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Bounds for the solutions of $\Delta \omega \geq P(r)f(w)$


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by

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Let $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \ldots + \partial^2/\partial x_n^2$ be the $n$-dimensional Laplace operator and let $D_r$ and $S_r$ denote the open sphere $x_1^2 + x_2^2 + \ldots + x_n^2 < r^2$ ($r > 0$) and its boundary $x_1^2 + x_2^2 + \ldots + x_n^2 = r^2$, respectively. The aim of this paper is to find explicit bounds for the max. $\omega(Q)$ where $\omega(Q)$ satisfies the differential equation

(1) \[ \Delta \omega = P(r)f(\omega) \]

or, more generally, the differential inequality

(2) \[ \Delta \omega \geq P(r)f(\omega) \]

where $Q \in D_R$ with $R > r$ and $P(r)$ is positive, monotonically increasing and twice continuously differentiable. In theorem 1 an upper bound is obtained where $P(r)$ is either $C_1 e^{c'r^2}$ or $A r^n$ where $C_1$, $C_2$ and $A$ are arbitrary constants, $C_1$ and $A$ being positive and $C_2$ non-negative. However, in 2-dimensional case, an upper bound is given if $P(r)$ is $ar^\beta$ where $a$ and $\beta$ are arbitrary positive constants. In theorem 2 we find a lower bound in case $P(r)$ is $ar^\beta$ and $\Delta$ is a 2-dimensional Laplace operator. Also, the behaviour of these solutions at an isolated singularity is investigated.

Theorem 1. Let $f(\omega)$ be positive, non-decreasing, differentiable function in $(-\infty, \infty)$, for which

$$\int_{-\infty}^{\infty} \frac{dt}{f(t)} \quad (\omega > -\infty)$$

exists and

(3) \[ f'(\omega) \int_{-\infty}^{\infty} \frac{dt}{f(t)} \leq 1. \]

If

(G) \[ u(r) = \sup_{Q \in S_r} \omega(Q) \]

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where $\omega(Q)$ ranges over all functions of class $C^2$ in $D$, which satisfy (2), then

$$
\frac{C_1 e^{Cr^3}(R^2-r^2)^2}{4nR^2} \leq \int_{u(r)}^{\infty} \frac{dt}{f(t)}
$$

provided $P(r) = C_1 e^{Cr^3}$ ($C_1 > 0$ and $C_2 \geq 0$) and

$$
\frac{Ar^n(R^2-r^2)^2}{4nR^2} \leq \int_{u(r)}^{\infty} \frac{dt}{f(t)}
$$

in case $P(r) = Ar^n$ ($A > 0$). Also if $\Delta$ is a 2-dimensional Laplace operator and $P(r) = \alpha r^\delta$, then

$$
\frac{\alpha r^\delta(R^2-r^2)^2}{8R^2} \leq \int_{u(r)}^{\infty} \frac{dt}{f(t)}
$$

$\alpha$ and $\beta$ being arbitrary positive constants. The inequalities (4), (5) and (6) are sharp.

PROOF: Consider the function $g = g(r)$ defined by

$$
\frac{C(R^2-r^2)^2}{R^2} P(r) = \int_{u(r)}^{\infty} \frac{dt}{f(t)}
$$

where $C$ is a positive constant to be chosen later. Denoting by $x$ one of the variables $x_k$ and differentiating with respect to $x$, we have

$$
\frac{-4C x P(r)(R^2-r^2)}{R^2} + \frac{2C x \dot{P}(r)(R^2-r^2)}{R^2} = -\frac{g_x}{f(g)}
$$

where the dot denotes differentiation with respect to $r^2$. Differentiating again with respect to $x$

$$
\frac{-4C P(r)(R^2-r^2)}{R^2} + \frac{8C x^2 P(r)}{R^2} - \frac{16C x^2 \dot{P}(r)(R^2-r^2)}{R^2}
$$

$$
+ \frac{2C \ddot{P}(r)(R^2-r^2)^2}{R^2} + \frac{4C x^2 \ddot{P}(r)(R^2-r^2)}{R^2} = -\frac{g_{xx}}{f(g)} + \frac{g_x^2}{f^2(g)} f'(g).
$$

With the help of (7) and (8), we obtain,

$$
\frac{g_{xx}}{P(r)f(g)} = -\frac{8C x^2}{R^2} + \frac{4C(R^2-r^2)}{R^2} + \frac{16C x^2 \ddot{P}(r)(R^2-r^2)}{R^2 P(r)} + \frac{4C x^2}{R^2} \cdot C P(r)(R^2-r^2)^2 f'(g) \left[2 - \frac{\ddot{P}(r)(R^2-r^2)}{P(r)}\right]^2
$$

$$
- \frac{2}{P^2(r)} (2x^2 \ddot{P}(r) + \dddot{P}(r)) \int_{u(r)}^{\infty} \frac{dt}{f(t)}.
$$
Summing over all $x_k$, we have

\[
\frac{\Delta g}{P(r)f(g)} = -\frac{8Cr^2}{R^2} + \frac{4nC(R^2-r^2)}{R^2} + \frac{16Cr^2\dot{P}(r)(R^2-r^2)}{R^2P(r)} + \frac{4Cr^2}{R^2} \cdot \frac{CP(r)(R^2-r^2)\dot{f}(g)}{R^2} \left[2 - \frac{\dot{P}(r)(R^2-r^2)}{P(r)}\right]^2 \]

\[-\frac{2}{P^2(r)} (2r^2\ddot{P}(r) + \dot{P}(r)n) \int_{0}^{\infty} \frac{dt}{f(t)}.\]

Using (3) it reduces to

(9) \[
\frac{\Delta g}{P(r)f(g)} \leq 4C \left[ n - \frac{r^2}{R^2} (n-2) \right] - \frac{2C(R^2-r^2)^2}{R^2} \left( 2r^2\ddot{P}(r) + n\dot{P}(r) \frac{\dot{P}(r)}{P(r)} - \frac{2r^2\dot{P}^2(r)}{P^2(r)} \right).
\]

Now we consider three cases:

**Case I:** Choose $P(r)$ such that $\dot{P}(r) - \dot{P}^2(r)/P(r) = 0$ or $P = C_1 e^{C_2r^2}$ where $C_1 > 0$ and $C_2 \geq 0$ are arbitrary constants. Then (9) reduces to

\[
\frac{\Delta g}{C_1 e^{C_2r^2}f(g)} \leq 4C \left[ n - \frac{r^2}{R^2} (n-2) \right].
\]

If, $n \geq 2$ and $C = 1/4n$, it follows that

\[
\Delta g \leq C_1 e^{C_2r^2}f(g).
\]

Since $g(0) = 0$ and $g(r)$ increases to $\infty$ as $r \to R$ the proof of (4) will follow from the following lemma:

**Lemma:** Let $f(t)$ be monotonically increasing continuous function defined for all $t$. Suppose the functions $g$ and $\omega$ are subject to the inequalities

\[
\Delta g \leq P(r)f(g)
\]

and

\[
\Delta \omega \geq P(r)f(\omega)
\]

respectively, for $0 \leq r_0 < r < R$. If $g \to \infty$ for $r \to R$, then

\[
\omega \leq g
\]

for $r_0 < r < R$.

A proof of this lemma (for $r_0 = 0$) can be found in [3]. The changes required to provide it for $r_0 > 0$ are obvious.
Case II. Assume $P(r)$ to satisfy
\[
\frac{\dot{P}(r)}{P(r)} \left( 2r^2 \frac{\dot{P}(r)}{P(r)} - n \right) = 0.
\]

(i) If $\dot{P}(r)/P(r) = 0$ then $P = k$ where $k$ is an arbitrary positive constant. This case is implied by Case I if we choose $C_1 = k$ and $C_2 = 0$.

(ii) If $2r^2 \dot{P}(r)/P(r) - n = 0$ or, $P = Ar^n$ ($A$ being an arbitrary positive constant) the inequality (9) becomes
\[
\frac{\Delta g}{Ar^{nf(g)}} \leq 4C \left( n - \frac{r^2}{R^2} (n-2) \right).
\]
Again, if $n \geq 2$ and $C = 1/4n$, we have
\[
\Delta g \leq Ar^{nf(g)}.
\]
Now the proof of (5) will follow from the above lemma.

Case III: Choose $P$ such that $2r^2P(r)\dot{P}(r)+nP(r)\dot{P}(r)-2r^2\dot{P}^2(r)=0$ or, $P = \alpha r^\beta$ where $\alpha$ and $\beta$ are arbitrary positive constants and $n = 2$. Then (9) gives
\[
\frac{\Delta g}{\alpha r^\beta f(g)} \leq 8C.
\]
If $C = 1/8$, it follows that
\[
\Delta g \leq \alpha r^\beta f(g).
\]
Again, with the help of the Lemma, we get (6).

This completes the proof of theorem 1. Now, we derive the following corollaries.

**Corollary 1.** In case of a function $\omega$ satisfying
\[
\Delta \omega = C_1 e^{\omega + \alpha r^2}
\]
which is regular in $D_R$
\[
\omega \leq 2 \log \frac{2 \sqrt{n R}}{\sqrt{C_1 (R^2 - r^2)}} - C_2 r^2 \quad (n \geq 2)
\]
where $C_1 > 0$ and $C_2 \geq 0$ are arbitrary constants.

Indeed, setting $f(t) = e^t$ in (4), we get, where $\omega = u$
\[
\frac{C_1 e^{\alpha r^2 (R^2 - r^2)^2}}{4n R^2} \leq e^{-\omega}.
\]
Taking logarithm on both sides
\[ \omega \leq 2 \log \frac{2\sqrt{nR}}{\sqrt{C_1(R^2 - r^2)}} - C_2r^2. \]

**Remark:** Nehari [2] proved that in case of a solution \( u \) of
\[ \Delta u = e^u \]
which is regular in \( D_R \)
\[ \phi(r) \leq 2 \log \frac{2\sqrt{nR}}{(R^2 - r^2)} \]
where \( \phi(r) = \sup_{Q \in S_r} u(Q) \).

If we take \( C_1 = 1 \) and \( C_2 = 0 \) the above result becomes a particular case of this corollary.

**Corollary 2:** If \( \omega \) satisfies the equation
\[ \Delta \omega = Ar^n e^\omega \]
where \( A \) is an arbitrary positive constant, it is subject to the inequality
(10)
\[ \omega \leq 2 \log \frac{2\sqrt{nR}}{\sqrt{A r^{n/2}(R^2 - r^2)}}. \]

At an isolated singularity of \( \omega \), the behaviour of \( \omega \) is such that
\[ \lim_{r \to 0} \left( \frac{\omega}{\log \frac{1}{r}} \right) \leq n. \]

Set \( f(t) = e^t \) in (5). With \( u = \omega \), we obtain
\[ e^{-\omega} \geq \frac{Ar^n (R^2 - r^2)^2}{4nR^2}. \]

Taking logarithm
\[ \omega \leq 2 \log \frac{2\sqrt{nR}}{\sqrt{A(R^2 - r^2)r^{n/2}}}. \]

Now (10) could be written
\[ \omega \leq \log \frac{4nR^2}{A(R^2 - r^2)^2} + n \log \frac{1}{r}. \]

Dividing by \( \log 1/r \) and letting \( r \to 0 \)
\[ \lim_{r \to 0} \left( \frac{\omega}{\log \frac{1}{r}} \right) \leq n. \]
Remark: This is a generalisation of and an improvement upon a result of the author [1], namely, if \( \omega = \omega(x_1, x_2, \ldots, x_n) \) is a solution of

\[
\Delta \omega \geq r^2 e^\omega
\]

which is regular for \( 0 < r < R \), then

\[
\omega \leq \log \frac{4(n+4)R^4}{r^4(R^2-r^2)^2}
\]

and at an isolated singularity of \( \omega \),

\[
\lim_{r \to 0} \left( \frac{\omega}{\log \frac{1}{r}} \right) \leq 4.
\]

Corollary 3. Every solution \( \omega \) of

\[
\Delta \omega = \alpha r^\beta e^\omega
\]

where \( \alpha \) and \( \beta \) are arbitrary positive constants and \( \Delta \) is 2-dimensional Laplace operator, satisfies

\[
\omega \leq 2 \log \frac{2\sqrt{2} R}{\sqrt{\alpha r^n} (R^2-r^2)}.
\]

At an isolated singularity of \( \omega \)

\[
\lim_{r \to 0} \left( \frac{\omega}{\log \frac{1}{r}} \right) \leq \beta.
\]

Setting \( f(t) = e^t \) in (6), it could be proved exactly as Corollary 2.

In the next theorem we find a lower bound for the maximum of the solutions of the differential inequality

\[
(11) \quad \Delta \omega \geq \alpha r^\beta f(\omega)
\]

where \( \Delta \) is a 2-dimensional Laplace operator and \( \alpha, \beta \) are arbitrary positive constants.

Theorem 2. Let \( f(\omega) \) satisfy the conditions of theorem 1 with (3) replaced by

\[
(3') \quad f'(\omega) \int_\omega^\infty \frac{dt}{f(t)} = 1.
\]

If

\[
(G') \quad v(r) = \sup_{Q \in \mathcal{S}_r} \omega(Q)
\]
where $\omega(Q)$ ranges over all functions of class $C^2$ in $D_r$ ($r^2 = x_1^2 + x_2^2$) and which satisfy the inequality (11), then

$$\int_1^\infty \frac{dt}{f(t)} \leq \frac{\alpha \rho^\beta (R^2 - r^2)}{4}.$$

**Proof:** Consider the function $h = h_\rho(r)$ defined by

$$\frac{\rho^2 - r^2}{4} = \frac{1}{P(r)} \int_0^\infty \frac{dt}{f(t)} \quad (\rho > R > r)$$

where $P(r)$ is positive, monotonically increasing and twice continuously differentiable. Clearly, $h_\rho(r)$ belongs to the class $C^2$ in $D_R$ and satisfies the differential inequality (11) if $P(r) = \alpha r^\beta$. Differentiating (12) twice with respect to $x = x_k (k = 1, 2)$, we obtain

$$- \frac{x}{2} = - \frac{hx}{f(h)P(r)} - \frac{2x\dot{P}(r)}{P^2(r)} \int_0^\infty \frac{dt}{f(t)}$$

$$- \frac{1}{2} = - \frac{h_{xx}}{f(h)P(r)} + \frac{2xh_x \dot{P}(r)}{f(h)P^2(r)} + \frac{h_x^2}{f(h)P(r)} \int_0^\infty \frac{dt}{f(t)} - \frac{2\dot{P}(r)}{P^2(r)} \int_0^\infty \frac{dt}{f(t)}$$

$$- \frac{4x^2}{P^2(r)} \ddot{P}(r) \int_0^\infty \frac{dt}{f(h)} + \frac{8x^2 \dot{P}(r)}{P^3(r)} \int_0^\infty \frac{dt}{f(t)} + \frac{2x\dot{P}(r)}{P^2(r)} \cdot h_x.$$

Using (13) and rearranging, we get,

$$- \frac{1}{2} = - \frac{h_{xx}}{P(r) f(h)} \frac{2x^2 \dot{P}(r)}{P(r)}$$

$$+ x^2 P(r) f'(h) \left[ \frac{\rho^2 - r^2}{2} \cdot \frac{\dot{P}(r)}{P(r)} - \frac{1}{2} \right]^2 - \frac{2x^2 \dot{P}(r)}{P(r)} \cdot \frac{\rho^2 - r^2}{2}.$$

Summing over both $x_k$, we have

$$\frac{\Delta h}{P(r) f(h)} = 1 + \frac{2r^2 \dot{P}(r)}{P(r)}$$

$$+ r^2 P(r) f'(h) \left[ \frac{\rho^2 - r^2}{2} \cdot \frac{\dot{P}(r)}{P(r)} - \frac{1}{2} \right]^2 - \frac{2r^2 \dot{P}(r)}{P(r)} + \frac{2 \dot{P}(r)}{P(r)} \cdot \frac{\rho^2 - r^2}{2}.$$

Since $f' > 0$, we obtain with the help of (3')

$$\frac{\Delta h}{P(r) f(h)} \geq 1 - \frac{2r^2 \ddot{P} P + 2 \dot{P} (P - 2r^2 \dot{P}^2)}{P^2(r)} \cdot \frac{\rho^2 - r^2}{2}.$$

Now choose $P(r)$ such that $2r^2 P(r) \ddot{P}(r) + 2P(r) \dot{P}(r) - 2r^2 \dot{P}^2(r) = 0$ or, $P = \alpha r^\beta$ where $\alpha$ and $\beta$ are arbitrary positive constants. Hence,

$$\Delta h \geq \alpha r^\beta f(h).$$
Consequently \((G')\) and \((14)\) imply
\[
h(r) \leq v(r).
\]
Since we can take \(\rho\) arbitrary, close to \(R\), we have
\[
\int_{v}^{\infty} \frac{dt}{f(t)} \leq \frac{\alpha r^\rho (R^2 - r^2)}{4}
\]
which proves theorem 2.

**Corollary 4:** If \(\omega\) satisfies the equation
\[
\Delta \omega = \alpha x^\rho e^\omega \quad \left( \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)
\]
and is regular in \(D \rho\), then
\[
\log \frac{4}{\alpha r^\rho (R^2 - r^2)} \leq \omega.
\]
Moreover, at an isolated singularity of \(\omega\), the behaviour of \(\omega\) is such that
\[
\lim_{r \to 0} \left( \frac{\omega}{\log \frac{1}{r}} \right) \geq \beta.
\]
Setting \(f(t) = e^t\) in \((15)\), this could be proved exactly as Corollary 2.

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**References**

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