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On the derivative of a G-function whose argument is a power of the variable

by

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1

In this paper we have established some formulae on the N-th order derivative of $G_{pq}^{ln}(\beta x^r|_{bs}^{aj})$. The Mellin-Barnes type integral [2. p. 207] which we have employed is

$$(1.1)\quad G_{pq}^{ln}\left(x \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right.\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{l} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\prod_{j=l+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s)} x^s ds$$

where an empty product is interpreted as 1, $0 \leqq l \leqq q$, $0 \leqq n \leqq p$ and the path L of integration runs from $-i\infty$ to $+i\infty$ so that all the poles of $\Gamma(b_j-s)$, $j = 1, 2, \dots l$ are to the right and all the poles of $\Gamma(1-a_j+s)$, $j = 1, 2, \dots n$ to the left of L. The formula is valid for $p+q < 2(1+n)$ and $|\arg x| < (l+n-\frac{1}{2}p-\frac{1}{2}q)\pi$. $a_j-b_h \neq 1, 2, \dots$ for $j = 1, \dots, n$ and $h = 1, \dots, l$. In the formulae (2.1), (2.2), (3.1), (4.1), (4.3)—(4.5) the conditions mentioned as (1.1) are tacitly supposed to be fulfilled. Although the well known technique is employed, the final result depends on the fact that in the formula

$$(1.2)\qquad \Gamma(mz) = (2\pi)^{(1-m)/2} m^{mz-\frac{1}{2}} \prod_{R=0}^{m-1} \Gamma\left(z+\frac{R}{m}\right) \qquad m = 2, 3 \dots$$

z, $z+1/m$, $x+2/m$, ... are in Arithmetical Progression. The other formulae used are

$$(1.3)\qquad z(z-1)\dots(z-\overline{N-1}) = \frac{\Gamma(z+1)}{\Gamma(z-\overline{N-1})},$$

$$(1.4)\qquad z(z+1)\dots(z+N-1) = \frac{\Gamma(z+N)}{\Gamma(z)}.$$

2

The first formula to be proved is

$$(2.1)\quad \begin{aligned} &\frac{d^N}{dx^N} x^{r(a_1-1)} G_{pq}^{ln}\left(\frac{\beta}{x^r}\middle| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix}\right) \\ &= (-n)^N x^{r(a_1-1)-N} G_{pq}^{ln}\left(\frac{\beta}{x^r}\middle| \begin{matrix} a_1 - N/r, \dots a_r - N/r, a_{r+1}, \dots a_p \\ b_1 \dots b_q \end{matrix}\right) \end{aligned}$$

provided $r < n$ and the parameters $a_1, a_2, \dots a_r$ are in A.P. with common difference $-1/r$.

PROOF:

Using (1.1) the L.H.S. of (2.1)

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{j} \Gamma(b_j - s) \prod_{j=r+1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=l+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} \cdot \beta^s \prod_{j=1}^{r} \Gamma(1 - a_j + s) \frac{d^N}{dx^N} x^{(a_1 - 1 - s)} ds$$

using (1.4) and (1.2) we get

$$= (-r)^N x^{r(a_1-1)-N}$$

$$\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{l} \Gamma(b_j - s) \prod_{j=1}^{r} \Gamma(1 - \overline{a_j - N/r} + s) \prod_{j=r+1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=l+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} \left(\frac{\beta}{x^r}\right)^s ds$$

$$= (-r)^N x^{(a_1-1)-N} G_{pq}^{ln}\left(\frac{\beta}{x^r}\middle| \begin{matrix} a_1 - N/r, \dots a_r - N/r, a_{r+1}, \dots a_p \\ b_1 \dots b_q \end{matrix}\right)$$

provided $r < n$ and the parameters $a_1, a_2, \dots a_r$ are in A.P. with common difference $-1/r$.

Putting $N = 1$ and $s = 1/x$ we get

(2.2)

$$x \frac{d}{dx} G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix}\right) = r G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 - 1/r, \dots a_r - 1/r, a_{r+1}, \dots a_p \\ b_1 \dots b_q \end{matrix}\right)$$

$$+ r(a_1 - 1) G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix}\right)$$

where $a_1, a_2, \dots a_r$ are in A.P. with common difference $-1/r$.

Putting $r = 1$ in (2.1) a result of Bhise [1] follows.

Putting $r = 1$ in (2.2) we get a known result (2. p. 210].

3

The second formula to be established is

$$
(3.1)\quad \begin{aligned} &\frac{d^N}{dx^N}\, x^{-rb_1} G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix}\right) \\ &= (-r)^N x^{-rb_1-N} G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1+N/r, \ldots b_r+N/r,\, b_{r+1}, \ldots b_q \end{matrix}\right) \end{aligned}
$$

provided $r < l$ and the parameters $b_1, b_2, \ldots b_r$ are in A.P. with common difference $1/r$.

This formula can be derived from (2.1) by using the well-known property

$$
G_{pq}^{ln}\left(x \middle| \begin{matrix} a_j \\ b_s \end{matrix}\right) = G_{qp}^{nl}\left(\frac{1}{x} \middle| \begin{matrix} 1-b_s \\ 1-a_j \end{matrix}\right).
$$

Putting $N = 1$ in (3.1) we get

$$
(3.2)\quad \begin{aligned} x\frac{d}{dx} G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix}\right) &= rb_1 G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix}\right) \\ &-rG_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1+1/r, \ldots b_r+1/r,\, b_{r+1} \ldots b_q \end{matrix}\right) \end{aligned}
$$

where $b_1, b_2, \ldots b_r$ are in A.P. with common difference $1/r$. Putting $r = 1$ in (3.1) and (3.2) two results of Bhise [1] follow.

4

The third formula sought to be established is

$$
(4.1)\quad \begin{aligned} &\frac{d^N}{dx^N}\, x^{r(a_{p-r+1}-1/r)} G_{pq}^{ln}\left(\frac{\beta}{x^r} \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix}\right) \\ &= r^N x^{r(a_{p-r+1}-1/r)-N} G_{pq}^{ln}\left(\frac{\beta}{x^r} \middle| \begin{matrix} a_1 \ldots a_{p-r},\, a_{p-r+1}-N/r, \ldots a_p-N/r \\ b_1 \ldots b_q \end{matrix}\right) \end{aligned}
$$

provided $p-r+1 > n$ and the parameters $a_{p-r+1}, \ldots a_p$ are in A.P. with common difference $1/r$.

PROOF: Using (1.1) the L.H.S. of (4.1) becomes

$$
(4.2)\qquad = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p-r}\Gamma(a_j-s)\prod_{j=p-r+1}^{p}\Gamma(a_j-s)} \cdot \beta^s \frac{d^N}{dx^N} x^{r(a_{p-r+1}-1/r-s)}\,ds.
$$

Using (1.3) and (1.2) we get after little simplification (4.2) to be

$$
= r^N x^{r(a_{p-r+1}-1/r)-N} G_{pq}^{ln}\left(\frac{\beta}{x^r}\middle| \begin{matrix} a_1 \cdot, \cdot a_{p-r},\ a_{p-r+1}-N/r, \cdot\cdot, a_p - N/r \\ b_1 \ldots b_q \end{matrix}\right).
$$

The fourth formula is

$$
(4.3)\qquad \begin{aligned} &\frac{d^N}{dx^N} x^{-r(b_{q-r+1}+1/r-1)} G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix}\right) \\ &= r^N x^{-r(b_{q-r+1}+1/r-1)} G_{pr}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_{q-r},\ b_{q-r+1+N/r} \ldots b_{q+N/r} \end{matrix}\right) \end{aligned}
$$

provided $q-r+1 > l$ and the parameters $b_{q-r+1} \ldots b_q$ are in A.P. with common difference $-1/r$.

The proof can be adduced on lines similar to (4.1).

Putting $N = 1$ and $x = 1/x$ in (4.2) we get

$$
(4.4)\qquad \begin{aligned} x\frac{d}{dx} G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix}\right) &= r\left(a_{p-r+1} - \frac{1}{r}\right) G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix}\right) \\ &-rG_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_{p-r},\ a_{p-r+1}-1/r, \ldots a_p - 1/r \\ b_1 \ldots b_q \end{matrix}\right) \end{aligned}
$$

Putting $N = 1$ in (4.3) we get

$$
(4.5)\qquad \begin{aligned} &x\frac{d}{dx} G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix}\right) \\ &= rG_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_{q-r},\ b_{q-r+1}+1/r, \ldots b_q + 1/r \end{matrix}\right) \\ &+r\left(b_{q-r+1} + \frac{1}{r} - 1\right) G_{pq}^{ln}\left(\beta x^r \middle| \begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix}\right). \end{aligned}
$$

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