COMPOSITIO MATHEMATICA

TAQDIR HUSAIN

Two tauberian theorems in Banach spaces

Compositio Mathematica, tome 18, nº 1-2 (1967), p. 87-93
<http://www.numdam.org/item?id=CM_1967__18_1-2_87_0>

Two Tauberian theorems in Banach spaces

by

Taqdir Husain

In this note, we show how the classical Tauberian theorems concerning Abel and Borel summable real or complex sequences can be proved for sequences in a Banach space. The classical proofs are well-known and are due to Hardy and Littlewood [1]. A function-theoretic proof using results from complex analysis is due to W. Jurkat [2] and [3]. We follow the latter, giving details only when necessary. We shall use notations of [1].

Let E be a complex Banach space and E' its dual. A sequence $\{x_n\}$ in E is said to be *weakly Borel summable* to $0 \in E$ if for each $f \in E'$, the series

$$\sum_{n=0}^{\infty} f(x_n) \frac{\lambda^n}{n!}$$

converges for all complex λ and is $o(e^{\lambda})$ for real $\lambda \to \infty$.

If

$$\sum_{n=0}^{\infty} x_n \frac{\lambda^n}{n!}$$

converges for all complex λ in the norm topology and $o(1)$ for real $\lambda \to \infty$, then $\{x_n\}$ is said to be *strongly Borel summable* to 0.

Similarly, $\{x_n\}$ is said to be *weakly* (or *strongly*) *Abel summable* to 0, if for each $f \in E'$ the series $(1-\lambda) \sum_{n=0}^{\infty} f(x_n)\lambda^n$ converges for $|\lambda| < 1$ (or $(1-\lambda) \sum_{n=0}^{\infty} x_n \lambda^n$ converges in the norm topology for $|\lambda| < 1$) and is $o(1/1-\lambda)$ as real $\lambda \to 1^-$.

It is easy to see that a strongly Borel (resp. Abel) summable sequence is weakly Borel (resp. Abel) summable.

We consider Borel summable sequences first and prove the following:

THEOREM 1. Let E be a complex Banach space and E' its dual. Let $\{x_n\}$ be a sequence in E which is weakly Borel summable to 0. Suppose $||x_n - x_{n-1}|| = O(1/\sqrt{n})$ as $n \to \infty$. Then $\{x_n\}$ converges weakly to 0, i.e., for each $f \in E'$, the sequence $\{f(x_n)\}$ of complex numbers converges to 0.

We need the following lemmas.

LEMMA 1. Let

$$e^{\lambda} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!},$$

$\lambda = re^{i\theta}$, $r \geqq 0$, $i = \sqrt{-1}$. Then

$$\sum_{n=0}^{\infty} \left(\frac{r^n}{n!}\right)^2 = O\left(\frac{e^{2r}}{\sqrt{r}}\right).$$

PROOF: Since

$$|e^{\lambda}|^2 = \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\right) \left(\overline{\sum_{m=0}^{\infty} \frac{\lambda^m}{m!}}\right)$$

is absolutely and uniformly convergent inside any finite circle and since $\cos\theta \leqq 1-\delta\theta^2$ $(\delta > 0)$, by integrating termwise we have

$$\sum_{n=0}^{\infty} \left(\frac{r^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2r\cos\theta} d\theta = O(e^{2r})\int_0^{2\pi} e^{-2r\delta\theta^2} d\theta$$

$$= O(e^{2r})\int_0^{\pi} e^{-t} \frac{dt}{\sqrt{\delta r t}} = O\left(\frac{e^{2r}}{\sqrt{r}}\right).$$

LEMMA 2. If $\{x_n\}$ in E is a sequence which is weakly Borel summable to 0 and if $||x_n - x_{n-1}|| = O(1/\sqrt{n})$ as $n \to \infty$, then $\{||x_n||\}$ is a bounded sequence.

PROOF: As in § 3, [3], we see that for all k, $n \geqq 0$,

$$||x_k - x_n|| \leqq K \frac{|k-n|}{\sqrt{n+1}}$$

for some $K > 0$. Hence for $f \in E'$,

$$e^{-n}\sum_{k=0}^{\infty} f(x_k)\frac{n^k}{k!} - f(x_n) = e^{-n}\sum_{k=0}^{\infty} \frac{n^k}{k!} f(x_k - x_n)$$

$$= O(e^{-n}||f||)\sum_{k=0}^{\infty} \frac{n^k}{k!} ||x_k - x_n||$$

$$= O\left(\frac{||f||e^{-n}}{\sqrt{n}}\right)\sum_{k=0}^{\infty} \frac{n^k}{k!} |k-n|$$

$$= O(||f||) \qquad \text{(cf. [3], p. 280).}$$

Since $\{x_n\}$ is weakly Borel summable, $e^{-n}\sum_{k=0}^{\infty} f(x_k)\, n^k/k! = o(1)$ as $n \to \infty$ and therefore $\{f(x_n)\}$ is a bounded sequence of complex numbers. In other words, $\{x_n\}$ is a weakly bounded sequence in E. Since weakly and norm bounded sets in E are the same, $\{||x_n||\}$ is bounded.

LEMMA 3. If $\{x_n\}$ is a weakly Borel summable (to 0) sequence in E such that $||x_n - x_{n-1}|| = O(1/\sqrt{n})$, then for each $f \in E'$,

$$g(\lambda) = e^{-\lambda}\sum_{n=0}^{\infty} f(x_n)\frac{\lambda^n}{n!}$$

represents an analytic function in the whole complex plane. Moreover, $g(\lambda)$ is bounded in each parabola $|\lambda| - \mathscr{R}\lambda < M$, $M > 0$ ($\mathscr{R}\lambda$ denotes the real part of λ); and $g(\lambda) \to 0$ uniformly as $|\lambda| \to \infty$ in $|\lambda| - \mathscr{R}\lambda < M'$, $0 < M' < M$.

PROOF: It is clear that $g(\lambda)$ is analytic. By Lemma 2, $\{f(x_n)\}$ is bounded. Hence

$$|g(\lambda)| \leqq |e^{-\lambda}|\sum_{n=0}^{\infty}\frac{|\lambda|^n}{n!} = e^{|\lambda| - \mathscr{R}\lambda}$$

shows that $g(\lambda)$ is bounded for $|\lambda| - \mathscr{R}\lambda < M$. The remainder follows from Hilfsatz 1, [3], where a well-known Montel theorem is used.

LEMMA 4. Let $g(\lambda)$ be as in Lemma 3. If $||x_n - x_{n-1}|| = O(1/\sqrt{n})$, then for each $f \in E'$, $\lambda = re^{i\theta}$,

$$\int_0^{2\pi} |\lambda g'(\lambda) e^{\lambda}|^2 d\theta = O(\sqrt{r}\, e^{2r}||f||),$$

where $g'(\lambda)$ is the derivative of $g(\lambda)$.

PROOF: By differentiating $g(\lambda)$, we obtain

$$e^{\lambda} g'(\lambda) = \sum_{n=1}^{\infty} f(x_n - x_{n-1})\frac{\lambda^{n-1}}{(n-1)!}.$$

Hence

$$\lambda e^{\lambda} g'(\lambda) = \sum_{n=1}^{\infty} f(nx_n - nx_{n-1})\frac{\lambda^n}{n!}.$$

Multiplying the last function by its conjugate and integrating termwise, we obtain

$$\int_0^{2\pi} |\lambda e^\lambda g'(\lambda)|^2 = 2\pi \sum_{n=1}^{\infty} |f(nx_n - nx_{n-1})|^2 \left(\frac{r^n}{n!}\right)^2$$
$$= O(||f||) \sum_{n=1}^{\infty} n^2 ||x_n - x_{n-1}||^2 \left(\frac{r^n}{n!}\right)^2$$
$$= O(||f||) \sum_{n=1}^{\infty} n \left(\frac{r^n}{n!}\right)^2$$
$$= O(r||f||) \sum_{n=1}^{\infty} \frac{n}{r} \left(\frac{r^n}{n!}\right)^2$$
$$= O(r||f||) \left\{ \sum_{n/r \leqq 1} \frac{n}{r} \left(\frac{r^n}{n!}\right)^2 + \sum_{1 < n/r < n^2/r^2} \frac{n}{r} \left(\frac{r^n}{n!}\right)^2 \right\}$$
$$= O(\sqrt{r}\, e^{2r} ||f||) \quad \text{by Lemma 1.}$$

Lemma 5. Let $\alpha = \alpha_1 + i\alpha_2 = re^{i\theta}$, $\alpha \ngeqq 0$ and for $n \geqq 2$, let

$$F_n(\alpha) = \int_L \frac{e^\lambda}{\lambda^{n+1}} d\lambda,$$

where L is a half line given by either $\lambda = \alpha t$, $\pi \geqq |\theta| \geqq \pi/2$ or $\lambda = \alpha_1 + i\alpha_2 t$, $0 < |\theta| \leqq \pi/2$, t real $\geqq 1$.

Then, for some $K > 0$,

$$|F_n(\alpha)| \leqq K \left| \frac{e^\alpha}{n\alpha^n \theta} \right|.$$

Proof: Suppose $\lambda = \alpha t$, $|\theta| \geqq \pi/2$. Clearly

$$|e^{\alpha(t-1)}| = e^{r\cos\theta(t-1)} \leqq 1,$$

for $\cos\theta \leqq 0$ (because $|\theta| \geqq \pi/2$) and $t \geqq 1$. Hence

$$|F_n(\alpha)| = \left| \frac{1}{\alpha^n} \int_1^\infty \frac{e^{\alpha t} dt}{t^{n+1}} \right| = \left| \frac{e^\alpha}{\alpha^n} \int_1^\infty \frac{e^{\alpha(t-1)}}{t^{n+1}} dt \right|$$
$$\leqq \left| \frac{e^\alpha}{\alpha^n} \right| \int_1^\infty \frac{dt}{t^{n+1}}$$
$$\leqq \left| \frac{e^\alpha}{n\alpha^n} \right| \leqq K \left| \frac{e^\alpha}{n\alpha^n \theta} \right|, \text{ since } |\theta| \leqq \pi.$$

In the second case, suppose $\lambda = \alpha_1 + i\alpha_2 t$, $0 < |\theta| \leqq \pi/2$, t real $\geqq 1$. Then

$$
\begin{aligned}
|F_n(\alpha)| &= \left| \int_1^\infty \frac{e^{\alpha_1 + i\alpha_2 t}}{(\alpha_1+i\alpha_2 t)^{n+1}} i\alpha_2 dt \right| \\
&= \left| \alpha_2 e^{\alpha} \int_1^\infty \frac{e^{i\alpha_2(t-1)}}{(\alpha_1+i\alpha_2 t)^{n+1}} dt \right| \\
&\leqq \left| \frac{e^\alpha}{\alpha_2} \right| \int_1^\infty \frac{\alpha_2^2 t dt}{(\alpha_1^2+\alpha_2^2 t^2)^{\frac{1}{2}(n+1)}} \\
&\leqq \left| \frac{e^\alpha}{\alpha_2} \right| \left[\frac{(\alpha_1^2+\alpha_2^2 t^2)^{\frac{1}{2}(-n+1)}}{-n+1} \right]_1^\infty \\
&\leqq K \left| \frac{e^\alpha}{n\alpha_2 \alpha^{n-1}} \right| \approx K \left| \frac{e^\alpha}{n\alpha^n \theta} \right|,
\end{aligned}
$$

because $\alpha_2 = |\alpha| \sin \theta \approx |\alpha|\,|\theta|$, if θ is small.

This establishes the lemma.

Proof of Theorem 1. To show that for each $f \in E'$, $\{f(x_n)\}$ converges to 0, we consider

$$
e^\lambda g(\lambda) = \sum_{n=0}^{\infty} f(x_n) \frac{\lambda^n}{n!} = o(e^\lambda)
$$

for $|\lambda| \to \infty$ in $|\lambda| - \mathscr{R}\lambda < M$, $M > 0$ (cf. Lemma 3).

Put $\lambda = ne^{i\theta}$. Then by the Cauchy integral formula, we have

$$
\begin{aligned}
f(x_n) &= \frac{n!}{2\pi i} \int_{|\lambda|=n} \frac{e^\lambda g(\lambda) d\lambda}{\lambda^{n+1}} \\
&= \frac{n!}{2\pi i} \left[\int_{|\theta|<m/\sqrt{n}} + \int_{|\theta| \geqq m/\sqrt{n}} \right] \frac{e^\lambda g(\lambda)}{\lambda^{n+1}} d\lambda \\
&= \frac{n!}{2\pi i} [I_1 + I_2], \text{ say.}
\end{aligned}
$$

Integrating I_2 by parts and putting $G_n(\lambda) = \int (e^\lambda/\lambda^{n+1}) d\lambda$, we have

$$
\begin{aligned}
I_2 &= [g(\lambda) G_n(\lambda)]_{\theta=m/\sqrt{n}}^{\theta=2\pi - m/\sqrt{n}} - \int_{m/\sqrt{n}}^{2\pi - m/\sqrt{n}} G_n(\lambda) g'(\lambda) d\lambda \\
&= A - B, \text{ say.}
\end{aligned}
$$

By Lemma 5, (observe that $|G_n(\lambda)| \leqq |F_n(\lambda)|$),

$$B = O(1)\int_{m/\sqrt{n}}^{2\pi-m/\sqrt{n}} \left|\frac{\lambda e^{\lambda} g'(\lambda)}{n\lambda^n \theta}\right| d\theta$$

$$= O\left(\frac{1}{n^{n+1}}\right)\int_{m/\sqrt{n}}^{2\pi-m/\sqrt{n}} \left|\frac{\lambda e^{\lambda} g'(\lambda)}{\theta}\right| d\theta \quad (|\lambda| = n.)$$

$$= O\left(\frac{1}{n^{n+1}}\right)\left\{\int_{m/\sqrt{n}}^{2\pi-m/\sqrt{n}} |\lambda e^{\lambda} g'(\lambda)|^2\right\}^{\frac{1}{2}}\left\{\int_{m/\sqrt{n}}^{\pi} \frac{d\theta}{\theta^2}\right\}^{\frac{1}{2}} \quad \text{(by Parsevals' formula)}$$

$$= O\left(\frac{1}{n^{n+1}}\right)\{\sqrt{n}\, e^{2n}||f||\}^{\frac{1}{2}}\left\{\frac{\sqrt{n}}{m}\right\}^{\frac{1}{2}} \quad \text{(by Lemma 4)}$$

$$= O\left(\frac{||f||^{\frac{1}{2}}}{n^n e^{-n}\sqrt{n}}\right)\left(\frac{1}{\sqrt{m}}\right) \quad \text{uniformly in } n \text{ and } m.$$

Hence, by Stirling's formula: $n! \sim n^n e^{-n}\sqrt{2\pi n}$, we have

$$\frac{n!}{2\pi} B = O\left(\frac{||f||^{\frac{1}{2}}}{\sqrt{m}}\right).$$

Given $\varepsilon > 0$ and $f \in E'$ we can choose m large enough such that

$$\left|\frac{n!}{2\pi i} B\right| < \frac{\varepsilon}{3} \quad \text{for all } n.$$

Further, for λ's inside the parabola $|\lambda| - \mathscr{R}\lambda < M$, $M > 0$, we have $g(\lambda) = o(1)$ by Lemma 3. Hence for the fixed m chosen above and such that λ lies in the parabola $|\lambda| - \mathscr{R}\lambda < M$, we have (using Lemma 5)

$$\frac{n!}{2\pi i} A = o(1)(n!)\,[G_n(\lambda)]_{\theta=m\sqrt{n}}^{\theta=2\pi-m/\sqrt{n}} = o(1)(n!)\left([F_n(\lambda)]_{\theta=m/\sqrt{n}}^{\theta=2\pi-m/\sqrt{n}}\right)$$

$$= o(1)(n!)\frac{e^n\sqrt{n}}{n^{n+1}}\,\frac{1}{m} = o(1)\frac{n^n e^{-n}\sqrt{n}\, e^n\sqrt{n}}{n^{n+1}} = o(1),$$

as $n \to \infty$.

Also since $g(\lambda) = o(1)$ in $|\lambda| - \mathscr{R}(\lambda) < M$, and since m is fixed

$$\left|\frac{n!}{2\pi i} I_1\right| = o(1)(n!)\left(\frac{e^n n}{n^{n+1}}\right) O\left(\frac{1}{\sqrt{n}}\right) = o(1).$$

Thus for sufficiently large n, we have

$$\left|\frac{n!}{2\pi i} A\right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left|\frac{n!}{2\pi i} I_1\right| < \frac{\varepsilon}{3}.$$

Hence $|f(x_n)| = |(n!/2\pi i)(I_1 + A - B)| < \varepsilon$ for sufficiently large n. This completes the proof.

THEOREM 2. Let E be a complex Banach space and E' its dual. Let $\{x_n\}$ be a sequence in E which is weakly Abel summable to 0. Suppose $||x_n - x_{n-1}|| = O(1/n)$ as $n \to \infty$. Then $\{x_n\}$ converges weakly to 0.

A simple way of proving this theorem is to show first that the Tauberian condition (T): $||x_n - x_{n-1}|| = O(1/n)$ implies that $||x_n|| = O(1)$ as $n \to \infty$. This follows exactly as in Lemma 2 by making appropriate changes. Next, we can show easily that if $\{x_n\}$ satisfies (T) and if for each $f \in E'$, $(1/n) \sum_{k=0}^{n} f(x_k) = o(1)$ as $n \to \infty$, then $\{x_n\}$ converges weakly to 0. Thus Theorem 2 would follow if we had showed that for any weakly Abel summable (to 0) sequence $\{x_n\}$ in E, $||x_n|| = O(1)$ as $n \to \infty$ implies that $\sum_{k=0}^{n} f(x_k) = o(n)$ as $n \to \infty$.

For this, as for Theorem 1, (cf. Lemma 3) we show that if $\{x_n\}$ is a bounded sequence $\{x_n\}$, then for each $f \in E'$,

$$h(\lambda) = (1-\lambda) \sum_{n=0}^{\infty} f(x_n)\lambda^n$$

is analytic for $|\lambda| < 1$, $h(\lambda) \to 0$ as $\lambda \to 1^-$ for λ inside $0 < |1-\lambda|/1-|\lambda|) < M$, $M > 0$.

Thus to prove Theorem 2, it is sufficient to prove the following so-called Abelian theorem.

THEOREM 3. If $\{x_n\}$ is a sequence in E such that $||x_n|| = O(1)$ as $n \to \infty$ and if for each $f \in E'$

$$h(\lambda) = \sum_{n=0}^{\infty} f(x_n)\lambda^n = o\left(\frac{1}{1-\lambda}\right)$$

for each $\lambda \to 1^-$ and $0 < (|1-\lambda|)/(1-|\lambda|) = O(1)$, then $f(x_n) = 0(n||f||)$.

The proof of Theorem 3 is exactly like that of Satz, 1[2] with appropriate changes and therefore omitted.

REFERENCES

G. H. HARDY

[1] Divergent Series; Clarendon Press, Oxford, 1946.

W. B. JURKAT,

[2] Ein funktionentheoretischer Beweis für O-Taubersätz bei Potenzreiben; Arch. Math. VII, 1956, pp. 122—125.

W. B. JURKAT,

[3] Ein funktionentheoretischer Beweis für O-Taubersätze bei den Verfahren von Borel und Euler-Knopp; Ibid, pp. 278—283.

(Oblatum 7-3-66) McMaster University, Hamilton, Ontario, Canada.