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by

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We shall be concerned throughout this paper with sequences $\omega = \{x_1, x_2, \ldots, x_n, \ldots\}$ of elements x_n from a compact Hausdorff space X satisfying the second axiom of countability. We are interested in some distribution properties of such sequences. In this context it is useful to regard sequences as elements of the countable product $\Omega = \prod_{n=1}^{\infty} X_n$, $X_n \equiv X$ ([3]). We denote by T the shift-transformation on Ω defined by

$$T\omega = T\{x_1, x_2, \ldots, x_n, \ldots\} = \{x_2, x_3, \ldots, x_{n+1}, \ldots\}.$$

T is a continuous mapping of the compact (in the usual Tychonoff-topology) Hausdorff space Ω onto itself. We call a subspace Y of Ω invariant, if $Ty \in Y$ for every $y \in Y$. An important example of a compact invariant subspace is the closure X_{ω_0} of the set $\{\omega_0, T\omega_0, \ldots, T^n\omega_0, \ldots\}$ for some $\omega_0 \in \Omega$. The shift transformation T induces then in a natural way a continuous transformation T on each invariant subspace Y.

For every compact invariant subspace Y of Ω we denote by $I_T(Y)$ the set of all invariant (regular) probability measures μ on Y, i.e. the set of all linear functionals μ on the space $C(Y)$ of all continuous complex valued functions on Y satisfying $\mu(f) \geqq 0$ for $f \geqq 0$ and

$$\int_Y f(Ty)d\mu(y) = \int_Y f(y)d\mu(y) \quad \text{for all } f \in C(Y).$$

It is easily shown that $I_T(Y)$ is a closed convex subset of the compact (in the weak-star-topology) space $M(Y)$ of all probability measures on Y. Let $I_T(X_{\omega_0}) = I_{\omega_0}$. Define $V_{\omega_0} \subseteqq I_{\omega_0}$ to be the set of all limits in $M(X_{\omega_0})$ of the sequence $\{\mu_n\}$ defined by

$$\mu_n(f) = \frac{f(\omega_0)+f(T\omega_0)+ \ldots +f(T^n\omega_0)}{n+1}$$

for all $f \in C(X_{\omega_0})$.

We consider now the map $\pi : X_{\omega_0} \to X$ defined by

$$\pi\omega = \pi\{x_1, x_2, \ldots, x_n, \ldots\} = x_1, \quad \omega \in X_{\omega_0}.$$

Without loss of generality we may suppose that

$$\omega_0 = \{x_1^0, x_2^0, \ldots, x_n^0, \ldots\}$$

is dense in X or equivalently that π is epimorphic. We may then identify $C(X)$ with the subalgebra

$$A = \{f' \in C(X_{\omega_0}) : f'(\omega) = f(\pi(\omega)), \ f \in C(X)\}.$$

Let now $\hat{\pi} : I_{\omega_0} \to M(X)$ be defined by $(\hat{\pi}\mu)(f) = \mu(f \circ \pi)$, $f \in C(X)$, i.e. let $\hat{\pi}$ be the restriction of $\mu \in I_{\omega_0}$ to the subalgebra A of $C(X_{\omega_0})$ and identify A with $C(X)$.

We say that ω_0 is m-uniformly distributed in X, $m \in M(X)$, if $\hat{\pi}V_{\omega_0} = \{m\}$, i.e. if the image of V_{ω_0} with respect to $\hat{\pi}$ consists of the single measure m. (This definition is of course equivalent to the usual one, comp. [1]. This holds also for the other concepts used).

There are two important special cases:

1) V_{ω_0} itself consists of a single measure. If this measure coincides with the product measure $\mu = m \times m \times m \times \ldots$ we call ω_0 completely m-uniformly distributed in X. In this case $X_{\omega_0} = \Omega$ and $\hat{\pi}(I_{\omega_0}) = M(X)$.

2) $\hat{\pi}(I_{\omega_0}) = \{m\}$.

We shall show that this occurs if and only if ω_0 is m-well-distributed in X, i.e. for every $f \in C(X)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n \leq N} f(x_{n+k}^0) = m(f) \quad \text{uniformly in } k.$$

More precisely we prove the

Theorem: The following four conditions are mutually equivalent:

1) $\hat{\pi}(I_{\omega_0}) = \{m\}$.

2) $\hat{\pi}(V_\omega) = \{m\}$ for all $\omega \in X_{\omega_0}$.

3) $\lim\limits_{N\to\infty} \sup\limits_{\omega \in X_{\omega_0}} \left| \frac{1}{N} \sum\limits_{n \leq N} f'(T^n\omega) - m(f') \right| = 0$ for all $f' \in A$.

4) $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n \leq N} f(x_{n+k}^0) = m(f)$ uniformly in k for all $f \in C(X)$.

Proof:

1) $\Rightarrow$ 3)

Suppose 3) does not hold. Then there exists $f_0' \in A$ and $\varepsilon_0 > 0$ such that for infinitely many N'

$$\left|\frac{1}{N'} \sum_{n \leqq N'} f_0'(T^n \omega_{N'}) - m(f_0')\right| \geqq \varepsilon_0 \text{ for some } \omega_{N'} \in X_{\omega_0}.$$

Define now

$$\mu_{N'} \in M(X_{\omega_0}) \text{ by } \mu_{N'}(\varphi) = \frac{1}{N'} \sum_{n \leqq N'} \varphi(T^n \omega_{N'})$$

for all $\varphi \in C(X_{\omega_0})$.

Let ν be any limit of the sequence $\mu_{N'}$ in $M(X_{\omega_0})$. Then $\nu \in I_{\omega_0}$ and $\nu(f_0') \neq m(f_0')$ which contradicts $\hat{\pi}\nu = m$.

3) $\Rightarrow$ 4):

4) may be written in the following form:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leqq N} f'(T^n T^k \omega_0) = m(f') \text{ for all } f' \in A \text{ uniformly in } k.$$

But $T^k \omega_0 \in X_{\omega_0}$ and so 4) is a special case of 3).

4) $\Rightarrow$ 3):

By 4) we have

$$\left|\frac{1}{N} \sum_{n \leqq N} f'(T^n T^k \omega_0) - m(f')\right| < \varepsilon/2 \text{ for } N \geqq N_0$$

and $k = 1, 2, 3, \ldots$. For every $\omega \in X_{\omega_0}$ there is a number k_N such that

$$\left|\frac{1}{N} \sum_{n \leqq N} f'(T^n \omega) - \frac{1}{N} \sum_{n \leqq N} f'(T^n T^{k_N} \omega_0)\right| < \varepsilon/2,$$

because $\{T^n \omega_0\}$ is dense in X_{ω_0} and f' and T are continuous.

Therefore for $N \geqq N_0$ we have

$$\left|\frac{1}{N} \sum_{n \leqq N} f'(T^n \omega) - m(f')\right| \leqq \left|\frac{1}{N} \sum_{n \leqq N} f'(T^n \omega) - \frac{1}{N} \sum_{n \leqq N} f'(T^{n+k_N} \omega_0)\right|$$
$$+ \left|\frac{1}{N} \sum_{n \leqq N} f'(T^{n+k_N} \omega_0) - m(f')\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

i.e. 3).

3) $\Rightarrow$ 2): trivial.

2) $\Rightarrow$ 1): Let $\nu \in I_{\omega_0}$. Then for every $f' \in A$ and $N \geqq 1$

$$\nu(f') = \int_{X_{\omega_0}} \left(\frac{1}{N} \sum_{n \leqq N} f'(T^n\omega)\right) d\nu(\omega)$$
$$= \int_{X_{\omega_0}} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n \leqq N} f'(T^n\omega)\right) d\nu(\omega) = \int_{X_{\omega_0}} m(f')d\nu(\omega)$$
$= m(f')$. This implies $\mathfrak{K}(I_{\omega_0}) = \{m\}$.

Remarks and examples:

a) If $I_T(X_{\omega_0})$ consists of exactly one measure μ, then T is strictly ergodic on X_{ω_0}. In this case our theorem is well known ([5]).

Denote e.g. by T^k the k-dimensional torusgroup i.e. the quotient group R^k/Z^k of the additive group of k-tupels of real numbers by the discrete group Z^k of integer-valued k-tupels in the usual topology. Let $\vartheta \in T^1$ be irrational (i.e. a generator of the monothetic group T^1), then the sequence $\omega_0 = \{n^k\vartheta\}$ is λ-well-distributed in T^1 ($\lambda =$ Haar measure), $X_{\omega_0} =$ (isomorphic to) T^k and $I_T(X_{\omega_0}) =$ Haar measure on T^k. The same holds for every sequence of the form $\omega_0 = \{p(n)\}$, $p(x) = a_k x^k + \ldots + a_0$; a_k irrational.

b) Some negative examples:

Let $\omega_0 = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence on X satisfying

$$\lim_{k\to\infty} x_{n_k+h} = \lim_{k\to\infty} x_{n_k} = x$$

for some convergent subsequence and every $h = 1, 2, 3, \ldots$.

Then ω_0 cannot be m-well-distributed for any $m \in M(X)$ not concentrated in a single point. For in this case

$$\omega_1 = \{x, x, x, \ldots\} \in X_{\omega_0} \quad \text{and} \quad \mathfrak{K}(V_{\omega_1}) \neq m$$

which contradicts 2).

Consider now the special case $X = T^1$, $m = \lambda$, $\omega = \{f(n)\}$ where $f(n+h) - f(n) \to 0$ mod 1. These sequences cannot be λ-well-distributed mod 1. (This generalizes results of [4], [6]).

The same conclusion holds for sequences of the form $\omega_0 = \{q_1 q_2 \ldots q_n \alpha\}$ on T^1, if $q_n \geqq 1$ is a bounded sequence of integers. For suppose ω_0 to be λ-well-distributed mod 1, then it is a fortiori λ-uniformly-distributed and therefore dense in T^1. Choose now a subsequence n_k such that $q_1 q_2 \ldots q_{n_k}\alpha \to 0$ (mod 1). Then also $q_1 \ldots q_{n_k} q_{n_k+1} \ldots q_{n_k+h}\alpha \to 0$.

c) G. Helmberg and A. Paalman-de Miranda [2] proved the following fact: The set $\{\omega \in \Omega : \omega$ is m-well-distributed$\}$ has μ-measure 0, if m is not concentrated in a single point and $\mu = m \times m \times \ldots$. This follows at once from our theorem because the set of all m-completely uniformly distributed sequences has μ-measure 1 by the individual ergodic theorem. Furthermore $X_{\omega_0} = \Omega$ and $\mathfrak{K}(I_{\omega_0}) = M(X)$ for every such sequence (we assumed ω_0 to be dense in X, otherwise replace X by the least compact subspace containing ω_0). The set of all m-well-distributed sequences is therefore disjoint from the set of m-completely-uniformly distributed sequences and is therefore a null set.

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