudo-inverse A^{-1} of A on its range $A(X)$ has been given (Definition 3.2). The operator A^{-1} belongs to $\mathscr{I}(X)$ if and only if it is bounded and defined on all of X. Therefore, A is (pseudo-) regular in the sense of Definition II-6.3 if and only if it admits a bounded (pseudo-) inverse and $A(X) \equiv X$. In such a case, we shall say that A admits a (pseudo-) inverse A^{-1}, without any mention of the range of A.

Let $A_\lambda \equiv A-\lambda I$. By Definition II-6.7, λ belongs to the spectrum $\sigma(A)$ of A if and only if A_λ is singular.

Theorems 6.6, 6.8 and 6.9 of Part II apply to the V-algebra $\mathscr{I}(X)$, (with $x \in X$ replaced by $A \in \mathscr{I}(X)$ and e replaced by I). The formulation of these three theorems for bounded linear operators on a V-space should be compared with similar theorems for bounded linear operators on Banach spaces: see [36], Theorems 4.1-C and 4.1-D, page 164, and Theorem 5.1-A, page 256.

NOTE: As in Theorem 5.1-A of [36], we can add to the statement of Theorem II-6.9 the following precision: Let $A \in \mathscr{I}(X)$, $|A| > 1$; if for some scalar $\mu \in F$, A_μ has a pseudo-inverse A_μ^{-1} on its range $A_\mu(X)$ and $|A_\mu^{-1}| < 1$, then, for all $\lambda \in F$, A_λ has a pseudo-inverse on its range $A_\lambda(X)$ and the topological closure of the range of A_λ is not a proper subset of the topological closure of the range of A_μ.

The proof is identical to that given in [36], p. 256. Our modification of Riesz's Lemma (Theorem I-6.1) must be used.

6. Complete spectral decompositions

The scalar λ is called an eigenvalue of $A \in \mathscr{I}(X)$ if for some $x_\lambda \in X, |x_\lambda| \neq 0, Ax_\lambda = \lambda x_\lambda$. The point x_λ is called an eigenelement associated with λ. The set

$$X_\lambda = \{x \in X : Ax = \lambda x\}$$

is a closed subspace of X and is called the eigenspace associated with λ.

DEFINITION 6.1. An operator $A \in \mathscr{I}(X)$ is said to have the *complete spectral decomposition* $\{(\lambda_i, h_i) : i \in J\}$ if for each i in the index set J, $Ah_i = \lambda_i h_i$, not all λ_i are equal to 0 and the set of eigenelements $H = \{h_i : i \in J\}$ is a distinguished basis of X.

THEOREM 6.2. If A has a complete spectral decomposition $\{(\lambda_i, h_i) : i \in J\}$, then:

(i) $|A| = 1$;

(ii) For all $\lambda \notin \{\lambda_i : i \in J\}$, A_λ is pseudo-regular and $|A_\lambda| = |A_\lambda^{-1}| = 1$.

(iii) If $\lambda_i \neq 0$ for each $i \in J$, then A is an isometry on X, i.e. $|Ax| = |x|$ for all $x \in X$.

PROOF: (i) The operator A satisfies (IV.7) with $M = 1$. Thus, by Theorem 4.1, $|A| = 1$.

(ii) Let x be an arbitrary point in X. It admits a non-increasing expansion in terms of H:

$$x = \sum_{n=0}^{\infty} \alpha_n h_n, \quad \alpha_n \in F, \quad \alpha_0 \neq 0, \quad h_n \in H. \tag{IV.14}$$

By Lemma I-5.5, $|x| = |h_0|$.

Then,

$$A_\lambda x = \sum_{n=0}^{\infty} \alpha_n(\lambda_n - \lambda) h_n. \tag{IV.15}$$

If $\lambda \notin \{\lambda_i : i \in J\}$, $|A_\lambda x| = |h_0| = |x|$. It follows from Theorem 4.1, that the operator A_λ^{-1}, defined on H by

$$A_\lambda^{-1} h_i = \frac{1}{\lambda_i - \lambda} h_i, \quad h_i \in H,$$

is a pseudo-inverse of A_λ. If x is given by (IV.14),

$$A_\lambda^{-1} x = \sum_{n=0}^{\infty} \alpha_n \frac{1}{\lambda_n - \lambda} h_n.$$

Thus, $|A_\lambda^{-1} x| = |h_0| = |x|$.

(iii) follows from (IV.14) and (IV.15) with $\lambda = 0$, $\lambda_n \neq 0$ for each $n \in J$.

Corollary 6.3. If $A \in \mathscr{I}(X)$ admits a complete spectral decomposition, then the cardinality of the set of its eigenvalues cannot exceed the dimension of the space.

Proof: Let $\{(\lambda_i, h_i) : i \in J\}$ be a complete spectral decomposition of A. If the cardinality of the set of eigenvalues exceeds the dimension of the space, i.e. the cardinality of the distinguished basis $\{h_i : i \in J\}$, there exists an eigenvalue λ which does not belong to $\{\lambda_i : i \in J\}$. Since λ is an eigenvalue, A_λ is singular. This contradicts (ii) of Theorem 6.2.

In the following Lemma 6.4 and Theorem 6.5, the assumptions and notations are as follows:

$A \in \mathscr{I}(X)$ admits a complete spectral decomposition

$$\{(\lambda_i, h_i) : i \in J\}, \quad H = \{h_i : i \in J\}.$$

For an arbitrary scalar λ,

$$J_\lambda = \{i \in J : \lambda_i = \lambda\}, \quad H_\lambda = \{h_i \in H : \lambda_i = \lambda\}.$$

X_λ denotes the closed subspace generated by H_λ. Clearly, if λ is not an eigenvalue, by Theorem 6.2(ii), $\lambda \neq \lambda_i$ for each $i \in J$ and, therefore, J_λ, H_λ and X_λ are empty. If λ is an eigenvalue, then $\lambda = \lambda_i$ for some $i \in J$ and X_λ is the non-empty eigenspace associated with λ.

Let P_λ denote a linear operator from X to X, defined on H by

$$P_\lambda h = \begin{cases} h & \text{if } h \in H_\lambda, \\ \theta & \text{if } h \in H \backslash H_\lambda. \end{cases}$$

By Theorem 4.1, $|P_\lambda| = 1$ if X_λ is not empty and $|P_\lambda| = 0$ if X_λ is empty.

Lemma 6.4. For all $x \in X$ and all scalars λ:

$$|x - P_\lambda x| = |Ax - \lambda x|.$$

PROOF: Given $x \in X$, x admits an expansion of the form

$$x = \sum_{h_i \in H_\lambda} (x, h_i)_H h_i + \sum_{h_i \in H \backslash H_\lambda} (x, h_i)_H h_i$$

(For notation, see Section 5 of Part I). Thus,

$$|x - P_\lambda x| = \Big| \sum_{h_i \in H \backslash H_\lambda} (x, h_i)_H h_i \Big|$$

$$|Ax - \lambda x| = \Big| \sum_{h_i \in H \backslash H_\lambda} (\lambda_i - \lambda)(x, h_i)_H h_i \Big|$$

By Lemma I-5.5,

$$|x - P_\lambda x| = |Ax - \lambda x| = \begin{cases} 0 \text{ if } (x, h_i)_H = 0 \text{ for all } h_i \in H \backslash H_\lambda, \\ \sup\limits_{\substack{h_i \in H \backslash H_\lambda \\ (x, h_i) \neq 0}} |h_i| \text{ otherwise.} \end{cases}$$

Lemma 6.4 is the equivalent, in V-spaces, of a theorem of C. A. Swanson, valid for Hilbert spaces: Theorem 1 of [34], Theorem 2 of [35]. This lemma is used to prove the following comparison theorem:

THEOREM 6.5. Let $B \in \mathscr{I}(X)$ and suppose that λ is an eigenvalue of B, with the associated eigenspace Y_λ. If $|B - A| < 1$, then:
(i) λ is also an eigenvalue of A,
(ii) the dimension of Y_λ is less than or equal to the dimension of X_λ.

PROOF: Let H'_λ be a distinguished basis for Y_λ. By Lemma 6.4, we have for each $h' \in H'_\lambda$:

$$|h' - P_\lambda h'| = |Ah' - \lambda h'| = |Ah' - Bh'| < |h'|.$$

Therefore,

$$|P_\lambda h'| = |h'| \neq 0.$$

Hence, X_λ is non-trivial and (i) is proved.

By Theorem II-2.4 (Paley-Wiener Theorem), the set $P_\lambda H'$ is a distinguished subset of X_λ and, by Corollary II-2.3, the cardinality of a distinguished basis of X_λ is greater than or equal to the cardinality of $P_\lambda H'$. (ii) follows.

The following corollaries are immediate:

COROLLARY 6.6. If both A and B admit complete spectral decompositions and $|B - A| < 1$, then
(i) A and B have the same eigenvalues,
(ii) for each eigenvalue λ, the associated eigenspaces for A and for B have the same dimensions.

COROLLARY 6.7. Suppose that A admits a complete spectral decomposition and $|B-A| < 1$. If λ is an eigenvalue of A but is not an eigenvalue of B, then B does not admit a complete spectral decomposition.

EXAMPLE 1. This example shows that the converse of Theorem 6.5(i) is not true, i.e. if A has a complete spectral decomposition and $|B-A| < 1$, an eigenvalue of A is not necessarily an eigenvalue of B.

Let $H = \{h_i : i = 0, 1, 2, \ldots\}$ be a distinguished basis of a V-space X, with $|h_i| > |h_{i+1}|$ for all $i \geqq 0$. Define A and B by their values on H (Theorem 4.1):

$$Ah_i = h_i \text{ if } i \text{ is even, } Ah_i = \theta \text{ if } i \text{ is odd,}$$
$$Bh_i = h_i \text{ if } i \text{ is even, } Bh_i = h_{i+2} \text{ if } i \text{ is odd.}$$

A admits a complete spectral decomposition and, by Theorem 6.2, its only eigenvalues are 0 and 1.

Let $x \in X$. Then for some $N = N(x)$:

$$x = \sum_{i \geqq N} (\alpha_i h_{2i} + \beta_i h_{2i+1}), \quad \alpha_i \in F, \quad \beta_i \in F, \quad |\alpha_N| + |B_N| \neq 0,$$
$$|x| > |h_{2N+2}|,$$
$$Ax = \alpha_N h_{2N} + \sum_{i \geqq N+1} \alpha_i h_{2i},$$
$$Bx = \alpha_N h_{2N} + \sum_{i \geqq N+1} \alpha_i h_{2i} + \sum_{i \geqq N} \beta_i h_{2i+3},$$
$$|Ax - Bx| = |\sum_{i \geqq N} \beta_i h_{2i+3}| \leqq |h_{2N+2}|.$$

Thus, $|A-B| < 1$.

In accordance with Theorem 6.5(i), the eigenvalue 1 of B is also an eigenvalue of A. It is easily verified that the eigenvalue 0 of A is not an eigenvalue of B. It follows from Corollary 6.7 that B does not have a complete spectral decomposition.

EXAMPLE 2. The following are linear operators on the space $\mathscr{P}_k$ of III-4, for $k \geqq 0$.

(i) $$(\mathscr{L}x)(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{-t/\lambda} x(t) dt.$$

$\mathscr{L}$ admits the complete spectral decomposition $\{(n!, \varphi_n); n = k, k+1, k+2, \ldots\}$ since $\mathscr{L}\varphi_n = n!\varphi_n$.

$\mathscr{L}$ has arbitrarily large eigenvalues and was studied by T. E. Hull [15].

Compare $\mathscr{L}$ with the Laplace Transform ([8], Vol. I):

$$x(\lambda) \to \int_0^\infty e^{-t\lambda} x(\lambda) d\lambda.$$

(ii) For $\mu > 0$,

$$(\mathscr{M}_\mu x)(\lambda) = \int_0^\lambda (\lambda - t)^{\mu-1} \frac{x(t)}{t^\mu} dt.$$

$\mathscr{M}_\mu$ admits the complete spectral decomposition

$$\left\{\left(\frac{\Gamma(\mu)\Gamma(n+1)}{\Gamma(\mu+n+1)}, \varphi_n\right); \ n = k, \ k+1, \ k+2, \ldots\right\}.$$

Compare $\mathscr{M}_\mu$ with the Riemann-Liouville fractional integral ([8], Vol. II):

$$x(\lambda) \to [\Gamma(\mu)]^{-1} \int_0^t (t-\lambda)^{\mu-1} x(\lambda) d\lambda.$$

(iii) For $K \geqq 1$,

$$(\mathscr{C}x)(\lambda) = \int_0^\lambda J_0(\lambda - t) \frac{x(t)}{t} dt.$$

$\mathscr{C}$ admits the complete spectral decomposition $\{(1/n, J_n) : n = K, K+1, K+2, \ldots\}$. Concerning this convolution product, see, for example, Mikusiński [23], pp. 174—178 and p. 456.

Since none of the above operators $\mathscr{L}$, $\mathscr{M}_\mu$, $\mathscr{C}$ has eigenvalue 0, they are isometries:

$$|\mathscr{L}x| = |\mathscr{M}_\mu x| = |\mathscr{C}x| = |x| \text{ for all } x \in \mathscr{P}_k.$$

From Theorem 6.2, they have no other eigenvalues than those given in their respective spectral decompositions above.

7. Note on projections

A. F. Monna [24], [25] has introduced a notion of projection in non-Archimedean normed linear spaces. In the special case of V-spaces we have the following:

DEFINITION 7.1. Let Y be a closed subspace of a V-space X. An operator $P \in \mathscr{I}(X)$ is called a projection on Y if for all $x \in X$, $Px \in Y$ and

(IV.16) $$|x - Px| \leqq |x - y| \text{ for all } y \in Y.$$

Theorems on projections and comparisons with projections in Hilbert space theory [7], [36] will be found in Monna [24], Part IV and [25], Part I.

The existence and non-uniqueness of projections on a given subspace Y of X were proved by Monna. The proofs of Monna do not involve explicitly the use of distinguished bases. We give here an alternate and simple proof.

Let $H(Y)$ be a distinguished basis of Y and H be an arbitrary extension of $H(Y)$ to all of X. Denote by Z the closed subspace generated by $H \backslash H(Y)$.

Define the linear operator P on X by its values on H (Theorem 4.1):

$$Ph \equiv h \text{ if } h \in H(Y), \quad Ph \equiv \theta \text{ if } h \in H \backslash H(Y).$$

By Theorem II-4.4 and Corollary II-3.4, the spaces Y and Z are distinguished complements of one another. Therefore for each $x \in X$, there exist $y_x \in Y$ and $z_x \in Z$ such that $x \equiv y_x + z_x$. Since the restriction of P to Y is the identity mapping and its restriction to Z is the 0-operator:

$$Px = Py_x + Pz_x = y_x \in Y,$$

and

$$|x - Px| = |x - y_x| = |z_x|.$$

For an arbitrary $y \in Y$, $y - y_x \in Y$ and, since Y and Z are distinguished subsets of X:

$$|x - y| = |(y_x - y) + z_x| = \text{Max}\,\{|y - y_x|, |z_x|\}.$$

Hence (IV.16) is satisfied. This proves that P is a projection on Y.

The non-uniqueness of the projections on Y is a consequence of the non-uniqueness of the extensions H of $H(Y)$.

Remark: The operator P_λ of Lemma 6.4 is a projection on X_λ.

(Oblatum 18–3–66)