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Bounds for the solutions of $\Delta\omega \geq P(r)f(\omega)$

by

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Let $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \dots + \partial^2/\partial x_n^2$ be the n -dimensional Laplace operator and let D_r and S_r denote the open sphere $x_1^2 + x_2^2 + \dots + x_n^2 < r^2$ ($r > 0$) and its boundary $x_1^2 + x_2^2 + \dots + x_n^2 = r^2$, respectively. The aim of this paper is to find explicit bounds for the max. $\omega(Q)$ where $\omega(Q)$ satisfies the differential equation

$$(1) \quad \Delta\omega = P(r)f(\omega)$$

or, more generally, the differential inequality

$$(2) \quad \Delta\omega \geq P(r)f(\omega)$$

where $Q \in D_R$ with $R > r$ and $P(r)$ is positive, monotonically increasing and twice continuously differentiable. In theorem 1 an upper bound is obtained where $P(r)$ is either $C_1 e^{C_2 r^2}$ or $A r^n$ where C_1, C_2 and A are arbitrary constants, C_1 and A being positive and C_2 non-negative. However, in 2-dimensional case, an upper bound is given if $P(r)$ is αr^β where α and β are arbitrary positive constants. In theorem 2 we find a lower bound in case $P(r)$ is αr^β and Δ is a 2-dimensional Laplace operator. Also, the behaviour of these solutions at an isolated singularity is investigated.

THEOREM 1. *Let $f(\omega)$ be positive, non-decreasing, differentiable function in $(-\infty, \infty)$, for which*

$$\int_{\omega}^{\infty} \frac{dt}{f(t)} \quad (\omega > -\infty)$$

exists and

$$(3) \quad f'(\omega) \int_{\omega}^{\infty} \frac{dt}{f(t)} \leq 1.$$

If

$$(G) \quad u(r) = \sup_{Q \in S_r} \omega(Q)$$

where $\omega(Q)$ ranges over all functions of class C^2 in D_r which satisfy (2), then

$$(4) \quad \frac{C_1 e^{c_2 r^2} (R^2 - r^2)^2}{4nR^2} \leq \int_{u(r)}^\infty \frac{dt}{f(t)}$$

provided $P(r) = C_1 e^{c_2 r^2}$ ($C_1 > 0$ and $C_2 \geq 0$) and

$$(5) \quad \frac{Ar^n (R^2 - r^2)^2}{4nR^2} \leq \int_{u(r)}^\infty \frac{dt}{f(t)}$$

in case $P(r) = Ar^n$ ($A > 0$). Also if Δ is a 2-dimensional Laplace operator and $P(r) = \alpha r^\beta$, then

$$(6) \quad \frac{\alpha r^\beta (R^2 - r^2)^2}{8R^2} \leq \int_{u(r)}^\infty \frac{dt}{f(t)}$$

α and β being arbitrary positive constants. The inequalities (4), (5) and (6) are sharp.

PROOF: Consider the function $g = g(r)$ defined by

$$(7) \quad \frac{C(R^2 - r^2)^2}{R^2} P(r) = \int_\sigma^\infty \frac{dt}{f(t)}$$

where C is a positive constant to be chosen later. Denoting by x one of the variables x_k and differentiating with respect to x , we have

$$(8) \quad -\frac{4CxP(r)(R^2 - r^2)}{R^2} + \frac{2Cx\dot{P}(r)(R^2 - r^2)^2}{R^2} = -\frac{g_x}{f(g)}$$

where the dot denotes differentiation with respect to r^2 . Differentiating again with respect to x

$$\begin{aligned} & -\frac{4CP(r)(R^2 - r^2)}{R^2} + \frac{8Cx^2P(r)}{R^2} - \frac{16Cx^2\dot{P}(r)(R^2 - r^2)}{R^2} \\ & + \frac{2C\dot{P}(r)(R^2 - r^2)^2}{R^2} + \frac{4Cx^2\ddot{P}(r)(R^2 - r^2)^2}{R^2} = -\frac{g_{xx}}{f(g)} + \frac{g_x^2}{f^2(g)} f'(g). \end{aligned}$$

With the help of (7) and (8), we obtain,

$$\begin{aligned} \frac{g_{xx}}{P(r)f(g)} &= -\frac{8Cx^2}{R^2} + \frac{4C(R^2 - r^2)}{R^2} + \frac{16Cx^2\dot{P}(r)(R^2 - r^2)}{R^2P(r)} + \frac{4Cx^2}{R^2} \\ & \cdot \frac{CP(r)(R^2 - r^2)^2 f'(g)}{R^2} \cdot \left[2 - \frac{\dot{P}(r)(R^2 - r^2)}{P(r)} \right]^2 \\ & - \frac{2}{P^2(r)} (2x^2\ddot{P}(r) + \dot{P}(r)) \int_\sigma^\infty \frac{dt}{f(t)}. \end{aligned}$$

Summing over all x_k , we have

$$\begin{aligned} \frac{\Delta g}{P(r)f(g)} = & -\frac{8Cr^2}{R^2} + \frac{4nC(R^2-r^2)}{R^2} + \frac{16Cr^2\dot{P}(r)(R^2-r^2)}{R^2P(r)} + \frac{4Cr^2}{R^2} \\ & \cdot \frac{CP(r)(R^2-r^2)^2f'(g)}{R^2} \cdot \left[2 - \frac{\dot{P}(r)(R^2-r^2)}{P(r)}\right]^2 \\ & - \frac{2}{P^2(r)} (2r^2\ddot{P}(r) + \dot{P}(r)n) \int_0^\infty \frac{dt}{f(t)}. \end{aligned}$$

Using (3) it reduces to

$$(9) \quad \frac{\Delta g}{P(r)f(g)} \leq 4C \left[n - \frac{r^2}{R^2} (n-2) \right] - \frac{2C(R^2-r^2)^2}{R^2} \left\{ \frac{2r^2\ddot{P}(r) + n\dot{P}(r)}{P(r)} - \frac{2r^2\dot{P}^2(r)}{P^2(r)} \right\}.$$

Now we consider three cases:

Case I: Choose $P(r)$ such that $\ddot{P}(r) - \dot{P}^2(r)/P(r) = 0$ or $P = C_1 e^{c_2 r^2}$ where $C_1 > 0$ and $C_2 \geq 0$ are arbitrary constants. Then (9) reduces to

$$\frac{\Delta g}{C_1 e^{c_2 r^2} f(g)} \leq 4C \left\{ n - \frac{r^2}{R^2} (n-2) \right\}.$$

If, $n \geq 2$ and $C = 1/4n$, it follows that

$$\Delta g \leq C_1 e^{c_2 r^2} f(g).$$

Since $g(0) = 0$ and $g(r)$ increases to ∞ as $r \rightarrow R$ the proof of (4) will follow from the following lemma:

LEMMA: *Let $f(t)$ be monotonically increasing continuous function defined for all t . Suppose the functions g and ω are subject to the inequalities*

$$\Delta g \leq P(r)f(g)$$

and

$$\Delta \omega \geq P(r)f(\omega)$$

respectively, for $0 \leq r_0 < r < R$. If $g \rightarrow \infty$ for $r \rightarrow R$, then

$$\omega \leq g$$

for $r_0 < r < R$.

A proof of this lemma (for $r_0 = 0$) can be found in [3]. The changes required to provide it for $r_0 > 0$ are obvious.

Case II. Assume $P(r)$ to satisfy

$$\frac{\dot{P}(r)}{P(r)} \left(\frac{2r^2 \dot{P}(r)}{P(r)} - n \right) = 0.$$

(i) If $\dot{P}(r)/P(r) = 0$ then $P = k$ where k is an arbitrary positive constant. This case is implied by Case I if we choose $C_1 = k$ and $C_2 = 0$.

(ii) If $2r^2 \dot{P}(r)/P(r) - n = 0$ or, $P = Ar^n$ (A being an arbitrary positive constant) the inequality (9) becomes

$$\frac{\Delta g}{Ar^n f(g)} \leq 4C \left\{ n - \frac{r^2}{R^2} (n-2) \right\}.$$

Again, if $n \geq 2$ and $C = 1/4n$, we have

$$\Delta g \leq Ar^n f(g).$$

Now the proof of (5) will follow from the above lemma.

Case III: Choose P such that $2r^2 P(r) \ddot{P}(r) + nP(r) \dot{P}(r) - 2r^2 \dot{P}^2(r) = 0$ or, $P = \alpha r^\beta$ where α and β are arbitrary positive constants and $n = 2$. Then (9) gives

$$\frac{\Delta g}{\alpha r^\beta f(g)} \leq 8C.$$

If $C = \frac{1}{8}$, it follows that

$$\Delta g \leq \alpha r^\beta f(g).$$

Again, with the help of the Lemma, we get (6).

This completes the proof of theorem 1. Now, we derive the following corollaries.

COROLLARY 1. In case of a function ω satisfying

$$\Delta\omega = C_1 e^{\omega + c_1 r^2}$$

which is regular in D_R

$$\omega \leq 2 \log \frac{2\sqrt{n}R}{\sqrt{C_1}(R^2 - r^2)} - C_2 r^2 \quad (n \geq 2)$$

where $C_1 > 0$ and $C_2 \geq 0$ are arbitrary constants.

Indeed, setting $f(t) = e^t$ in (4), we get, where $\omega = u$

$$\frac{C_1 e^{c_1 r^2} (R^2 - r^2)^2}{4nR^2} \leq e^{-\omega}.$$

Taking logarithm on both sides

$$\omega \leq 2 \log \frac{2\sqrt{n}R}{\sqrt{C_1}(R^2-r^2)} - C_2 r^2.$$

REMARK: Nehari [2] proved that in case of a solution u of

$$\Delta u = e^u$$

which is regular in D_R

$$\phi(r) \leq 2 \log \frac{2\sqrt{n}R}{(R^2-r^2)}$$

where $\phi(r) = \sup_{Q \in S_r} u(Q)$.

If we take $C_1 = 1$ and $C_2 = 0$ the above result becomes a particular case of this corollary.

COROLLARY 2: *If ω satisfies the equation*

$$\Delta \omega = Ar^n e^\omega$$

where A is an arbitrary positive constant, it is subject to the inequality

$$(10) \quad \omega \leq 2 \log \frac{2\sqrt{n}R}{\sqrt{A}r^{n/2}(R^2-r^2)}.$$

At an isolated singularity of ω , the behaviour of ω is such that

$$\overline{\lim}_{r \rightarrow 0} \left(\frac{\omega}{\log \frac{1}{r}} \right) \leq n$$

set $f(t) = e^t$ in (5). With $u = \omega$, we obtain

$$e^{-\omega} \geq \frac{Ar^n(R^2-r^2)^2}{4nR^2}.$$

Taking logarithm

$$\omega \leq 2 \log \frac{2\sqrt{n}R}{\sqrt{A}(R^2-r^2)r^{n/2}}.$$

Now (10) could be written

$$\omega \leq \log \frac{4nR^2}{A(R^2-r^2)^2} + n \log \frac{1}{r}.$$

Dividing by $\log 1/r$ and letting $r \rightarrow 0$

$$\overline{\lim}_{r \rightarrow 0} \left(\frac{\omega}{\log \frac{1}{r}} \right) \leq n.$$

REMARK: This is a generalisation of and an improvement upon a result of the author [1], namely, if $\omega = \omega(x_1, x_2, \dots, x_n)$ is a solution of

$$\Delta\omega \geq r^2 e^\omega$$

which is regular for $0 < r < R$, then

$$\omega \leq \log \frac{4(n+4)R^4}{r^4(R^2-r^2)^2}$$

and at an isolated singularity of ω ,

$$\overline{\lim}_{r \rightarrow 0} \left(\frac{\omega}{\log \frac{1}{r}} \right) \leq 4.$$

COROLLARY 3. Every solution ω of

$$\Delta\omega = \alpha r^\beta e^\omega$$

where α and β are arbitrary positive constants and Δ is 2-dimensional Laplace operator, satisfies

$$\omega \leq 2 \log \frac{2\sqrt{2}R}{\sqrt{\alpha r^\beta} (R^2-r^2)}.$$

At an isolated singularity of ω

$$\overline{\lim}_{r \rightarrow 0} \left(\frac{\omega}{\log \frac{1}{r}} \right) \leq \beta.$$

Setting $f(t) = e^t$ in (6), it could be proved exactly as Corollary 2.

In the next theorem we find a lower bound for the maximum of the solutions of the differential inequality

$$(11) \quad \Delta\omega \geq \alpha r^\beta f(\omega)$$

where Δ is a 2-dimensional Laplace operator and α, β are arbitrary positive constants.

THEOREM 2. Let $f(\omega)$ satisfy the conditions of theorem 1 with (3) replaced by

$$(3') \quad f'(\omega) \int_\omega^\infty \frac{dt}{f(t)} = 1.$$

If

$$(G') \quad v(r) = \sup_{Q \in S_r} \omega(Q)$$

where $\omega(Q)$ ranges over all functions of class C^2 in D_r ($r^2 = x_1^2 + x_2^2$) and which satisfy the inequality (11), then

$$\int_{\nu}^{\infty} \frac{dt}{f(t)} \leq \frac{\alpha r^{\beta}(R^2 - r^2)}{4}.$$

PROOF: Consider the function $h = h_{\rho}(r)$ defined by

$$(12) \quad \frac{\rho^2 - r^2}{4} = \frac{1}{P(r)} \int_h^{\infty} \frac{dt}{f(t)} \quad (\rho > R > r)$$

where $P(r)$ is positive, monotonically increasing and twice continuously differentiable. Clearly, $h_{\rho}(r)$ belongs to the class C^2 in D_R and satisfies the differential inequality (11) if $P(r) = \alpha r^{\beta}$. Differentiating (12) twice with respect to $x = x_k$ ($k = 1, 2$), we obtain

$$(13) \quad -\frac{x}{2} = -\frac{hx}{f(h)P(r)} - \frac{2x\dot{P}(r)}{P^2(r)} \int_h^{\infty} \frac{dt}{f(t)}$$

$$-\frac{1}{2} = -\frac{h_{xx}}{f(h)P(r)} + \frac{2xh_x\dot{P}(r)}{f(h)P^2(r)} + \frac{h_x^2}{f^2(h)P(r)} f'h - \frac{2\dot{P}(r)}{P^2(r)} \int_h^{\infty} \frac{dt}{f(t)}$$

$$- \frac{4x^2}{P^2(r)} \ddot{P}(r) \int_h^{\infty} \frac{dt}{f(t)} + \frac{8x^2\dot{P}(r)}{P^3(r)} \int_h^{\infty} \frac{dt}{f(t)} + \frac{2x\dot{P}(r)}{P^2(r)} \cdot \frac{h_x}{f(h)}.$$

Using (13) and rearranging, we get,

$$-\frac{1}{2} = -\frac{h_{xx}}{P(r)f(h)} + \frac{2x^2\dot{P}(r)}{P(r)}$$

$$+ x^2P(r)f'(h) \left[\frac{\rho^2 - r^2}{2} \cdot \frac{\dot{P}(r)}{P(r)} - \frac{1}{2} \right]^2 - \frac{2x^2\ddot{P}(r) + \dot{P}(r)}{P(r)} \cdot \frac{\rho^2 - r^2}{2}.$$

Summing over both x_k , we have

$$\frac{\Delta h}{P(r)f(h)} = 1 + \frac{2r^2\dot{P}(r)}{P(r)}$$

$$+ r^2P(r)f'(h) \left[\frac{\rho^2 - r^2}{2} \cdot \frac{\dot{P}(r)}{P(r)} - \frac{1}{2} \right]^2 - \frac{2r^2\ddot{P}(r) + 2\dot{P}(r)}{P(r)} \cdot \frac{\rho^2 - r^2}{2}.$$

Since $f' > 0$, we obtain with the help of (3')

$$\frac{\Delta h}{P(r)f(h)} \geq 1 - \frac{2r^2\ddot{P}P + 2P\dot{P} - 2r^2\dot{P}^2}{P^2(r)} \cdot \frac{\rho^2 - r^2}{2}.$$

Now choose $P(r)$ such that $2r^2P(r)\ddot{P}(r) + 2P(r)\dot{P}(r) - 2r^2\dot{P}^2(r) = 0$ or, $P = \alpha r^{\beta}$ where α and β are arbitrary positive constants. Hence,

$$(14) \quad \Delta h \geq \alpha r^{\beta} f(h).$$

Consequently (G') and (14) imply

$$h(r) \leq v(r).$$

Since we can take ρ arbitrary, close to R , we have

$$(15) \quad \int_v^\infty \frac{dt}{f(t)} \leq \frac{\alpha r^\beta (R^2 - r^2)}{4}$$

which proves theorem 2.

COROLLARY 4: *If ω satisfies the equation*

$$(16) \quad \Delta\omega = \alpha r^\beta e^\omega \quad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)$$

and is regular in D_r , then

$$(17) \quad \log \frac{4}{\alpha r^\beta (R^2 - r^2)} \leq \omega.$$

Moreover, at an isolated singularity of ω , the behaviour of ω is such that

$$\overline{\lim}_{r \rightarrow 0} \left(\frac{\omega}{\log \frac{1}{r}} \right) \geq \beta.$$

Setting $f(t) = e^t$ in (15), this could be proved exactly as Corollary 2.

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