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FU CHENG HSIANG

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# On an extension of a theorem of O. Szász

by

Fu Cheng Hsiang

## 1

Suppose that  $f(x)$  is integrable in Lebesgue's sense and periodic with period  $2\pi$ . Let its Fourier series be

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and let

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} \bar{A}_n(x)$$

be the conjugate series of the Fourier series of  $f(x)$ . Write

$$\bar{S}_n(x) = \sum_{\nu=1}^n (b_\nu \cos \nu x - a_\nu \sin \nu x) \equiv \sum_{\nu=1}^n \bar{A}_\nu(x).$$

Let  $\bar{\sigma}_n^\alpha(x)$  be the  $n$ -th Cesàre mean of order  $\alpha$  of the sequence  $\{\bar{S}_n(x)\}$ . O. Szász [3] has established the following

**THEOREM A.** *At a given point  $x$ , if there exists a number  $D(x)$ , such that*

$$\begin{aligned} \text{(i)} \quad \Psi(t) &= \int_0^t \psi(u) du \\ &\equiv \int_0^t \{f(x+u) - f(x-u) - D(x)\} du \\ &= o(t), \end{aligned}$$

$$\text{(ii)} \quad \Psi^*(t) = \int_0^t |\psi(u)| du = O(t)$$

as  $t \rightarrow +0$ , then

$$\lim_{n \rightarrow \infty} \{\bar{\sigma}_{2n}^1(x) - \bar{\sigma}_n^1(x)\} = \frac{1}{\pi} D(x) \log 2.$$

*It should be noted that the corresponding conditions:*

$$\begin{aligned}
 \text{(iii)} \quad \Phi(t) &= \int_0^t \varphi(u) du \\
 &= \int_0^t \{f(x+u) + f(x-u) - 2f(x)\} du \\
 &= o(t),
 \end{aligned}$$

$$\text{(iv)} \quad \Phi^*(t) = \int_0^t |\varphi(u)| du = O(t)$$

give Lebesgue's  $(C, 1)$  summability criterion for the Fourier series of  $f(x)$  at  $x$ .

On the other hand, we have a  $(C, \alpha)$  summability criterion due to Hahn [1].

**THEOREM B.** (iii) is not sufficient for  $(C, 1)$  summability of the Fourier series of  $f(x)$  at  $x$ , though it implies  $(C, \alpha)$  summability for every  $\alpha > 1$ .

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In this note, by applying Hahn's condition to Szász's theorem, we extend Theorem A as follows.

**THEOREM.** Let  $\lambda \geq 2$  be any positive integer. If (i) holds, then

$$\lim_{n \rightarrow \infty} \{\bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x)\} = \frac{1}{\pi} D(x) \log \lambda$$

for every  $\alpha > 1$ .

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The following lemmas are used.

**LEMMA 1** [2]. If  $\alpha > -1$  and  $\bar{\tau}_n^\alpha(x)$  denotes the  $n$ -th Cesàro mean of order  $\alpha$  of the sequence  $\{n\bar{A}_n(x)\}$ , then

$$\begin{aligned}
 \bar{\tau}_n^\alpha(x) &= n\{\bar{\sigma}_n^\alpha(x) - \bar{\sigma}_{n-1}^\alpha(x)\}, \\
 \bar{\tau}_n^{\alpha+1}(x) &= (\alpha+1)\{\bar{\sigma}_n^\alpha(x) - \bar{\sigma}_{n-1}^{\alpha+1}(x)\}.
 \end{aligned}$$

**LEMMA 2** [2]. If  $g_n^\alpha(t)$  denotes the  $n$ -th Cesàro mean of order  $\alpha$  of the sequence  $\{g_n(t)\}$ , where  $g_n(t) = \cos nt$  ( $n \geq 1$ ),  $g_0(t) = \frac{1}{2}$ , then, for  $\alpha > 0$ ,  $0 < t < \pi$ ,  $k = 0, 1, 2, \dots$ ,

$$\left| \left( \frac{d}{dt} \right)^k g_n^\alpha(t) \right| \leq \begin{cases} An^k & (k \geq 0), \\ An^{-2}t^{-k-2} & (k \leq \alpha-2), \\ An^{k-\alpha}t^{-\alpha} & (k > \alpha-2). \end{cases} \quad ^1$$

<sup>1</sup> Through this paper,  $A$  denotes an absolute constant not necessarily the same at each occurrence.

This lemma can easily be proved with a similar argument used by Zygmund [4].

LEMMA 3. *If*

$$h_n^\alpha(t) = \sum_{\nu=n+1}^{\lambda n} \frac{1}{\nu} g_\nu^\alpha(t),$$

then, for  $\alpha > 0$ ,  $0 < t < \pi$ ,  $k = 1, 2, \dots$ ,

$$\left| \left( \frac{d}{dt} \right)^k h_n^\alpha(t) \right| \leq \begin{cases} An^k & (k \geq 0), \\ An^{-2}t^{-k-2} & (k \leq \alpha-1), \\ An^{-k-\alpha-1}t^{-\alpha-1} & (k > \alpha-1). \end{cases}$$

By Lemma 1, with  $g_n(t)$  in place of  $\bar{S}_n(t)$ , we have

$$(\alpha+1) \frac{1}{n} g_n^\alpha(t) = (\alpha+1) \frac{1}{n} g_n^{\alpha+1}(t) + g_n^{\alpha+1}(t) - g_{n-1}^{\alpha+1}(t).$$

Thus,

$$\begin{aligned} (\alpha+1) \left| \left( \frac{d}{dt} \right)^k h_n^\alpha(t) \right| &\leq (\alpha+1) \sum_{\nu=n+1}^{\lambda n} \frac{1}{\nu} \left| \left( \frac{d}{dt} \right)^k g_\nu^{\alpha+1}(t) \right| \\ &\quad + \left| \left( \frac{d}{dt} \right)^k g_{\lambda n}^{\alpha+1}(t) \right| + \left| \left( \frac{d}{dt} \right)^k g_n^{\alpha+1}(t) \right|. \end{aligned}$$

The lemma follows from Lemma 2.

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Now, we are in a position to prove the theorem. Since

$$\begin{aligned} n\bar{A}_n(x) &= \frac{n}{\pi} \int_0^\pi \{f(x+t) - f(x-t)\} \sin ntdt \\ &= -\frac{1}{\pi} \int_0^\pi \{f(x+t) - f(x-t)\} \frac{d}{dt} \cos ntdt, \end{aligned}$$

from this, we get

$$\bar{\tau}_n^\alpha(x) = -\frac{1}{\pi} \int_0^\pi \{f(x+t) - f(x-t)\} \frac{d}{dt} g_n^\alpha(t) dt.$$

Now,

$$\begin{aligned} \bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x) &= \sum_{\nu=n+1}^{\lambda n} \frac{1}{\nu} \bar{\tau}_\nu^\alpha(x) \\ &= -\frac{1}{\pi} \int_0^\pi \{f(x+t) - f(x-t)\} \frac{d}{dt} h_n^\alpha(t) dt. \end{aligned}$$

Denote

$$\omega_n = -\frac{1}{\pi} \int_0^\pi \frac{d}{dt} h_n^\alpha(t) dt.$$

Then

$$\begin{aligned} -\pi\{\bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x) - \omega_n D(x)\} &= \int_0^\pi \psi(t) \frac{d}{dt} h_n^\alpha(t) dt \\ &= \int_0^{l/n} + \int_{l/n}^\pi \\ &= I_1 + I_2, \end{aligned}$$

say, where  $0 < l < n$ . Let  $\beta = \min(\alpha, 2)$ , then, by Lemma 3, we have

$$\begin{aligned} |I_2| &\leq \left| \left\{ \Psi(t) \frac{d}{dt} h_n^\alpha(t) \right\}_{l/n}^\pi \right| + \left| \int_{l/n}^\pi \Psi(t) \left( \frac{d}{dt} \right)^2 h_n^\alpha(t) dt \right| \\ &\leq An^{-\beta} \left\{ |\Psi(\pi)| \pi^{-\beta-1} + \left| \Psi \left( \frac{l}{n} \right) \left| \left( \frac{l}{n} \right)^{-\beta-1} \right| \right\} + An^{-\beta+1} \int_{l/n}^\pi |\Psi(t)| t^{-\beta-1} dt \\ &\leq An^{-\beta+1} + Al^{-\beta+1}, \end{aligned}$$

and

$$\begin{aligned} I_1 &= \left\{ \Psi(t) \frac{d}{dt} h_n^\alpha(t) \right\}_0^{l/n} - \int_0^{l/n} \Psi(t) \left( \frac{d}{dt} \right)^2 h_n^\alpha(t) dt \\ &= o\left(\frac{1}{n}\right) O(n) + o\left(n^2 \int_0^{l/n} t dt\right) \\ &= o(1) \end{aligned}$$

as  $n \rightarrow \infty$ . Hence,

$$\limsup_{n \rightarrow \infty} \pi |\bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x) - \omega_n D(x)| \leq Al^{-\beta-1}$$

for all  $l > 0$ . Since  $\beta \geq \alpha > 1$ , it follows that

$$\lim_{n \rightarrow \infty} \{\bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x) - \omega_n D(x)\} = 0.$$

Now,

$$\begin{aligned} \omega_n &= -\frac{1}{\pi} \{h_n^\alpha(\pi) - h_n^\alpha(0)\} \\ &= -\frac{1}{\pi} \sum_{\nu=n+1}^{\lambda n} \frac{1}{\nu A_\nu^\alpha} \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{\alpha-1} \{(-1)^\mu - 1\}, \end{aligned}$$

where

$$A_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

Since the sequence  $\{(-1)^n - 1\}$  is summable  $(C, \alpha)$  to  $-1$  for every  $\alpha > 0$ , we may write

$$\frac{1}{A_\nu^\alpha} \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{\alpha-1} \{(-1)^\mu - 1\} = -1 + \varepsilon_\nu,$$

where  $\varepsilon_\nu \rightarrow 0$  as  $\nu \rightarrow \infty$ . Therefore,

$$\omega_n = -\frac{1}{\pi} \sum_{\nu=n+1}^{\lambda n} \frac{-1 + \varepsilon_\nu}{\nu} = \frac{1}{\pi} \log \lambda + o(1)$$

as  $n \rightarrow \infty$ . I.e.,

$$\lim_{n \rightarrow \infty} \{\bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x)\} = \frac{D(x)}{\pi} \log \lambda.$$

This completes the proof of the theorem.

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National Taiwan University  
Taipei, Formosa, China