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Some formulae by means of fractional derivatives

by

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1

The object of this paper is to obtain some formulae, believed to be new, by using the conception of fractional derivatives as defined by

$$(1)\qquad D_w^\lambda[w^{\mu-1}] = \frac{d^\lambda w^{\mu-1}}{dw^\lambda} = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\, w^{\mu-\lambda-1}.$$

2

To start with, let us consider the elementary identity

$$[1-u(x+y-xy)]^{-a}(1-uz)^{-\lambda} = (1-u\,\overline{x+y}+uxy)^{-a}(1-uz)^{-\lambda}$$

or

$$\sum_{m,n=0}^{\infty} \frac{(a)_m(\lambda)_n}{m!n!}(x+y-xy)^m z^n u^{m+n}$$
$$= \sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^r(a)_{r+m}(\lambda)_n}{r!m!n!}(xy)^r(x+y)^m z^n u^{r+m+n}$$

Now we multiply both the sides by u^{b-1} and then apply the operator, D_u^{b-c}, so that

$$\sum_{m,n=0}^{\infty} \frac{(a)_m(\lambda)_n}{m!n!}(x+y-xy)^m z^n D_u^{b-c}[u^{b+m+n-1}]$$
$$= \sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^r(a)_{r+m}(\lambda)_n}{r!m!n!}(xy)^r(x+y)^m z^n D_u^{b-c}[u^{b+r+m+n-1}].$$

Using the definition (1) it can be easily seen that

$$D_u^{b-c}[u^{b+m+n-1}] = \frac{\Gamma(b+m+n)}{\Gamma(c+m+n)}\, u^{c+m+n-1}$$

and

$$D_u^{b-c}[u^{b+r+m+n-1}] = \frac{\Gamma(b+r+m+n)}{\Gamma(c+r+m+n)}\, u^{c+r+m+n-1}.$$

Thus we arrive at

$$F_1[b; a, \lambda; c; u(x+y-xy), uz] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r!\,(c)_r} (-uxy)^r$$
$$\times F_1[b+r; a+r, \lambda; c+r; u(x+y), uz]$$

or, putting $u = 1$,

$$F_1(b; a, \lambda; c; x+y-xy, z] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r!\,(c)_r} (-xy)^r \tag{2}$$
$$\times F_1(b+r; a+r, \lambda; c+r; x+y, z).$$

In case we put $z = x+y-xy$ and use the formula [2, p. 239]

$$F_1(\alpha; \beta, \beta'; \gamma; x, x) = {}_2F_1(\alpha, \beta+\beta'; \gamma; x), \tag{3}$$

(2) reduces to

$${}_2F_1(a+\lambda, b; c; x+y-xy) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r!\,(c)_r} (-xy)^r \tag{4}$$
$$\times F_1(b+r; a+r, \lambda; c+r; x+y, x+y-xy).$$

On the other hand, if we put $z = x+y$, it follows from (2) that

$$F_1(b; a, \lambda; c; x+y-xy, x+y) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r!\,(c)_r} (-xy)^r \tag{5}$$
$$\times {}_2F_1(a+\lambda+r, b+r; c+r; x+y).$$

For $\lambda = 0$, (5) reduces to the known result due to Burchnall and Chaundy [1].

In order to find an inverse of (2), we consider

$$[1-u(x+y)]^{-a}(1-uz)^{-\lambda} = [1-u(x+y-xy)-uxy)^{-a}(1-uz)^{-\lambda}$$

or

$$\sum_{m,\,n=0}^{\infty} \frac{(a)_m (\lambda)_n}{m!\,n!} (x+y)^m z^n u^{m+n}$$
$$= \sum_{r,\,m,\,n=0}^{\infty} \frac{(a)_{r+m} (\lambda)_n}{r!\,m!\,n!} (xy)^r (x+y-xy)^m z^n u^{r+m+n}.$$

We multiply both the sides by u^{b-1} and apply the operator D_u^{b-c}, thus having

$$\sum_{m,\,n=0}^{\infty} \frac{(a)_m (\lambda)_n}{m!\,n!} (x+y)^m z^n D_u^{b-c} [u^{b+m+n-1}]$$
$$= \sum_{r,\,m,\,n=0}^{\infty} \frac{(a)_{r+m} (\lambda)_n}{r!\,m!\,n!} (xy)^r (x+y-xy)^m z^n D_u^{b-c} [u^{b+r+m+n-1}],$$

which yields, on putting $u = 1$,

$$F_1(b;\, a, \lambda;\, c;\, x+y, z) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r!\,(c)_r} (xy)^r \times F_1(b+r;\, a+r, \lambda;\, c+r;\, x+y-xy, z). \tag{6}$$

Putting $z = x+y$ and using (3), (6) becomes

$${}_2F_1(a+\lambda, b;\, c;\, x+y) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r!\,(c)_r} (xy)^r \times F_1(b+r;\, a+r, \lambda;\, c+r;\, x+y-xy, x+y), \tag{7}$$

while for $z = x+y-xy$, we have

$$F_1(b;\, a, \lambda;\, c;\, x+y, x+y-xy) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r!\,(c)_r} (xy)^r \times {}_2F_1(a+\lambda+r, b+r;\, c+r;\, x+y-xy). \tag{8}$$

Again, for $\lambda = 0$, (8) yields the known formula due to Burchnall and Chaundy [1].

Similarly, if we consider the identity

$$[1-u(x+y-xy)]^{-b} = [(1-ux)(1-uy)+uxy(1-u)]^{-b}$$

or

$$\sum_{n=0}^{\infty} \frac{(b)_n}{n!} (x+y-xy)^n u^{a+n-1} = \sum_{r=0}^{\infty} \frac{(-1)^r (b)_r}{r!} (xy)^r \times \sum_{m,n,p=0}^{\infty} \frac{(b+r)_m (b+r)_n (-r)_p}{m!\,n!\,p!} x^m y^n u^{a+r+m+n+p-1}$$

and apply the operator D_u^{a-c}, we get

$${}_2F_1\,(a, b;\, c;\, u(x+y-xy)] = \sum_{r=0}^{\infty} \frac{(-1)^r (a)_r (b)_r}{r!\,(c)_r} (uxy)^r \times F_D(a+r;\, b+r, b+r, -r;\, c+r;\, ux, uy, u). \tag{9}$$

For $u = 1$, we get the formula due to Burchnall and Chaundy [1].

3

We take up another elementary result

$$(2-2x-2y)^{-\alpha} = [(1-2x)+(1-2y)]^{-\alpha}$$

or

$$\sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{m!\,n!} x^m y^n = 2^{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)_n}{n!} (1-2x)^{-\alpha-n} (1-2y)^n.$$

Now we multiply both the sides by $x^{\beta-1}y^{\beta'-1}$, so that

$$\sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{m!n!} x^{\beta+m-1}y^{\beta'+n-1} = 2^{\alpha}\sum_{n=0}^{\infty} \frac{(-1)^n(\alpha)_n}{n!} \times\sum_{r=0}^{\infty}\frac{(\alpha+n)_r}{r!}2^r x^{\beta+r-1}\sum_{k=0}^{n}\frac{(-n)_k}{k!}2^k y^{\beta'+k-1}$$

Applying the operators $D_x^{\beta-\gamma}$ and $D_y^{\beta'-\gamma'}$, we obtain

$$F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y) = 2^{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n}{n!} \times {}_2F_1(\alpha+n,\beta;\gamma;2x)\,{}_2F_1(-n,\beta';\gamma';2y). \tag{10}$$

Using Euler's formula

$${}_2F_1(a,b;c;x) = (1-x)^{-a}{}_2F_1\left(a,c-b;c;-\frac{x}{1-x}\right),$$

we rewrite (10) as

$$F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y) = 2^{\alpha}(1-2x)^{-\beta},$$
$$\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n}{n!}\,{}_2F_1\left(\gamma-\alpha-n,\beta;\gamma;-\frac{2x}{1-2x}\right){}_2F_1(-n,\beta';\gamma';2y),$$

which on putting $\gamma=\gamma'=\alpha$, $\beta'=\beta$, $2x/(2x-1)=y$ and using the relation [2, p. 238]

$$F_2(\alpha;\beta,\beta';\alpha,\alpha;x,y) = (1-x)^{-\beta}(1-y)^{-\beta'}{}_2F_1\left[\beta,\beta';\alpha;\frac{xy}{(1-x)(1-y)}\right],$$

becomes

$${}_2F_1\left[\beta,\beta;\alpha;-\frac{y^2}{(1-y)^2}\right] = 2^{\alpha}(1-y)^{2\beta}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n}{n!}[{}_2F_1(-n,\beta;\alpha;2y)]^2 \tag{11}$$

Dividing y by β and taking $\beta\to\infty$, we get from (11) that

$${}_0F_1(-;1+\alpha;-y^2) = 2^{1+\alpha}e^{-2y}\sum_{n=0}^{\infty}\frac{(-1)^n n!}{(1+\alpha)_n}[L_n^{(\alpha)}(2y)]^2 \tag{12}$$

or, if we prefer,

$$J_{\alpha}(y) = \frac{2y^{\alpha}}{\Gamma(1+\alpha)}e^{-y}\sum_{n=0}^{\infty}\frac{(-1)^n n!}{(1+\alpha)_n}[L_n^{\alpha}(y)]^2. \tag{13}$$

Next, we consider

$$[(1+x)(1-y)+(1-x)(1+y)]^n = 2^n(1-xy)^n$$

or

$$\sum_{r=0}^{n} \binom{n}{r} (1+x)^{n-r}(1-x)^r(1-y)^{n-r}(1+y)^r = 2^n \sum_{r=0}^{n} \frac{(-n)_r}{r!} x^r y^r.$$

Multiplying both the sides by $x^{\alpha-1}\, y^{\gamma-1}$ and applying the operators $D_x^{\alpha-\beta}$ and $D_y^{\gamma-\delta}$, we get

$$(14) \quad \sum_{r=0}^{n} \binom{n}{r} F_1(\alpha; -n+r, -r; \beta; -x, x) F_1(\gamma; -n+r, -r; \delta; y, -y) = 2^n\, {}_3F_2(-n, \alpha, \gamma; \beta, \delta; xy).$$

Lastly, we deal with the identity

$$[(1-y)-(1-x)]^n = (x-y)^n$$

or

$$\sum_{r=0}^{n} \frac{(-n)_r}{r!} (1-x)^r (1-y)^{n-r} = \sum_{r=0}^{n} \frac{(-n)_r}{r!} x^{n-r} y^r$$

or

$$\sum_{r=0}^{n} \frac{(-n)_r}{r!} \sum_{k=0}^{r} \frac{(-r)_k}{k!} x^{a_1+k-1} \sum_{i=0}^{n-r} \frac{(-n+r)_i}{i!} y^{\alpha_1+i-1} = \sum_{r=0}^{n} \frac{(-n)_r}{r!} x^{a_1+n-r-1} y^{\alpha_1+r-1}.$$

Applying the operators $D_x^{a_1-b_1}$ and $D_y^{\alpha_1-\beta_1}$, it yields

$$(15) \quad \sum_{r=0}^{n} \frac{(-n)_r}{r!}\, {}_2F_1(-r, a_1; b_1; x)\, {}_2F_1(-n+r, \alpha_1; \beta_1; y) = \frac{(a_1)_n x^n}{(b_1)_n}\, {}_3F_2\left(-n, 1-b_1-n, \alpha_1; 1-a_1-n, \beta_1; \frac{y}{x}\right).$$

If we continue this process of performing operators p times with respect to x and l times with respect to y and then suppress some parameters, we arrive at

$$(16) \quad \sum_{r=0}^{n} \frac{(-n)_r}{r!}\, {}_{1+p}F_q \left[\begin{matrix} -r, a_1, a_2, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix}\ x\right] {}_{1+l}F_m \left[\begin{matrix} -n+r, \alpha_1, \ldots, \alpha_l; \\ \beta_1, \ldots, \beta_m; \end{matrix}\ y\right]$$
$$= \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} x^n\, {}_{1+q+l}F_{p+m} \left[\begin{matrix} -n, 1-b_1-n, \ldots, 1-b_q-n, \alpha_1, \ldots, \alpha_l; \\ 1-a_1-n, \ldots, 1-a_p-n, \beta_1, \ldots, \beta_m \end{matrix}\ (-1)^{p+q} \frac{y}{x}\right].$$

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