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On shifting iterated convolutions I

by

A. J. Stam

1. Introduction

Throughout this paper P, Q, R , with or without indices, denote probability measures on the Borel sets of the real line, PQ denotes the convolution of P and Q and P^n the n^{th} iterated convolution of P . So $U_a P^n$, where U_a is the probability measure degenerate at a , is the n^{th} convolution of P , shifted to the right over a distance a .

The problem considered in this paper is to describe the set L_0 of those values a for which

$$(1.1) \quad \lim_{n \rightarrow \infty} \|P^n - U_a P^n\| = 0.$$

Here $\|M\|$, for any finite signed measure M , is the total variation of M . It is well known that, for any two finite signed measures M and N ,

$$(1.2) \quad \|M+N\| \leq \|M\| + \|N\|,$$

$$(1.3) \quad \|MN\| \leq \|M\| \|N\|,$$

MN denoting convolution as before.

In section 5 we consider the following property, weaker than (1.1):

$$(1.4) \quad \lim_{n \rightarrow \infty} \|P^n Q - U_a P^n Q\| = 0$$

for every absolutely continuous Q . This holds for every a if P is not a lattice distribution.

Our main results on (1.1) are the following. The limit in (1.1) always exists and is either 0 or 2. The set L_0 is the real line if and only if P^n for some n has an absolutely continuous component. If P is purely discrete, L_0 is the additive group generated by the set of differences of those y for which $P(\{y\}) > 0$.

For the case that every P^n is purely singular, the author only found examples of a countable L_0 and an uncountable L_0 .

The restriction to probability measures is essential. If $\|P\| < 1$, the problem is trivial since then $\lim_{n \rightarrow \infty} \|P^n\| = 0$. If P is a measure with $P(-\infty, +\infty) > 1$, we may expect $L_0 = \{0\}$, since for probability measures the convergence in (1.1) and (1.4), if present, is of order $n^{-\frac{1}{2}}$ (see lemma 6 below).

2. Preliminary results

LEMMA 1. *The set of all a for which*

$$(2.1) \quad \lim_{n \rightarrow \infty} \|P^n R - U_a P^n R\| = 0,$$

is an additive group.

PROOF. The additivity is immediate by (1.2). Moreover, if (2.1) holds for a , the same is true for $-a$.

LEMMA 2. *The sequence $\|P^n R - U_a P^n R\|$, $n = 1, 2, \dots$, is non-increasing.*

PROOF. The assertion follows from (1.3) since $\|P\| = 1$.

LEMMA 3. *Let Q be any probability measure on the real line. Then $\|Q - U_a Q\| < 2$ if and only if there exist probability measures Q_0 and Q_1 and real numbers α, β with $\alpha > 0$, $\beta \geq 0$, $\alpha + \beta = 1$, such that*

$$(2.2) \quad Q = \alpha(\tfrac{1}{2}U_0 + \tfrac{1}{2}U_a)Q_0 + \beta Q_1.$$

PROOF. That (2.2) is sufficient follows from the inequality

$$\begin{aligned} \|Q - U_a Q\| &= \|\tfrac{1}{2}\alpha U_0 Q_0 + \beta Q_1 - \tfrac{1}{2}\alpha U_{2a} Q_0 - \beta U_a Q_1\| \\ &\leq \tfrac{1}{2}\alpha + \beta + \tfrac{1}{2}\alpha + \beta = 1 + \beta < 2. \end{aligned}$$

To prove necessity, let A, B be a Hahn decomposition of $(-\infty, +\infty)$ with respect to $Q - U_a Q$. (Halmos [1], § 29). Then for every Borel set E we have, putting $R \triangleq U_a Q$:

$$Q(E) = M_1(E) + M_0(E), \quad R(E) = M_2(E) + M_0(E),$$

with

$$\begin{aligned} M_1(E) &\triangleq Q(AE) - R(AE), \\ M_2(E) &\triangleq R(BE) - Q(BE), \\ M_0(E) &\triangleq Q(BE) + R(AE). \end{aligned}$$

By definition of a Hahn decomposition, M_1 and M_2 are (non-negative) measures. The measure M_0 does not vanish, since this

would imply $Q(B) = R(A) = 0$ in contradiction with the assumption $\|Q - R\| < 2$.

From $Q = M_1 + M_0$ and $Q = U_{-a}R = U_{-a}M_2 + U_{-a}M_0$ it follows that

$$Q = (\tfrac{1}{2}U_0 + \tfrac{1}{2}U_a)U_{-a}M_0 + \tfrac{1}{2}(M_1 + U_{-a}M_2).$$

Since M_0, M_1, M_2 are measures and M_0 does not vanish, (2.2) holds with $Q_0 = U_{-a}M_0/\|M_0\|$ and Q_1 either vanishing or equal to $(M_1 + U_{-a}M_2)/\|M_1 + M_2\|$.

LEMMA 4. *If $P = P_1P_2$ and $\lim_{n \rightarrow \infty} \|P_1^n R - U_a P_1^n R\| = 0$, then*

$$\lim_{n \rightarrow \infty} \|P^n R - U_a P^n R\| = 0.$$

PROOF. Since $\|P_2\| = 1$, the lemma follows immediately by (1.8) and the relation

$$P^n R - U_a P^n R = P_2^n (P_1^n R - U_a P_1^n R).$$

LEMMA 5. *For some m let*

$$(2.8) \quad P^m = \alpha P_1 + \beta P_2,$$

with P_1 and P_2 probability measures and α, β constants with $\alpha > 0$, $\beta \geq 0$, $\alpha + \beta = 1$. If P_1 satisfies (2.1), the same is true for P . In fact, we have

$$(2.4) \quad \lim_{n \rightarrow \infty} \|P^n R - U_a P^n R\| \leq \lim_{n \rightarrow \infty} \|P_1^n R - U_a P_1^n R\|.$$

PROOF. By lemma 2, with $Q \stackrel{\text{df}}{=} P^m$

$$\lim_{n \rightarrow \infty} \|P^n R - U_a P^n R\| = \lim_{n \rightarrow \infty} \|Q^n R - U_a Q^n R\|.$$

Since the case $\beta = 0$ is trivial, we assume $\alpha < 1$.

By (1.2) and (1.3)

$$\begin{aligned} \|Q^n R - U_a Q^n R\| &= \left\| \sum_{k=0}^n \binom{n}{k} \alpha^k \beta^{n-k} P_1^k P_2^{n-k} (R - U_a R) \right\| \\ &\leq \sum_{k=0}^n \binom{n}{k} \alpha^k \beta^{n-k} \|P_1^k R - P_1^k U_a R\|. \end{aligned}$$

Now $\lim_{k \rightarrow \infty} \|P_1^k R - P_1^k U_a R\|$ exists, so by the Toeplitz theorem (Loève [2], § 16.3, p. 238) the relation (2.4) follows.

Lemma 5 will be fundamental in our proofs. If (2.3) holds, we will say that P^m contains P_1 .

LEMMA 6. Let $P = \frac{1}{2}U_b + \frac{1}{2}U_{a+b}$. Then, for $n \rightarrow \infty$,

$$(2.5) \quad \|P^n - U_a P^n\| \sim cn^{-\frac{1}{2}}.$$

PROOF. Since P^n is a binomial distribution concentrated in the points $nb + ka$, $k = 0, 1, \dots, n$,

$$\begin{aligned} \|P^n - U_a P^n\| &= \binom{n}{0} 2^{-n} + \sum_{k=1}^n \left| \binom{n}{k} - \binom{n}{k-1} \right| 2^{-n} + \binom{n}{n} 2^{-n} \\ &= \frac{4}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} 2^{-n-1} \left| k - \frac{n+1}{2} \right| \\ &= \frac{4}{n+1} \int |x| dB_{n+1}(x) = 2(n+1)^{-\frac{1}{2}} \int |y| dB_{n+1}(\tfrac{1}{2}y\sqrt{n+1}), \end{aligned}$$

where B_m is the distribution function of the binomial distribution $b(\frac{1}{2}, m)$ centered at zero. Since $B_n(\frac{1}{2}y\sqrt{n})$ converges completely to the distribution function of $N(0, 1)$ and has second moment bounded with respect to n , we have (see Loève [2], § 11.4)

$$\lim_{n \rightarrow \infty} \int |y| dB_{n+1}(\tfrac{1}{2}y\sqrt{n+1}) = (2\pi)^{-\frac{1}{2}} \int |y| \exp(-\tfrac{1}{2}y^2) dy,$$

which concludes the proof.

3. The set L_0

In this section we consider the set L_0 of those a for which (1.1) holds.

THEOREM 1. The value of $\lim_{n \rightarrow \infty} \|P^n - U_a P^n\|$ is either 0 or 2.

PROOF. Obviously the limit is in $[0, 2]$. If it is not 2, then for some n

$$\|P^n - U_a P^n\| < 2,$$

and P^n by lemma 3 contains a probability measure of the form $(\frac{1}{2}U_0 + \frac{1}{2}U_a)Q_0$. So by applying lemma 6, 4 and 5 respectively, we see that $\lim_{n \rightarrow \infty} \|P^n - U_a P^n\| = 0$.

THEOREM 2. The set L_0 is the real line if and only if P^n for some n has an absolutely continuous component.

PROOF. Sufficiency: If P is absolutely continuous with density $p(x)$, then

$$\lim_{a \rightarrow 0} \|P - U_a P\| = \lim_{a \rightarrow 0} \int |p(x) - p(x-a)| dx = 0,$$

so that $\|P - U_a P\| < 2$ if $a \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Therefore $L_0 \supset (-\varepsilon, \varepsilon)$ by lemma 2 and theorem 1. It follows from lemma 1 that $L_0 = (-\infty, +\infty)$.

If P^n has an absolutely continuous component, the assertion $L_0 = (-\infty, +\infty)$ follows from lemma 5 and what has been shown above. Necessity: Let Q be any absolutely continuous probability measure with density $q(y)$. Then A_n, B_n being a Hahn decomposition for $P^n - QP^n$, we have

$$\begin{aligned} \|P^n - QP^n\| &= P^n(A_n) - QP^n(A_n) + QP^n(B_n) - P^n(B_n) \\ (3.1) \quad &= \int q(y) \{P^n(A_n) - U_y P^n(A_n) + U_y P^n(B_n) - P^n(B_n)\} dy \\ &\leq 2 \int q(y) \|P^n - U_y P^n\| dy. \end{aligned}$$

Here $\|P^n - U_y P^n\|$ is a Borel function of y . This is seen by the following relation, $F(x)$ being the distribution function of P^n :

$$\|P^n - U_y P^n\| = \sup \sum_{i=1}^{N-1} |F(b_{i+1}) - F(b_{i+1} - y) - F(b_i) + F(b_i - y)|,$$

where the supremum is taken over $N = 2, 3, \dots$ and rational $b_1, \dots, b_N$, since $F(x)$ is continuous from the left.

By our assumption and the Lebesgue dominated convergence theorem the right hand side of (3.1) tends to zero for $n \rightarrow \infty$. So $\|P^n - QP^n\| < 2$ for $n \geq n_1$ and, since QP^n is absolutely continuous, P^n for $n \geq n_1$ must have an absolutely continuous component.

THEOREM 3. *If P is purely discrete, L_0 is the additive group generated by the difference set of the set J of all those x with $P(\{x\}) > 0$.*

PROOF. Let $J = \{c_1, c_2, \dots\}$. Then P^n is restricted to the set of all x of the form

$$x = \sum_{k=1}^n c_{i_k},$$

where some or all i_k may be equal. In order that $\|P^n - U_a P^n\| < 2$ for some n , it is necessary that

$$a = \sum_{k=1}^n c_{i_k} - \sum_{k=1}^n c_{j_k} = \sum_{k=1}^n (c_{i_k} - c_{j_k})$$

for some $i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n$, which shows that L_0 is a subset of the additive group generated by the $c_i - c_j$.

On the other hand, if $x \in J, y \in J$, the measure P contains the measure $\frac{1}{2}U_x + \frac{1}{2}U_y$, so that $x - y \in L_0$ by lemma 6 and lemma 5. So by lemma 1 the additive group generated by the $c_i - c_j$ is a subset of L_0 .

THEOREM 4. *The set L_0 is an F_σ .*

PROOF. If P is purely discrete, L_0 is a countable set by theorem 3. Assume, then, that P has a nondiscrete component. Writing D_n, C_n for the discrete and nondiscrete component of P^n , we have

$$(3.2) \quad \begin{aligned} & | \|P^n - U_a P^n\| - \|C_n - U_a C_n\| | \\ & \leq \|P^n - U_a P^n - (C_n - U_a C_n)\| \\ & = \|D_n - U_a D_n\| \leq 2 \|D_n\| = 2 \|D_1\|^n, \end{aligned}$$

with $\|D_1\| < 1$. Let

$$V_n(x) \stackrel{\text{df}}{=} \|C_n - U_x C_n\|, \quad n = 1, 2, \dots, \quad -\infty < x < \infty.$$

By (3.2) and theorem 1

$$L_0 = \bigcup_{n=n_0}^{\infty} \{x : V_n(x) \leq 1\}.$$

Here n_0 is chosen so that $2 \|D_1\|^{n_0} < \frac{1}{2}$, say. Let $G_n(y)$ denote the distribution function of C_n . Then

$$(3.3) \quad V_n(x) = \sup \sum_{i=1}^{N-1} |G_n(b_{i+1}) - G_n(b_{i+1} - x) - G_n(b_i) + G_n(b_i - x)|,$$

where the supremum is taken over $N = 2, 3, \dots$ and $b_1, b_2, \dots, b_N$:

$$V_n(x) = \sup_{\alpha} V_{n,\alpha}(x), \quad -\infty < x < \infty, \quad n = 1, 2, \dots,$$

where the $V_{n,\alpha}(x)$ are of the form occurring in (3.3). The $V_{n,\alpha}(x)$ are continuous functions of x . So the sets $\{x : V_{n,\alpha}(x) \leq 1\}$ are closed, and

$$L_0 = \bigcup_{n=n_0}^{\infty} \bigcap_{\alpha} \{x : V_{n,\alpha}(x) \leq 1\}$$

is an F_σ .

4. Examples of singular distributions

If P^n is purely singular for every n , the problem of characterizing the set L_0 is still open. Here we present two examples of purely singular P^n , $n = 1, 2, \dots$, where L_0 is countable and where L_0 has the power of the continuum, respectively.

Example 1. For P we take the probability distribution of the random variable

$$(4.1) \quad x \stackrel{\text{df}}{=} \sum_{n=1}^{\infty} x_n 3^{-n^2},$$

where the x_n are independent nonnegative integer valued random variables. Moreover it is assumed that there exist natural numbers n_1 and m such that the x_k for $k \geq n_1$ have the same distribution restricted to $\{0, 1, \dots, m\}$ with $P\{x_k = j\} > 0$, $j = 0, 1, \dots, m$.

As shown by (4.1), the range of x is an uncountable set W and for every $c \in W$ we have $P\{x = c\} = 0$. So P cannot have a discrete component and the same then is true for all P^n . It will be shown below, from the conditions on P stated above, that $\|P - U_a P\| = 2$, except for countably many a . But then this must hold also for every P^n , since, as is easily seen, P^n is of the same type as P . So L_0 is a countable set. By theorem 2 no P^n can have an absolutely continuous component, so every P^n is purely singular.

To prove our assertion on $\|P - U_a P\|$ we show that

$$P\{x + a \in W\} = 0,$$

which implies mutual singularity of P and $U_a P$, for all but countably many a . It is no restriction to assume $a \geq 0$. Let

$$a = \sum_{n=1}^{\infty} a_n 3^{-n^2},$$

where the a_n are chosen so that

$$(4.2) \quad a_n < 3^{n^2 - (n-1)^2} = 3^{2n-1}, \quad n = 2, 3, \dots$$

The event $\{x + a \in W\}$ implies the existence of (random) integers $b_1, b_2, \dots$ such that

$$(4.3) \quad \sum_{n=1}^{\infty} (a_n + x_n) 3^{-n^2} = \sum_{n=1}^{\infty} b_n 3^{-n^2},$$

$$(4.4) \quad 0 \leq b_n \leq m, \quad n \geq n_1$$

It will be shown that (4.3) and (4.4), for all but countably many a , imply the occurrence of a sequence of events $\{x_{\nu_k} \in A\}$, with $\nu_1 < \nu_2 < \dots$ and $P(x_{\nu_k} \in A) < 1$, $k = 1, 2, \dots$, from which follows, by the independence and equidistribution of the x_n for $n \geq n_1$, that $P\{x + a \in W\} = 0$.

First we note that there is n_2 such that for $n \geq n_2$ the carry c_n from the n^{th} to the $(n-1)^{\text{th}}$ place in the addition in (4.3) is at most 1.

We now distinguish the following cases:

a. There is an infinite sequence $\nu_1 < \nu_2 < \dots$ such that

$$1 \leq a_{\nu_k} \leq 3^{2\nu_k-1} - m - 2, \quad k = 1, 2, \dots$$

Since for $\nu_k \geq \max(n_1, n_2)$

$$a_{\nu_k} + x_{\nu_k} + c_{1+\nu_k} \leq 3^{2\nu_k-1} - m - 2 + m + 1 < 3^{2\nu_k-1},$$

$c_{\nu_k} = 0$ for $\nu_k \geq \max(n_1, n_2)$, and for (4.3) and (4.4) to hold we must have

$$x_{\nu_k} + a_{\nu_k} + c_{1+\nu_k} \leq m,$$

implying $x_{\nu_k} \leq m-1$ and we may take $A = \{0, 1, \dots, m-1\}$.

b. There is n_3 such that $a_n = 0$ or $a_n \geq 3^{2n-1} - m - 1$ for $n \geq n_3$.

b1. All but a finite number of the a_n are zero. The corresponding a form a countable set.

b2. There is n_4 with $a_n \geq 3^{2n-1} - m - 1$ for $n \geq n_4$. To satisfy (4.3) and (4.4) we must have $c_n > 0$ for $n \geq \max(n_1, n_4)$, so

$$\begin{aligned} x_n + a_n + c_{n+1} &\geq 3^{2n-1}, \\ x_n &\geq 3^{2n-1} - 1 - a_n, \end{aligned}$$

which by (4.2) implies $x_n \geq 1$ for infinitely many n , except if $a_n = 3^{2n-1} - 1$ for all but a finite number of n . But the set of a satisfying the latter condition is countable.

b3. The sets of n with $a_n = 0$ and with $a_n \geq 3^{2n-1} - m - 1$ are both infinite. Then we may select a sequence $\nu_1 < \nu_2 < \dots$ with

$$a_{\nu_k} \geq 3^{2\nu_k-1} - m - 1, \quad a_{1+\nu_k} = 0, \quad k = 1, 2, \dots$$

To satisfy (4.3) and (4.4) we must have $c_{\nu_k} > 0$, $k = 1, 2, \dots$, or, since $c_{1+\nu_k} = 0$ for $k \geq k_1$,

$$x_{\nu_k} + a_{\nu_k} \geq 3^{2\nu_k-1}, \quad k \geq k_1,$$

which by (4.2) implies the events $\{x_{\nu_k} \geq 1\}$, $k \geq k_1$.

Example 2. This example is taken from a paper by Wiener and Young [4], section 7. Let $n_1, n_2, \dots$ be an increasing sequence of natural numbers, such that

$$(4.5) \quad \sum_{k=1}^{\infty} n_k^{-1} < \infty,$$

and consider the expansion of $x \in (0, 1)$:

$$(4.6) \quad x = \frac{m_1}{n_1} + \frac{m_2}{n_1 n_2} + \frac{m_3}{n_1 n_2 n_3} + \dots,$$

the m_i being nonnegative integers with $m_i < n_i$, ambiguity being removed by taking the terminating expansion whenever possible. The n_k are assumed even, $n_k = 2r_k$, $k = 1, 2, \dots$. Let $F(x)$ be defined by

$$F(x) = 0, \quad x \leq 0, \quad F(x) = 1, \quad x \geq 1,$$

$$F(x) = \frac{m_1/2}{r_1} + \frac{m_2/2}{r_1 r_2} + \frac{m_3/2}{r_1 r_2 r_3} + \dots,$$

if every m_k in (4.6) is even, and

$$F(x) = \frac{m_1/2}{r_1} + \frac{m_2/2}{r_1 r_2} + \dots + \frac{m_{n-1}/2}{r_1 r_2 \dots r_{n-1}} + \frac{[m_n/2] + 1}{r_1 r_2 \dots r_n},$$

if m_n is the first odd m_k in (4.6).

It was shown by Wiener and Young, that $F(x)$ is the distribution function of a purely singular probability measure P and that the set of a with $\|P - U_a P\| < 2$ has the power of the continuum. So by our lemma 2 and theorem 1 the set L_0 for this P has the power of the continuum. For the sake of our example we only have to show that P^n for every n is purely singular. To this end we note that F is the distribution function of the random variable

$$(4.7) \quad x = \sum_{k=1}^{\infty} x_k (n_1 n_2 \dots n_k)^{-1},$$

where the x_k are independent and

$$(4.8) \quad P\{x_k = j\} = r_k^{-1}, \quad j = 0, 2, \dots, n_k - 2, \quad k = 1, 2, \dots$$

Clearly P^n for every n is a convergent infinite convolution of discrete distributions. By a theorem of Wintner, [5], p. 89, no. 148, such a distribution is of pure type. Since P is not discrete, it is sufficient to show that P^n is not purely absolutely continuous. This will follow from the fact that

$$(4.9) \quad \limsup_{u \rightarrow \infty} |\varphi(u)| > 0,$$

where $\varphi(u)$ denotes the characteristic function of P , since, if P^n were absolutely continuous, its characteristic function $\varphi^n(u)$ would tend to zero for $|u| \rightarrow \infty$ by the Riemann-Lebesgue lemma.

From (4.7) and (4.8) we have

$$(4.10) \quad \varphi(u) = \prod_{k=1}^{\infty} \varphi_k(u), \quad -\infty < u < \infty,$$

with

$$(4.11) \quad \varphi_k(u) = \frac{1}{r_k} \sum_{h=0}^{r_k-1} \exp\left(\frac{2\pi i u h}{n_1 n_2 \dots n_k}\right),$$

so that

$$(4.12) \quad \varphi_k(n_1 n_2 \dots n_H \pi) = 1, \quad k = 1, 2, \dots, H,$$

$$(4.13) \quad \varphi_{H+1}(n_1 n_2 \dots n_H \pi) = 2r_{H+1}^{-1} \{1 - \exp(2\pi i/n_{H+1})\}^{-1},$$

$$\varphi_{H+m}(n_1 n_2 \dots n_H \pi) = \frac{1}{r_{H+m}} \sum_{h=0}^{r_{H+m}-1} \exp\left(\frac{2\pi i h}{n_{H+1} \dots n_{H+m}}\right),$$

$$m = 2, 3, \dots,$$

$$\prod_{m=2}^M \varphi_{H+m}(n_1 n_2 \dots n_H \pi) = \frac{1}{r_{H+2} \dots r_{H+M}} \sum_{h_2=0}^{r_{H+2}-1} \dots \sum_{h_M=0}^{r_{H+M}-1} A(h_2, h_3, \dots, h_M),$$

with

$$A(h_2, h_3, \dots, h_M) = \exp\left(2\pi i \sum_{m=2}^M \frac{h_m}{n_{H+1} \dots n_{H+m}}\right).$$

Now

$$|1 - A(h_2, \dots, h_M)| \leq 2\pi \sum_{m=2}^M \frac{h_m}{n_{H+1} \dots n_{H+m}} \leq \pi \sum_{m=2}^M \frac{1}{n_{H+1} \dots n_{H+m-1}},$$

so that

$$(4.14) \quad |1 - \prod_{m=2}^M \varphi_{H+m}(n_1 n_2 \dots n_H \pi)| \leq \pi \sum_{m=2}^M \frac{1}{n_{H+1} \dots n_{H+m-1}}.$$

From (4.10)–(4.14) and $\lim_{H \rightarrow \infty} n_H = +\infty$ it follows that

$$\lim_{H \rightarrow \infty} \varphi(n_1 n_2 \dots n_H \pi) = -2/\pi i,$$

which proves (4.9).

5. The relation (1.4)

For fixed probability measure P let

$$(5.1) \quad D_n(x, Q) \stackrel{\text{df}}{=} \|P^n Q - U_x P^n Q\|, \quad n = 1, 2, \dots,$$

with Q absolutely continuous,

$$(5.2) \quad D(x, Q) \stackrel{\text{df}}{=} \lim_{n \rightarrow \infty} D_n(x, Q),$$

the limit existing by lemma 2, and

$$(5.3) \quad D(x) \stackrel{\text{df}}{=} \sup D(x, Q),$$

the supremum being taken over all absolutely continuous probability measures Q .

LEMMA 7. $D_n(x, Q)$ and $D(x, Q)$ are continuous functionals of Q , uniformly in x and n , in fact

$$|D_n(x, Q_1) - D_n(x, Q_2)| \leq 2 \|Q_1 - Q_2\|,$$

$$|D(x, Q_1) - D(x, Q_2)| \leq 2 \|Q_1 - Q_2\|.$$

PROOF. By (1.2) and (1.3)

$$|D_n(x, Q_1) - D_n(x, Q_2)| \leq \|P^n Q_1 - U_x P^n Q_1 - (P^n Q_2 - U_x P^n Q_2)\|$$

$$\leq \|P^n(Q_1 - Q_2)\| + \|U_x P^n(Q_1 - Q_2)\| \leq 2 \|Q_1 - Q_2\|.$$

LEMMA 8. Let $Q_k, k = 1, 2, \dots$, be a sequence of probability measures with densities $q_k(y), k = 1, 2, \dots$, such that

$$q_k(y) = kq_1(ky), \quad -\infty < y < \infty, \quad k = 1, 2, \dots$$

Then

$$D(x) = \sup_k D(x, Q_k).$$

PROOF. By definition of $D(x)$

$$(5.4) \quad S(x) \stackrel{\text{df}}{=} \sup_k D(x, Q_k) \leq D(x), \quad -\infty < x < \infty.$$

For any Q we have by (1.3)

$$D_n(x, QQ_k) \leq D_n(x, Q_k),$$

so, for $n \rightarrow \infty$,

$$(5.5) \quad D(x, QQ_k) \leq D(x, Q_k) \leq S(x), \quad k = 1, 2, \dots$$

Since Q is absolutely continuous, $\|Q - QQ_k\|$ tends to zero for $k \rightarrow \infty$, so from (5.5) and lemma 7 it follows that $D(x, Q) \leq S(x)$ for every absolutely continuous Q , implying

$$(5.6) \quad D(x) \leq S(x),$$

and the lemma follows from (5.4) and (5.6).

THEOREM 5. If P is not a lattice distribution,

$$(5.7) \quad D(x) = 0, \quad -\infty < x < \infty.$$

If P is a lattice distribution with span c ,

$$(5.8) \quad \begin{aligned} D(x) &= 0, & x = nc, & n \text{ integer,} \\ D(x) &= 2, & \text{elsewhere.} \end{aligned}$$

By a lattice distribution is meant here a discrete distribution concentrated in a subset of $\{a + nd, n \text{ integer}\}$ for some a and d , the span being the largest value that may be taken for d .

PROOF OF (5.7). By lemma 8 it is sufficient to prove that

$$(5.9) \quad \lim_{n \rightarrow \infty} D_n(x, Q_k) = 0, \quad k = 1, 2, \dots,$$

for a suitable sequence Q_k , $k = 1, 2, \dots$, of the form considered in lemma 8. We choose Q_1 in such a way that it is symmetric about zero and has finite second moment, that its density $q_1(y)$ belongs to L_2 and its characteristic function $\vartheta_1(u)$ satisfies

$$(5.10) \quad \vartheta_1(u) = 0, \quad |u| \geq 1.$$

This may be accomplished by taking

$$q_1(y) = \alpha(4y^{-1} \sin \frac{1}{4}y)^4, \quad -\infty < y < \infty,$$

with α a norming constant, as will be seen by applying the Fourier inversion formula to the characteristic function of the fourfold convolution of the uniform distribution on $[-\frac{1}{4}, \frac{1}{4}]$.

By lemma 5 it is no restriction to assume that P has finite second moment. We also center P at its first moment.

For fixed k let $p_n(x)$ and $r_n(x)$ be the densities of $P^n Q_k$ and $U_a P^n Q_k$, respectively. Then

$$D_n(a, Q_k) = \int |p_n(x) - r_n(x)| dx,$$

$$D_n(a, Q_k) \leq \int_{-A}^A |p_n(x) - r_n(x)| dx + 2 \int_{|x| \geq A-a} p_n(x) dx.$$

Here A is allowed to depend on n . By the inequality between arithmetic and quadratic mean and by Chebychev's inequality

$$D_n(a, Q_k) \leq \left[2A \int_{-\infty}^{+\infty} \{p_n(x) - r_n(x)\}^2 dx \right]^{\frac{1}{2}} + 2 \frac{nd^2 + v^2 k^{-2}}{(A-a)^2},$$

where d^2 is the variance of P and v^2 the variance of Q_1 . Since $q_1 \in L_2$, also $q_k \in L_2$ and therefore $p_n \in L_2$, $r_n \in L_2$. So by Parseval's formula

$$D_n(a, Q_k) \leq \left[\frac{A}{\pi} \int_{-\infty}^{+\infty} |(1 - e^{iua}) \varphi^n(u) \vartheta_k(u)|^2 du \right]^{\frac{1}{2}} + 2 \frac{nd^2 + v^2 k^{-2}}{(A-a)^2},$$

where $\varphi(u)$ denotes the characteristic function of P and $\vartheta_k(u) = \vartheta_1(u/k)$ the characteristic function of Q_k . Making use of (5.10) and the inequality $|\vartheta_k(u)| \leq 1$, and putting $A = Cn^{\frac{1}{2}}$, we find for n suitably large

$$D_n(a, Q_k) \leq \left[\frac{Cn^{\frac{1}{2}}}{\pi} \int_{-k}^k |\varphi(u)|^{2n} |1 - e^{iua}|^2 du \right]^{\frac{1}{2}} + 3d^2 C^{-2}.$$

Since P is not degenerate and has finite second moment, there are $\varepsilon \in (0, 1)$ and $\beta \in (0, 1)$ such that

$$|\varphi(u)|^2 \leq 1 - \beta u^2, \quad |u| \leq \varepsilon.$$

Moreover, P being not a lattice distribution, there is a constant $\gamma \in [0, 1)$ so that

$$|\varphi(u)| \leq \gamma, \quad \varepsilon \leq |u| \leq k.$$

(See Lukacs [3], theorem 2.1.4). So, since also

$$\begin{aligned} |1 - e^{iua}|^2 &\leq a^2 u^2 \leq a^2 |u| \text{ for } |u| \leq \varepsilon, \\ \limsup_n D_n(a, Q_k) &\leq 3d^2 C^{-2} + \limsup_n \left[\frac{2a^2 C n^{\frac{1}{2}}}{\pi} \int_0^\varepsilon (1 - \beta u^2)^n u du \right]^{\frac{1}{2}} \\ &= 3d^2 C^{-2} + \limsup_n \left[\frac{a C n^{\frac{1}{2}}}{\beta \pi (n+1)} \{1 - (1 - \beta \varepsilon^2)^{n+1}\} \right]^{\frac{1}{2}} = 3d^2 C^{-2}. \end{aligned}$$

Since this holds for every $C > 0$, the proof of (5.9) is concluded.

PROOF OF (5.8) That $D(x) = 0$ for $x = nc$, follows from theorem 8. That $D(x) = 2$ for $x \neq nc$, is seen by taking for Q the uniform distribution on $[-h, h]$, with h so small that no intervals $[nc-h, nc+h]$ and $[mc+x-h, mc+x+h]$, m, n integer, overlap.

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