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On the derivative of a $G$-function whose argument is a power of the variable

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On the derivative of a $G$-function
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by

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In this paper we have established some formulae on the $N$-th order derivative of $G_{pq}^{ln}(\beta x^{|a|})$. The Mellin-Barnes type integral [2. p. 207] which we have employed is

\[
G_{pq}^{ln}(x|a_1 \ldots a_q, b_1 \ldots b_p) = \frac{1}{2\pi i} \int_{L} \frac{\prod_{j=1}^{l} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\prod_{j=1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s)} x^s ds
\]

where an empty product is interpreted as 1, $0 \leq l \leq q$, $0 \leq n \leq p$ and the path $L$ of integration runs from $-i\infty$ to $+i\infty$ so that all the poles of $\Gamma(b_j-s)$, $j = 1, 2, \ldots l$ are to the right and all the poles of $\Gamma(1-a_j+s)$, $j = 1, 2, \ldots n$ to the left of $L$. The formula is valid for $p + q < 2(1 + n)$ and $|\arg x| < (l + n - \frac{1}{2}p - \frac{1}{2}q)\pi$. $a_j - b_h \neq 1, 2, \ldots$ for $j = 1, \ldots, n$ and $h = 1, \ldots, l$. In the formulae (2.1), (2.2), (3.1), (4.1), (4.8)–(4.5) the conditions mentioned as (1.1) are tacitly supposed to be fulfilled. Although the well known technique is employed, the final result depends on the fact that in the formula

\[
\Gamma(mz) = (2\pi)^{(1-m)/2}m^{mz-1} \prod_{R=0}^{m-1} \Gamma\left(z + \frac{R}{m}\right) \quad m = 2, 3 \ldots
\]

$z, z+1/m, z+2/m, \ldots$ are in Arithmetical Progression. The other formulae used are

\[
z(z-1) \ldots (z-N-1) = \frac{\Gamma(z+1)}{\Gamma(z-N-1)},
\]

\[
z(z+1) \ldots (z+N-1) = \frac{\Gamma(z+N)}{\Gamma(z)}.
\]
The first formula to be proved is

\[
\frac{d^N}{dx^N} x^{(a_1-1)} G_{pq}^n \left( \frac{\beta x}{b_1 \ldots b_q} \right) = \left( -n \right)^N x^{(a_1-1)-N} G_{pq}^n \left( \frac{\beta x}{b_1 \ldots b_q} \right)
\]

provided \( r < n \) and the parameters \( a_1, a_2, \ldots, a_r \) are in A.P. with common difference \(-1/r\).

**PROOF:**

Using (1.1) the L.H.S. of (2.1)

\[
= \frac{1}{2\pi i} \int \left( \prod_{j=1}^{i} \Gamma(b_j-s) \prod_{j=r+1}^{r} \Gamma(1-a_j+s) \right) \left( \prod_{j=1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s) \right) \beta^s \prod_{j=1}^{r} \Gamma(1-a_j+s) \frac{d^N}{dx^N} x^{(a_1-1-s)} ds
\]

using (1.4) and (1.2) we get

\[
= \left( -r \right)^N x^{(a_1-1)-N} \left( \prod_{j=1}^{i} \Gamma(b_j-s) \prod_{j=1}^{r} \Gamma(1-a_j-N/r+s) \right) \left( \prod_{j=1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s) \right) \beta^s \prod_{j=1}^{r} \Gamma(1-a_j-N/r+s) \frac{d^N}{dx^N} x^{(a_1-1-s)} ds
\]

provided \( r < n \) and the parameters \( a_1, a_2, \ldots, a_r \) are in A.P. with common difference \(-1/r\).

Putting \( N = 1 \) and \( s = 1/x \) we get

\[
x \frac{d}{dx} G_{pq}^n \left( \frac{\beta x}{b_1 \ldots b_q} \right) = r G_{pq}^n \left( \frac{\beta x}{b_1 \ldots b_q} \right) a_1 \ldots a_p
\]

where \( a_1, a_2, \ldots, a_r \) are in A.P. with common difference \(-1/r\).

Putting \( r = 1 \) in (2.1) a result of Bhise [1] follows.

Putting \( r = 1 \) in (2.2) we get a known result (2. p. 210).
The second formula to be established is

\[
\frac{d^N}{dx^N} x^{-r\beta_1} G^{ln}_{pq}(\beta x^r| \begin{array}{c} a_1 \cdots a_p \\ b_1 \cdots b_n \end{array})
\]

\[
= (-r)^N x^{-r\beta_1-N} G^{ln}_{pq}(\beta x^r| \begin{array}{c} a_1 \cdots a_p \\ b_1+N/r, \ldots b_r+N/r, b_{r+1}, \ldots b_q \end{array})
\]

provided \( r < l \) and the parameters \( b_1, b_2, \ldots b_r \) are in A.P. with common difference \( 1/r \).

This formula can be derived from (2.1) by using the well-known property

\[
G^{ln}_{pq}(x| \begin{array}{c} a_j \\ b_j \end{array}) = G^{ln}_{pq}(\frac{1}{x}| \begin{array}{c} 1-b_j \\ 1-a_j \end{array}).
\]

Putting \( N = 1 \) in (3.1) we get

\[
x \frac{d}{dx} G^{ln}_{pq}(\beta x^r| \begin{array}{c} a_1 \cdots a_p \\ b_1 \cdots b_q \end{array}) = rb_1 G^{ln}_{pq}(\beta x^r| \begin{array}{c} a_1 \cdots a_p \\ b_1 \cdots b_q \end{array})
\]

\[
-\frac{r}{N} G^{ln}_{pq}(\beta x^r| \begin{array}{c} a_1 \cdots a_p \\ b_1+1/r, \ldots b_r+1/r, b_{r+1}, \ldots b_q \end{array})
\]

where \( b_1, b_2, \ldots b_r \) are in A.P. with common difference \( 1/r \).

Putting \( r = 1 \) in (3.1) and (3.2) two results of Bhise [1] follow.

The third formula sought to be established is

\[
\frac{d^N}{dx^N} x^{-r(a_{p-r+1}-1/r)} G^{ln}_{pq}(\beta | \begin{array}{c} a_1 \cdots a_p \\ x^r b_1 \cdots b_q \end{array})
\]

\[
= r^N x^{r(a_{p-r+1}-1/r)-N} G^{ln}_{pq}(\beta | \begin{array}{c} a_1 \cdots a_{p-r} \cdots a_{p-r+1}-N/r, \ldots a_p-N/r \\ x^r b_1 \cdots b_q \end{array})
\]

provided \( p-r+1 > n \) and the parameters \( a_{p-r+1}, \ldots a_p \) are in A.P. with common difference \( 1/r \).

**Proof:** Using (1.1) the L.H.S. of (4.1) becomes
Using (1.3) and (1.2) we get after little simplification (4.2) to be

\[ r^N x^{(a_p-r+1-1/r)-N} G_{pq}^{|x|^N} \left( \beta x^r \left| a_1 \ldots a_p \right. \right. \left. \left. b_1 \ldots b_q \right) \]

The fourth formula is

\[ \frac{d^N}{dx^N} x^{-r(b_{q-r+1}+1/r-1)} G_{pq}^{|x|^N} \left( \beta x^r \left| a_1 \ldots a_p \right. \right. \left. \left. b_1 \ldots b_q \right) \]

\[ = r^{N} x^{-(b_{q-r+1}+1/r-1)} G_{pq}^{|x|^N} \left( \beta x^r \left| a_1 \ldots a_p \right. \right. \left. \left. b_1 \ldots b_{q-r}, b_{q-r+1+N/r} \ldots b_{q+N/r} \right) \]

provided \( q-r+1 > l \) and the parameters \( b_{q-r+1} \ldots b_q \) are in A.P. with common difference \(-1/r\).

The proof can be adduced on lines similar to (4.1).

Putting \( N = 1 \) and \( x = 1/x \) in (4.2) we get

\[ x \frac{d}{dx} G_{pq}^{|x|^N} \left( \beta x^r \left| a_1 \ldots a_p \right. \right. \left. \left. b_1 \ldots b_q \right) \]

\[ = r \left( a_{p-r+1} - \frac{1}{r} \right) G_{pq}^{|x|^N} \left( \beta x^r \left| a_1 \ldots a_p \right. \right. \left. \left. b_1 \ldots b_q \right) \]

\[ - r G_{pq}^{|x|^N} \left( \beta x^r \left| a_1 \ldots a_{p-r}, a_{p-r+1} - 1/r, \ldots a_p - 1/r \right. \right. \left. \left. b_1 \ldots b_q \right) \]

Putting \( N = 1 \) in (4.3) we get

\[ x \frac{d}{dx} G_{pq}^{|x|^N} \left( \beta x^r \left| a_1 \ldots a_p \right. \right. \left. \left. b_1 \ldots b_q \right) \]

\[ = r G_{pq}^{|x|^N} \left( \beta x^r \left| a_1 \ldots a_p \right. \right. \left. \left. b_1 \ldots b_{q-r}, b_{q-r+1+1/r}, \ldots b_q+1/r \right) \]

\[ + r \left( b_{q-r+1} + \frac{1}{r} - 1 \right) G_{pq}^{|x|^N} \left( \beta x^r \left| a_1 \ldots a_p \right. \right. \left. \left. b_1 \ldots b_q \right) \]

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REFERENCES

Bhise, V. M.,

Erdelyi, A.,

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