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Fundamental notions in the theory of seminearrings

by

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1. Introduction

As far as the authors know, there does not exist a theory of algebraic systems which combine the vices of semirings and nearrings, i.e. of systems with addition and multiplication that lack subtraction and one distributive law.

Such systems were introduced in [6] (cf. also [7]) under the name of seminearring (Fasthalbring). In [6] however only a very special type of seminearrings was considered and the question arose whether it is possible to develop a more general theory of seminearrings. In this paper we lay down a number of notions and some results which make clear that indeed it is possible to develop such a theory, extending known theories of nearrings and semirings. A further development of the theory of seminearrings as presented in this paper will be contained in the forthcoming thesis of the first author. The theory of seminearrings has several applications in other domains of mathematics and it is natural that more systems will turn up, that can be subsumed under the theory of seminearrings. We return to these applications on another occasion.

2. Seminearrings and S-semigroups

Definition 1: *A seminearring is a system* $(S, +, .)$ *such that*

1. $(S, +)$ *is a semigroup with unit* **0**.
2. $(S, .)$ *is a semigroup with zero* **0**.
3. $x(y+z) = xy+xz$ *for all* x, y, z *in* S.

DEFINITION 2: *A morphism of a seminearring S into a seminearring S' is a mapping ϕ such that*

$$\phi(x+y) = \phi(x)+\phi(y)$$
$$\phi(xy) = \phi(x)\phi(y)$$

for all $x, y \in S$.
$\phi S \subset S'$ is a seminearring with $\phi(0)$ as unit of $(\phi S, +)$ and zero of $(\phi S, .)$. The element $\phi(0)$ may be different from the zero of S'. *The kernel of a seminearring morphism is called ideal.*

Any morphism ϕ of a seminearring gives rise in a natural way to a factor structure, which again is a seminearring by introducing the following two-sidedly stable equivalence:

$$x \sim y \leftrightarrow \phi(x) = \phi(y)$$

and mapping every element onto its equivalence class. This factor seminearring will be denoted by S/ϕ.

DEFINITION 3: *An S-semigroup is a semigroup $(\Gamma, +)$ with unit ω, for which there is a seminearring S and a mapping $\Gamma \times S \to \Gamma$, $(\alpha, x) \to \alpha x$, such that for all $\alpha \in \Gamma$ and $x, y \in S$:*

1. $\alpha(x+y) = \alpha x + \alpha y$
2. $\alpha(xy) = (\alpha x)y$
3. $\alpha 0 = \omega$

REMARK 1: In $x+y$ the sign $+$ denotes addition in S, and in $\alpha x + \alpha y$ it denotes addition in Γ (αx and αy are elements of Γ).

For any S-semigroup Γ we have $\omega x = \omega$ for all $x \in S$, viz. $\omega x = (\omega 0)x = \omega(0x) = \omega 0 = \omega$.

DEFINITION 4: *An S-subsemigroup Δ of an S-semigroup Γ is a subsemigroup for which $\Delta S \subset \Delta$.*

REMARK 2: We conform to the convention that a subsemigroup is a non-void set, therefore an S-subsemigroup has a unit, notably ω (the unit of Γ).

DEFINITION 5: *An S-morphism of the S-semigroup Γ is a morphism ϕ of Γ into an S- semigroup Γ' such that $\phi(\alpha x) = \phi(\alpha)x$ for all $\alpha \in \Gamma$ and $x \in S$.*
An important, though simple, observation is that any S-morphism ϕ: $\Gamma \to \Gamma'$ maps ω onto ω' (the unit of Γ'), viz. $\phi(\omega) = \phi(\omega 0) = \phi(\omega)0 = \omega'$.

DEFINITION 6: *The complete preimage of $\phi(\omega)$ under the S-morphism ϕ: $\Gamma \to \Gamma'$ is called an S-kernel of Γ.*

Obviously if $(S, +,.)$ is a seminearring, $(S, +)$ is an S-semigroup. *The S-kernels of the S-semigroup $(S, +)$ are called right ideals of S.* Any S-semigroup Γ contains the two trivial S-kernels $\{\omega\}$ and Γ; the first one being the kernel of the identity mapping of Γ and the second one the kernel of the null-morphism.

Since an S-morphism of Γ is a morphism of Γ it follows that the S-kernels of Γ are normal subsemigroups of Γ (cf. e.g. [5], p. 278). Observe that normal subsemigroups of semigroups with unit contain that unit. It may further be noted that S-kernels are S-subsemigroups. For if $\Delta \subset \Gamma$ is S-kernel, an S-morphism ϕ exists with kernel Δ. Then $\phi(\delta x) = \phi(\delta)x = \omega' x = \omega'$ for all $x \in S$ and $\delta \in \Delta$, hence $\delta x \in \Delta$, so $\Delta S \subset \Delta$.

Definition 7: *A right admissible morphism of the S-semigroup Γ is a morphism ϕ of Γ such that for all $\alpha, \beta \in \Gamma$* and $x \in S$ *from $\phi(\alpha) = \phi(\beta)$ follows $\phi(\alpha x) = \phi(\beta x)$.*

Theorem 1: *If Δ is the kernel of the right admissible morphism ϕ of the S-semigroup Γ then*

$$\phi((\alpha+\delta+\beta)x) = \phi((\alpha+\beta)x)$$

for all $\alpha, \beta \in \Gamma$, $\delta \in \Delta$ and $x \in S$.

Proof. Evidently $\phi(\alpha+\delta+\beta) = \phi(\alpha+\beta)$ for all $\alpha, \beta \in \Gamma$ and $\delta \in \Delta$, hence by right admissibility

$$\phi((\alpha+\delta+\beta)x) = \varphi((\alpha+\beta)x)$$

for all $\alpha, \beta \in \Gamma$, $\delta \in \Delta$ and $x \in S$.

A description of S-kernels of Γ in terms of right admissible morphisms is contained in the following.

Theorem 2: *The S-kernels of the S-semigroup Γ are the kernels of right admissible morphisms of Γ.*

Proof. If Δ is kernel of the S-morphism ϕ, then ϕ is right admissible morphism of Γ with kernel Δ, for if $\phi(\alpha) = \phi(\beta)$ then $\phi(\alpha x) = \phi(\alpha)x = \phi(\beta)x = \phi(\beta x)$ for all $x \in S$. Conversely if Δ is kernel of a right admissible morphism ϕ of Γ then the definition $\phi(\alpha)\cdot x = \phi(\alpha x)$ turns ϕ into an S-morphism of Γ with kernel Δ.

In the following we use the transitive closure r''_Δ of the two-sidedly stable reflexive and symmetric relation r_Δ associated with a normal subsemigroup Δ of the semigroup Γ (with unit) defined by

$$\alpha \sim \beta(r_\Delta) \text{ if and only if } \alpha, \beta \in \{\xi+\Delta+\eta\}$$

for some ξ and η, being elements of Γ (cf. e.g. [5], p. 41 and 279 where this relation is introduced for arbitrary semigroups).

The morphism corresponding to r''_Δ will be denoted by λ_Δ. λ_Δ is the greatest common right divisor of the morphisms ϕ with $\Delta \subset \mathrm{Ker}(\phi)$ (cf. e.g. [5], p. 279).

Definition 8: *A normal subsemigroup Δ of the S-semigroup Γ is said to have property Q if the condition*

$$Q(\Delta)\colon \quad \text{for all } \alpha, \beta \in \Gamma,\ \delta \in \Delta \text{ and } x \in S$$
$$(\alpha+\delta+\beta)x \sim (\alpha+\beta)x \qquad (r''_\Delta)$$

holds.

Theorem 3: *The normal subsemigroup Δ of the S-semigroup Γ has property Q if and only if the morphism λ_Δ is right admissible.*

Proof. If λ_Δ is right admissible then Δ has property Q by theorem 1. Conversely suppose Δ has property Q. The morphism λ_Δ has kernel Δ.

Suppose $\lambda_\Delta(\alpha) = \lambda_\Delta(\beta)$, i.e. $\alpha \sim \beta(r''_\Delta)$, then there exist $\gamma_1 = \alpha, \gamma_2, \ldots, \gamma_p, \gamma_{p+1} = \beta$, such that

$$\gamma_i \sim \gamma_{i+1}(r_\Delta) \text{ for } i = 1, \ldots, p.$$

Hence there exist $\rho_i, \sigma_i \in \Gamma$ and $\delta_{i1}, \delta_{i2} \in \Delta$ such that

$$\gamma_i = \rho_i+\delta_{i1}+\sigma_i$$

and

$$\gamma_{i+1} = \rho_i+\delta_{i2}+\sigma_i \qquad \text{for } i = 1, \ldots, p.$$

So for all $x \in S$:

$$\gamma_i x = (\rho_i+\delta_{i1}+\sigma_i)x$$

and

$$\gamma_{i+1} x = (\rho_i+\delta_{i2}+\sigma_i)x$$

Then by property Q it follows

$$\gamma_i x \sim (\rho_i+\sigma_i)x \quad (r''_\Delta)$$

and

$$\gamma_{i+1} x \sim (\rho_i+\sigma_i)x \quad (r''_\Delta)$$

Hence

$$\gamma_i x \sim \gamma_{i+1} x \quad (r''_\Delta) \text{ for } i = 1, \ldots, p,$$

from which

$$ax \sim \beta x \quad (r''_\Delta)$$

which means that λ_Δ is right admissible.

COROLLARY 1: *If the normal subsemigroup Δ of the S-semigroup Γ has property Q then Δ is S-kernel of Γ.*

Property Q is not characteristic of S-kernels, i.e. normal subsemigroups Δ of the S-semigroup Γ may fail to satisfy $Q(\Delta)$ and still be kernels of S-morphisms (different from λ_Δ) as the following example shows.

EXAMPLE 1:

Let $(S, +,.)$ be the seminearring defined by

$$S = \{0, a, b, c, d\}$$

and

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	b	b
d	0	a	d	d	d

If $\phi(S, +)$ is defined by

+	{0, a}	{b, c, d}
{0, a}	{0, a}	{b, c, d}
{b, c, d}	{b, c, d}	{b, c, d}

then ϕ is an S-morphism of $(S, +)$ with kernel $D = \{0, \text{a}\}$, so D is S-kernel. However it does not have property Q, since λ_D, defined by $\lambda_D(0) = \lambda_D(a) = \{0, \text{a}\}$, $\lambda_D(\text{b}) = \{\text{b}\}$ and $\lambda_D(c) = \lambda_D(d) = \{c, \text{d}\}$ is not right admissible, because $\lambda_D(c) = \lambda_D(d)$, while $\lambda_D(cb) = \lambda_D(b) = \{\text{b}\} \neq \lambda_D(db) = \lambda_D(d) = \{\text{c, d}\}$.
On the other hand a normal subsemigroup Δ of an S-semigroup Γ, satisfying $Q(\Delta)$, may be kernel of a morphism that is not right admissible.

EXAMPLE 2:

Let $(\Gamma, +)$ be defined by $\Gamma = \{\omega, \alpha, \beta, \gamma, \delta\}$ and the same addition table as $(S, +)$ in example 1.

Let now $(S, +,.)$ be defined by

+	0	a
0	0	a
a	a	a

.	0	a
0	0	0
a	0	a

A product which makes $(\Gamma, +)$ an S-semigroup is

	0	a
ω	ω	ω
α	ω	α
β	ω	α
γ	ω	γ
δ	ω	γ

$\Delta = \{\omega, \alpha\}$ is a normal subsemigroup of Γ, satisfying $Q(\Delta)$, while the morphism ϕ, defined by $\phi(\omega) = \phi(\alpha) = \{\omega, \alpha\}$, $\phi(\beta) = \phi(\gamma) = \phi(\delta) = \{\beta, \gamma, \delta\}$ has kernel Δ and is not right admissible, because $\phi(\beta) = \phi(\gamma)$ and $\phi(\beta a) = \Delta \neq \phi(\gamma a) = \phi(\gamma) = \{\beta, \gamma, \delta\}$.

REMARK 3: If the S-semigroup Γ is a group and $\Delta \subset \Gamma$ a normal subsemigroup, then Δ is a normal subgroup of Γ. The relation $\alpha \sim \beta(r''_\Delta)$ then is equivalent to $\alpha - \beta \in \Delta$ and $Q(\Delta)$ reduces to $(\delta + \alpha)x - \alpha x \in \Delta$ for all $\alpha \in \Gamma$, $\delta \in \Delta$ and $x \in S$. This is Betsch's condition 1.4.1 in [1] if at the same time we further restrict S to be a nearring. In that case it reduces to Blackett's condition in [2] if Δ is assumed to be a normal subgroup of $(S, +)$. In all these cases the set of morphisms with kernel Δ reduces to a singleton.

3. Ideals

DEFINITION 9: *A morphism ϕ of $(S, +)$ of the seminearring $(S, +, .)$ is called left admissible if for all $x, y, z \in S$ from $\phi(x) = \phi(y)$ follows $\phi(zx) = \phi(zy)$. It is called admissible if it is both left and right admissible.*

THEOREM 4: *The ideals of the seminearring $(S, +, .)$ are the kernels of admissible morphisms of $(S, +)$.*

PROOF. A morphism ϕ of $(S, +, .)$ is a morphism of $(S, +)$ with the same kernel. If $\phi(x) = \phi(y)$ then

$$\phi(xz) = \phi(x)\phi(z) = \phi(y)\phi(z) = \phi(yz)$$

and

$$\phi(zx) = \phi(z)\phi(x) = \phi(z)\phi(y) = \phi(zy),$$

hence ϕ is an admissible morphism of $(S, +)$.

Conversely let ϕ be an admissible morphism of $(S, +)$, then a product in ϕS is defined by

$$\phi(x) \cdot \phi(y) = \phi(xy).$$

Evidently ϕ is a seminearring morphism.

THEOREM 5: *A normal subsemigroup D of $(S, +)$ of the seminearring $(S, +, .)$ is left invariant if and only if λ_D is left admissible.*

PROOF. If λ_D is left admissible then for any $d \in D$ and $z \in S$ we have $\lambda_D(zd) = \lambda_D(z0) = \lambda_D(0)$, i.e. $zd \in D$ for any $z \in S$ and $d \in D$, in other words $SD \subset D$.

Conversely let D be left invariant.
Let $\lambda_D(x) = \lambda_D(y)$, i.e. there exist

$$u_1 = x, u_2, \ldots, u_p, u_{p+1} = y \qquad \text{such that}$$
$$u_i \sim u_{i+1}(r_D) \qquad (i = 1, \ldots, p)$$

Hence there exist $r_i, s_i \in S$ and $d_{i1}, d_{i2} \in D$, such that

$$u_i = r_i + d_{i1} + s_i$$
$$u_{i+1} = r_i + d_{i2} + s_i \qquad (i = 1, \ldots, p)$$

Hence, for all $z \in S$:

$$zu_i = zr_i + zd_{i1} + zs_i$$
$$zu_{i+1} = zr_i + zd_{i2} + zs_i \qquad (i = 1, \ldots, p)$$

Since D is left invariant, we have

$$zd_{i1} \in D \text{ and } zd_{i2} \in D \qquad (i = 1, \ldots, p)$$

so

$$zu_i \sim zu_{i+1}(r_D) \qquad (i = 1, \ldots, p)$$

from which $zx \sim zy \quad (r_D'')$,
i.e.

$$\lambda_D(zx) = \lambda_D(zy).$$

COROLLARY 2: *If D is a normal subsemigroup of $(S, +)$ of the seminearring $(S, +, .)$ then λ_D is admissible if and only if D is left invariant and has property Q.* (Cf. theorem 3.)

Beside ideals we consider weak ideals, i.e. left invariant right ideals and strong (right) ideals, i.e. (right) ideals with property Q.

Evidently every strong (right) ideal is an (right) ideal. The converse however is not true. In example 1 the set $D = \{0, a\}$ is an ideal, because ϕ is admissible. Since λ_D is not admissible, D does not satisfy $Q(D)$ (corollary 2), hence D is not a strong ideal.

The following example shows that not every proper right ideal (i.e. a right ideal which is not a weak ideal) is a strong right ideal.

EXAMPLE 3: Let the seminearring $(T, +, .)$ be such that $(T, +)$ is $(S, +)$ from example 1 and $(T, .)$ is

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	b	b	b	b
c	0	b	b	b	b
d	0	d	d	d	d

$D = \{0, a\}$ is right ideal (cf. the morphism ϕ of example 1) and not left invariant (e.g. $ba = b \notin D$).
Property Q does not hold, because λ_D is not right admissible (cf. theorem 3).

THEOREM 6: *Every ideal of the seminearring S is a weak ideal.*

PROOF. Let D be an ideal of S. Then for all $d \in D$ and $x \in S$ we have

$$\phi(xd) = \phi(x)\phi(d) = \phi(x)\phi(0) = \phi(0),$$

hence $xd \in D$, so D is left invariant. Since ϕ is admissible it is right admissible and so D is S-kernel of $(S, +)$ by theorem 2.

COROLLARY 3: Since a strong ideal is an ideal with property Q it follows from theorems 5 and 6 that *the strong ideals are the normal subsemigroups D of $(S, +)$ for which λ_D is admissible.*

It is further clear that weak ideals with property Q are strong ideals.

4. Sets of morphisms

In order to obtain further information about S-kernels and ideals we need some facts about sets of morphisms of semigroups.

To any non-void set Φ of two-sidedly stable equivalences of a semigroup Γ there exist the two-sidedly stable equivalences inf Φ and sup Φ, defined respectively by $(\alpha, \beta) \in \inf \Phi$ if and only if $(\alpha, \beta) \in \phi$ for all $\phi \in \Phi$ and $(\alpha, \beta) \in \sup \Phi$ if and only if there exist $\phi_1, \ldots, \phi_p$ in Φ and $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{p+1} = \beta$ in Γ, such that $(\alpha_i, \alpha_{i+1}) \in \phi_i$ for $i = 1, \ldots, p$ (cf. e.g. [5], p. 268). Evidently $\inf \Phi \subset \sup \Phi$.

REMARK 4: In the following the same symbol will be used to denote two-sidedly stable equivalences and there corresponding natural morphisms. So $(\alpha, \beta) \in \phi$ and $\phi(\alpha) = \phi(\beta)$ have the same meaning. Moreover Γ will be assumed to have a unit (ω).

THEOREM 7: $\mathrm{Ker}(\inf \Phi) = \cap\{\mathrm{Ker}(\phi);\ \phi \in \Phi\}$

PROOF. By definition of inf Φ.

THEOREM 8: $\sum \mathrm{Ker}\,(\phi_i) \subset \mathrm{Ker}\,(\sup \Phi)$ *for any finite sum* $\sum \mathrm{Ker}\,(\phi_i)$ *with* $\phi_i \in \Phi$.

PROOF. If $\alpha = \alpha_1 + \ldots + \alpha_p$ with $\alpha_i \in \mathrm{Ker}(\phi_i)$ for $i = 1, \ldots, p$, then

$$\phi_1(\alpha) = \phi_1(\alpha_2 + \ldots + \alpha_p),$$
$$\phi_2(\alpha_2 + \ldots + \alpha_p) = \phi_2(\alpha_3 + \ldots + \alpha_p),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\phi_p(\alpha_p) = \phi_p(\omega),$$

so $(\alpha, \omega) \in \sup \Phi$, in other words

$$\alpha \in \mathrm{Ker}(\sup \Phi).$$

COROLLARY 4:

$$\mathrm{Ker}\,(\inf \Phi) = \cap\{\mathrm{Ker}\,(\phi);\ \phi \in \Phi\} \subset \cup\{\mathrm{Ker}(\phi);\ \phi \in \Phi\} \subset \mathrm{Ker}\,(\sup \Phi).$$

THEOREM 9: *The morphisms ϕ of the non-void set Φ have common kernel Δ if and only if*

$$\mathrm{Ker}(\inf \Phi) = \mathrm{Ker}(\sup \Phi) = \Delta.$$

PROOF. If $\mathrm{Ker}(\inf \Phi) = \mathrm{Ker}(\sup \Phi) = \Delta$ then $\mathrm{Ker}(\phi) = \Delta$ for all $\phi \in \Phi$, by corollary 4.

Conversely, suppose $\mathrm{Ker}(\phi) = \Delta$ for all $\phi \in \Phi$.
Let $\alpha \in \mathrm{Ker}\,(\sup \Phi)$, then $\phi_1, \ldots, \phi_p$ in Φ and $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{p+1} = \omega$ in Γ exist, such that

$$(\alpha_i, \alpha_{i+1}) \in \phi_i \qquad (i = 1, \ldots, p)$$

Since $(\alpha_p, \omega) \in \phi_p$, we have

$$\alpha_p \in \mathrm{Ker}(\phi_p) = \Delta = \mathrm{Ker}(\phi_{p-1}),$$

so by

$$\phi_{p-1}(\alpha_{p-1}) = \phi_{p-1}(\alpha_p) = \phi_{p-1}(\omega)$$

it follows

$$\alpha_{p-1} \in \mathrm{Ker}(\phi_{p-1}) = \Delta = \mathrm{Ker}(\phi_{p-2}), \text{ etc.}$$

So finally we arrive at

$$\alpha = \alpha_1 \in \mathrm{Ker}(\phi_1) = \Delta.$$

Hence

$$\mathrm{Ker}(\sup \Phi) \subset \Delta = \mathrm{Ker}(\inf \Phi),$$

and with corollary 4 it follows

$$\mathrm{Ker}(\inf \Phi) = \mathrm{Ker}(\sup \Phi).$$

If K is a non-void set of normal subsemigroups of the semigroup Γ, the class of normal subsemigroups containing all members of K is non-void, in fact contains Γ. The non-void intersection of this class of normal subsemigroups is called the normal subsemigroup generated by K and is denoted by $[K]$. It is the smallest normal subsemigroup containing all members of K.

THEOREM 10: $\lambda_{[K]} = \sup\{\lambda_\Delta ; \Delta \in K\}$.

PROOF. $\mathrm{Ker}(\sup\{\lambda_\Delta ; \Delta \in K\})$ is a normal subsemigroup.
From $\lambda_\Delta \leqq \sup\{\lambda_\Delta ; \Delta \in K\}$ for all $\Delta \in K$ follows

$$\Delta \subset \mathrm{Ker}(\sup\{\lambda_\Delta ; \Delta \in K\}) \text{ for all } \Delta \in K,$$

so

$$[K] \subset \mathrm{Ker}(\sup\{\lambda_\Delta ; \Delta \in K\})$$

and

$$\lambda_{[K]} \leqq \sup\{\lambda_\Delta ; \Delta \in K\}.$$

Conversely: from

$$\Delta \subset [K] \qquad \text{for all } \Delta \in K$$

follows

$$\lambda_\Delta \leqq \lambda_{[K]} \qquad \text{for all } \Delta \in K,$$

so

$$\sup\{\lambda_\Delta ; \Delta \in K\} \leqq \lambda_{[K]},$$

hence

$$\lambda_{[K]} = \sup\{\lambda_\Delta ; \Delta \in K\}.$$

For non-void classes F of sets Φ of two-sidedly stable equivalences, the following two-sidedly stable equivalences exist:

$$\inf\{\inf \Phi; \Phi \in F\},\ \inf\{\sup \Phi; \Phi \in F\},$$
$$\sup\{\inf \Phi; \Phi \in F\},\ \sup\{\sup \Phi; \Phi \in F\}.$$

These equivalences are obviously related by

$$\inf\{\inf \Phi; \Phi \in F\} \leqq \inf\{\sup \Phi; \Phi \in F\} \leqq \sup\{\sup \Phi; \Phi \in F\}$$

and

$$\inf\{\inf \Phi; \Phi \in F\} \leqq \sup\{\inf \Phi; \Phi \in F\} \leqq \sup\{\sup \Phi; \Phi \in F\}.$$

We further notice the following theorems.

THEOREM 11: $\inf\{\inf \Phi; \Phi \in F\} = \inf\{\cup F\}$.

PROOF. $(\alpha, \beta) \in \inf \{\inf \Phi; \Phi \in F\}$ if and only if
$(\alpha, \beta) \in \inf \Phi$ for all $\Phi \in F$, i.e. if and only if
$(\alpha, \beta) \in \phi$ for all φ for which a $\Phi \in F$ exists with $\phi \in \Phi$,
i.e. if and only if

$$(\alpha, \beta) \in \inf \{\cup F\}.$$

THEOREM 12: $\sup \{\sup \Phi; \Phi \in F\} = \sup \{\cup F\}$.

PROOF. Suppose $(\alpha, \beta) \in \sup \{\cup F\}$. Then there exist $\phi_1, \ldots, \phi_p$ in $\cup F$ and $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{p+1} = \beta$ in Γ, such that

$$(\alpha_i, \alpha_{i+1}) \in \phi_i \qquad \text{for } i = 1, \ldots, p.$$

To any ϕ_i there exists a Φ_i such that $\phi_i \in \Phi_i$, hence by $\phi_i \subset \sup \Phi_i$, we also have

$$(\alpha_i, \alpha_{i+1}) \in \sup \Phi_i \qquad i = 1, \ldots, p$$

so

$$(\alpha, \beta) \in \sup \{\sup \Phi; \Phi \in F\}$$

and thus

$$\sup \{\cup F\} \subset \sup \{\sup \Phi; \Phi \in F\}.$$

Conversely suppose

$$(\alpha, \beta) \in \sup \{\sup \Phi; \Phi \in F\},$$

then there exist

$$\sup \Phi_1, \ldots, \sup \Phi_p \qquad \text{with } \Phi_i \in F$$

and

$$\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{p+1} = \beta \qquad \text{in } \Gamma$$

such that

$$(\alpha_i, \alpha_{i+1}) \in \sup \Phi_i \qquad \text{for } i = 1, \ldots, p.$$

By definition of sup there exist to any $i = 1, \ldots, p$ morphisms $\phi_1^i, \ldots, \phi_{j_i}^i$ in Φ_i and elements of Γ:

$$\alpha_i = \alpha_1^i, \alpha_2^i, \ldots, \alpha_{j_i+1}^i = \alpha_{i+1}$$

such that

$$(\alpha_k^i, \alpha_{k+1}^i) \in \phi_k^i \qquad \text{for } k = 1, \ldots, j_i.$$

Now the finite sequence

$$\alpha = \alpha_1 = \alpha_1^1, \ldots, \alpha_{j_1+1}^1 = \alpha_2, \alpha_1^2 = \alpha_2, \ldots, \alpha_{j_p+1}^p = \beta$$

is connected by means of the morphisms ϕ_k^i of $\cup F$, and so

$$(\alpha, \beta) \in \sup \{\cup F\},$$

thus

$$\sup \{\sup \Phi ; \Phi \in F\} \subset \sup \{\cup F\},$$

consequently

$$\sup \{\sup \Phi; \Phi \in F\} = \sup \{\cup F\}.$$

5. Sets of ideals

We are now in a position to make some more statements about S-kernels and ideals.

Since for any class Φ of (left-, right-) admissible morphisms, the morphisms inf Φ and sup Φ are likewise (left-, right-) admissible it follows by theorems 2 and 7 that the intersection of a class of S-kernels is an S-kernel and by theorems 4 and 7 that the intersection of a class of ideals is an ideal. Property Q, in general, is not preserved under intersection.

As a further application we notice:

THEOREM 13: *The annihilator $A(\Delta)$ of a non-void subset Δ of the S-semigroup Γ is a right ideal of S.*

PROOF. Since $A(\Delta) = \cap\{A(\delta); \delta \in \Delta\}$ it suffices to show that $A(\alpha)$ is S-kernel of $(S, +)$ for arbitrary $\alpha \in \Gamma$. Since Γ is an S-semigroup, the mapping $x \rightarrow \alpha x$ is a right admissible morphism of $(S, +)$.
Its kernel is $A(\alpha)$, so $A(\alpha)$ is S-kernel by theorem 2.

REMARK 5: It may further be noted that the annihilator $A(\Gamma)$ of the S-semigroup Γ is left invariant, so $A(\Gamma)$ is a weak ideal of S.

In general the sum of two normal subsemigroups of a semigroup Γ fails to be a normal subsemigroup. The same is true for right ideals of a seminearring, as the following example shows.

EXAMPLE 4: Let $(U, +,.)$ be the seminearring with $(U, +)$ equal to $(S, +)$ of example 1 and let $(U,.)$ be defined by

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	b	b	b	b
c	0	c	c	c	c
d	0	d	d	d	d

Then $D = \{0, a\}$ and $E = \{0, c\}$ are right ideals of U. However $D+E = \{0, a, c, d\}$ is not a normal subsemigroup of $(U, +)$.

Instead the notion of generated normal subsemigroup of a set of normal subsemigroups extends to sets of S-kernels and ideals. For if K is a non-void set of S-kernels (ideals) then to any $\Delta \in K$ there exists a right admissible (admissible) morphism ϕ_Δ with kernel Δ. Then Ker(sup ϕ_Δ; $\Delta \in K$) is an S-kernel (ideal) which contains all $\Delta \in K$. So the set of S-kernels (ideals) containing all $\Delta \in K$ is non-void and its non-void intersection is again an S-kernel (ideal) which contains all $\Delta \in K$. This S-kernel (ideal) is called the S-kernel (ideal) generated by K. In general it is not equal to Ker(sup $\{\phi_\Delta; \Delta \in K\}$).

For sets of strong ideals we can assert more.

THEOREM 14: *If K is a non-void set of strong ideals of the seminearring $(S, +, .)$ then its generated normal subsemigroup $[K]$ of $(S, +)$ is a strong ideal of S.*

PROOF. By theorem 10 we have

$$[K] = \text{Ker}(\sup\{\lambda_D; D \in K\})$$

Since all $D \in K$ are strong ideals the morphisms λ_D are admissible by corollary 3, hence

$$\lambda_{[K]} = \sup\{\lambda_D; D \in K\} \qquad \text{(theorem 10)}$$

is likewise admissible, so $[K]$ is a strong ideal.

REMARK 6: Using theorem 3 instead of corollary 3 one shows in the same way that the generated normal subsemigroup $[K]$ of a non-void set K of S-kernels with property Q is an S-kernel with property Q.

With respect to the operations of intersection and generation the class of normal subsemigroups of a semigroup, the class of S-kernels of an S-semigroup, the class of weak ideals of a seminearring and the class of ideals of a seminearring are complete lattices.

With respect to the operation [ ], defined by

$$[D, E] = [\{D, E\}]$$

for normal subsemigroups D, E, the classes of S-kernels with property Q and of strong ideals are complete upper semilattices by remark 6 and theorem 14.

REMARK 7: If $(\Gamma, +)$ is a group, then the normal subsemigroups of Γ are normal subgroups (cf. e.g. [5], p. 278) and the operation $[\Delta, E]$ reduces to $\Delta + E$. The same remark applies if $(\Gamma, +)$ is an S-group and S a nearring and *a fortiori* if in this case $(\Gamma, +) = (S, +)$ (cf. [1]).

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