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G-functions as self-reciprocal in an integral transform

by

Roop Narain Kesarwani

1

It has been proved [**1**, p. 298 and **2**, p. 396] that the function

$$(1.1)\quad K(x) = \gamma\mu^{\gamma/2}x^{(\gamma-1)/2}G^{q,p}_{2p,2q}\left((\mu x)^\gamma \middle| \begin{matrix} a_1, \ldots, a_p, & -a_1, \ldots, -a_p \\ b_1, \ldots, b_q, & -b_1, \ldots, -b_q \end{matrix}\right),$$

where G denotes a Meijer's G-function [**3**, p. 207], plays the role of a symmetric Fourier kernel. Here μ and γ are real constants, p and q are integers such that

$$p \geqq 0, \qquad q \geqq 1$$

and a_j $(j = 1, \ldots, p)$, b_j $(j = 1, \ldots, q)$ are complex numbers satisfying

$$a_j - b_h \neq 1, 2, 3, \ldots \qquad j = 1, \ldots, p;\ h = 1, \ldots, q$$

and obeying conditions mentioned in the following theorem due to Fox [**2**, p. 399]. See also [**4**, p. 277].

THEOREM 1. *If*

(i) $\gamma > 0$, $q - p > 0$
(ii) $\text{Re}\,(\frac{1}{2} - a_j) > 0$, $j = 1, \ldots, p$
(iii) $\text{Re}\,(\frac{1}{2} + b_j) > 0$, $j = 1, \ldots, q$
(iii) $f(x) \in L_2(0, \infty)$
(iv) $K_1(x) = \int_0^x K(x)dx$

then the formula

$$(1.2)\qquad \frac{d}{dx}\int_0^\infty K_1(xu)f(u)\frac{du}{u} = g(x)$$

defines almost everywhere the function $g(x) \in L_2(0, \infty)$. *Also the reciprocal formula*

$$(1.3)\qquad \frac{d}{dx}\int_0^\infty K_1(xu)g(u)\,\frac{du}{u} = f(x)$$

holds almost everywhere.

The kernel (1.1) is a very general one. It contains as its particular cases Fourier type kernels discovered by various authors from time to time. Some of them have been listed in an earlier paper of the author [**5**, pp. 957—58].

In case we can differentiate with respect to x under the sign of integral, the formulas (1.2) and (1.3) reduce to

$$\int_0^\infty K(xu)f(u)du = g(x),$$

$$\int_0^\infty K(xu)g(u)du = f(x).$$

The functions $f(x)$ and $g(x)$ may be called *G-transforms* of each other. If further, $f(x) = g(x)$ so that

$$(1.4)\qquad \int_0^\infty K(xu)f(u)du = f(u),$$

then $f(x)$ is called *self-reciprocal* for the kernel (1.1). For the construction of a self-reciprocal function Fox [**2**, p. 407] has proved the following theorem which we shall use in this paper to obtain a G-function self-reciprocal in the sense of a relation almost equivalent to (1.4).

THEOREM 2. *If*

(i) $\gamma > 0,\ q-p > 0$

(ii) $\mathrm{Re}\,(\frac{1}{2}-a_j) > 0,\ j = 1, \ldots, p$
$\mathrm{Re}\,(\frac{1}{2}+b_j) > 0,\ j = 1, \ldots, q$

(iii) $K_1(x) = \int_0^x K(x)\,dx$

[1]) (iv) $N(s) = \mu^{(\frac{1}{2}-s)/2}\prod^q \Gamma(\frac{1}{2}+b_j+(s-\frac{1}{2})/\gamma)\prod^p \Gamma(\frac{1}{2}-a_j-(s-\frac{1}{2})/\gamma)$

(v) $E(s) = E(1-s)$

(vi) $N(s)E(s) \in L_2(\frac{1}{2}-i\infty,\ \frac{1}{2}+i\infty)$

(vii) $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} N(s)\,E(s)\,x^{-s}ds,$

[1]) For the sake of printing convenience the products $\prod_{j=1}^p f(j)$ and the sums $\sum_{j=1}^q g(j)$ in which the product or the summation starts from $j = 1$ will be denoted by $\prod^p f(j)$ and $\sum^q g(j)$ respectively.

then

$$\int_0^x f(x)dx = \int_0^\infty K_1(xy)f(y)\frac{dy}{y}. \tag{1.5}$$

Here also if we can differentiate with respect to x under the integral sign, then (1.5) reduces to (1.4). Whether we can differentiate under the integral sign or not we shall, however, say that $f(x)$ is self-reciprocal for (1.1) whenever (1.5) is satisfied.

In the next section we construct a self-reciprocal function with a suitable choice of $E(s)$. In the last section we mention a few special cases of our general formula.

2

By using the definition of the G-function [**3**, p. 207], it is possible to write

$$K(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{N(s)}{N(1-s)} x^{-s}\, ds, \tag{2.1}$$

where $N(s)$ is given by (iv) of theorem 2 and the path of integration is a straight line parallel to the imaginary axis with indentations, if necessary, to avoid the poles of the integrand. The conditions (i) and (ii) of theorem 2 are sufficient for the convergence of (2.1).

Let us suppose that $-2\beta_j \neq 1, 2, 3, \ldots$ $(j = 1, \ldots, m)$ and take

$$E(s) = \frac{\prod^m \Gamma(\tfrac{1}{2}+\beta_j+(s-\tfrac{1}{2})/\gamma)\prod^m \Gamma(\tfrac{1}{2}+\beta_j-(s-\tfrac{1}{2})/\gamma)}{\prod^n \Gamma(\tfrac{1}{2}+\alpha_j+(s-\tfrac{1}{2})/\gamma)\prod^n \Gamma(\tfrac{1}{2}+\alpha_j-(s-\tfrac{1}{2})/\gamma)} \tag{2.2}$$

which obviously satisfies (v) of theorem 2 and defines a function $f(x)$ by (vii) of theorem 2 so that

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mu^{(\frac{1}{2}-s)/2}\,\frac{\prod^q \Gamma(\tfrac{1}{2}+b_j+(s-\tfrac{1}{2})/\gamma)\prod^p \Gamma(\tfrac{1}{2}-a_j-(s-\tfrac{1}{2})/\gamma)\prod^m \Gamma(\tfrac{1}{2}+\beta_j+(s-\tfrac{1}{2})/\gamma)\prod^m \Gamma(\tfrac{1}{2}+\beta_j-(s-\tfrac{1}{2})/\gamma)}{\prod^n \Gamma(\tfrac{1}{2}+\alpha_j+(s-\tfrac{1}{2})/\gamma)\prod^n \Gamma(\tfrac{1}{2}+\alpha_j-(s-\tfrac{1}{2})/\gamma)}\, x^{-s}ds \\ &= \frac{\gamma\mu^{\gamma/4}x^{(\gamma-1)/2}}{2\pi i} \\ &\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\prod^q \Gamma(b_j+S)\prod^p \Gamma(1-a_j-S)\prod^m \Gamma(\beta_j+S)\prod^m \Gamma(1+\beta_j-S)}{\prod^n \Gamma(\alpha_j+S)\prod^n \Gamma(1+\alpha_j-S)}(\sqrt{\mu}x)^{-\gamma S}\,dS \end{aligned} \tag{2.3}$$

(2.4)

$$= \gamma \mu^{\gamma/4} x^{(\gamma-1)/2} G^{q+m,\, p+m}_{p+m+n,\, q+m+n}\left((\sqrt{\mu x})^{\gamma} \middle| \begin{matrix} a_1, \ldots, a_p, -\beta_1, \ldots, -\beta_m, \alpha_1, \ldots, \alpha_n \\ b_1, \ldots, b_q, \beta_1, \ldots, \beta_m, -\alpha_1, \ldots, -\alpha_n \end{matrix}\right).$$

To prove that this function is self-reciprocal for the kernel (1.1) by virtue of theorem 2, it remains to show that its condition (vi) is satisfied.

We know [6, p. 272] that for large $|z|$, $|\arg z| < \pi - \delta$, $\delta > 0$,

$$\Gamma(z+a) = \exp\{(z+a-\tfrac{1}{2})\log z - z\}\{\sqrt{2\pi} + O(|z|^{-1})\}. \tag{2.5}$$

If we write

$$\alpha = p+q+2m-2n,$$

$$\lambda = \sum^{q} b_j - \sum^{p} a_j + 2\sum^{m} \beta_j - 2\sum^{n} \alpha_j,$$

we see, on using (2.5), that the absolute value of $N(\frac{1}{2}+it)E(\frac{1}{2}+it)$ is comparable with

$$\exp(-\tfrac{1}{2}\alpha\pi|t|/\gamma)|t/\gamma|^{\mathrm{Re}\,\lambda},$$

when $|t|$ is large. The condition (vi) of theorem 2 is clearly satisfied if $\alpha > 0$. In case $\alpha = 0$, we only need to assume that $\mathrm{Re}\,\lambda < -\frac{1}{2}$.

For the convergence of the integral in (2.3), we first write $s = \frac{1}{2}+it$ and $\sqrt{\mu x} = Re^{i\Phi}$, $R > 0$, Φ real. By (2.5) again, the absolute value of the integral is comparable with

$$\exp(-\tfrac{1}{2}\alpha\pi|t|/\gamma)|t/\gamma|^{\mathrm{Re}\,\lambda} R^{-1/2} e^{\Phi t},$$

when $|t|$ is large. There are two cases in which the integral in question is convergent.

First case: $\alpha > 0$. The integral converges absolutely for $|\Phi| < \alpha\pi/2\gamma$ and defines a function analytic in the sector $|\arg x| < \min(\pi, \alpha\pi/2\gamma)$. The point $x = 0$ is tacitly excluded.

Second case: $\alpha = 0$. The integral (2.3) does not converge for complex x. For $x > 0$, it converges absolutely if $\mathrm{Re}\,\lambda < -1$ and there exists an analytic function of x, defined over $|\arg x| < \pi$, whose values for positive x are given by (2.3).

3

Giving suitable values to the parameters in (2.4), we can deduce as particular cases a number of self-reciprocal functions [2]) under different Fourier type transforms. We give here a few examples

[2]) Reference of the known cases has been indicated as far as it has been possible.

of them. The notations for the various transcendental functions occurring below are the same as used by Watson [7, p. 789].

(a) With $p = 0$, $q = 1$, $b_1 = \nu/2$, $\gamma = 2$, $\mu = \frac{1}{2}$, the kernel (1.1) becomes

$$\sqrt{x}\, G_{0,2}^{1,0}\left(x^2/4 \,\middle|\, \nu/2,\ -\nu/2\right) \equiv \sqrt{x}\, J_\nu(x), \tag{3.1}$$

which is the kernel of the Hankel transform [8, p. 245] of order ν. With the above special values of the parameters (2.4) becomes

$$\sqrt{2x}\, G_{m+n,\, m+n+1}^{1+m,\, m}\left(x^2/2 \,\middle|\, \begin{matrix} -\beta_1, \ldots, -\beta_m, \alpha_1, \ldots, \alpha_n \\ \nu/2, \beta_1, \ldots, \beta_m, -\alpha_1, \ldots, -\alpha_n \end{matrix}\right) \tag{3.2}$$

which is self-reciprocal [9, p. 286] in the Hankel transform of order ν. Let R_ν denote [8, p. 245] the class of functions self-reciprocal in the Hankel transform of order ν. It is interesting to note the following functions which are some of the many self-reciprocal functions that can be obtained from (3.2) by specializing the parameters. In what follows A is only a constant factor having an appropriate value in different cases.

(i) The functions

$$\sqrt{2x}\, G_{1,2}^{2,1}\left(x^2/2 \,\middle|\, \begin{matrix} -\nu/2+\frac{1}{2} \\ \nu/2,\ \nu/2-\frac{1}{2} \end{matrix}\right) \equiv A x^{\nu-\frac{1}{2}} e^{x^2/4} D_{-2\nu}(x)$$

and

$$\sqrt{2x}\, G_{1,2}^{2,1}\left(x^2/2 \,\middle|\, \begin{matrix} -\nu/2-\frac{1}{2} \\ \nu/2,\ \nu/2+\frac{1}{2} \end{matrix}\right) \equiv A x^{\nu+\frac{1}{2}} e^{x^2/4} D_{-2\nu}(x)$$

both are of class R_ν [10, p. 12].

(ii) The self-reciprocal function

$$\sqrt{2x}\, G_{2,3}^{3,2}\left(x^2/2 \,\middle|\, \begin{matrix} -\mu/2,\ -(\nu-2)/6 \\ \nu/2,\ \mu/2,\ (\nu-2)/6 \end{matrix}\right) \in R_\nu$$

was obtained by Agarwal [11, p. 318] expressed as a sum of three hypergeometric functions of the type ${}_2F_2$.

(iii) $$\sqrt{2x}\, G_{1,2}^{2,1}\left(x^2/2 \,\middle|\, \begin{matrix} 2m-\nu/2 \\ \nu/2,\ \nu/2-2m \end{matrix}\right) \equiv A x^{\nu-2m-\frac{1}{2}} e^{x^2/4}\, W_{3m-\frac{1}{2}-\nu,\, m}(x^2/2) \in R_\nu$$

(iv) $$\sqrt{2x}\, G_{0,1}^{1,0}\left(\frac{x^2}{2} \,\middle|\, \nu/2\right) \equiv A x^{\nu+1/2} e^{-x^2/2}$$

is the familiar self-reciprocal function of the class R_ν [8, p. 260].

(b) With $p = 0$, $q = 2$, $b_1 = \mu/2$, $b_2 = \nu/2$, $\mu = \frac{1}{4}$, $\gamma = 2$, the kernel (1.1) reduces to

$$\frac{1}{2}\sqrt{x}\, G^{2,0}_{0,4}\left(x^2/16 \mid \mu/2, \nu/2, -\mu/2, -\nu/2\right) \equiv \varpi_{\mu,\nu}(x) \tag{3.3}$$

which is the kernel of the Watson transform [**12**, p. 308]. A detailed study of this transform has been made by Bhatnagar [**13**] who denotes by $R_{\mu,\nu}$ the class of functions self-reciprocal for the kernel $\varpi_{\mu,\nu}$. With the above choice of the parameters, the function (2.4) reduces to

$$\sqrt{x}\, G^{2+m,m}_{m+n,m+n+2}\left(x^2/4 \,\middle|\, \begin{matrix} -\beta_1, \ldots, -\beta_m, \alpha_1, \ldots, \alpha_n \\ \mu/2, \nu/2, \beta_1, \ldots, \beta_m, -\alpha_1, \ldots, -\alpha_n \end{matrix}\right) \tag{3.4}$$

which belongs to $R_{\mu,\nu}$. It is interesting to note the following particular cases.

(i) $\sqrt{x}\, G^{2,0}_{0,2}\left(x^2/4 \mid \mu/2, \nu/2\right) \equiv A x^{(\mu+\nu+1)/2} K_{(\mu-\nu)/2}(x) \in R_{\mu,\nu}$

(ii) $\sqrt{x}\, G^{2,0}_{0,2}\left(x^2/4 \mid \mu/2, \mu/2+n\right) \equiv A x^{\mu+n+1/2} K_n(x) \in R_{\mu,\mu+2n}$ [**13**, p. 111].

(iii) $$\sqrt{x}\, G^{2,0}_{1,3}\left(x^2/4 \,\middle|\, \begin{matrix} (\nu+\frac{1}{2})/2 \\ (3\nu-\frac{1}{2})/2, (\nu-\frac{1}{2})/2, -(\nu+\frac{1}{2})/2 \end{matrix}\right) \equiv A x^{\nu} J_\nu(x/2) Y_\nu(x/2) \in R_{3\nu-1/2,\nu-1/2}.$$

(iv) $$\sqrt{x}\, G^{4,2}_{2,4}\left(x^2/4 \,\middle|\, \begin{matrix} -(\mu+1)/2, -\mu/2 \\ \mu/2, (\mu-1)/2, (\mu+1)/2, \mu/2 \end{matrix}\right) \equiv A x^{2\mu-1} e^{x/2} W_{-3\mu-1/2, 1/2}(x) \in R_{\mu,\mu-1}.$$

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