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# On the absolute summability factors of Fourier series at a given point

by

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## 1.

A series  $\sum a_n$  is said to be absolutely summable ( $A$ ) or summable  $|A|$  if the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is of bounded variation in the interval  $(0,1)$  Let  $\sigma_n^\alpha$  denote the  $n$ -th Cesàro mean of order  $\alpha$  of the series  $\sum a_n$ , i.e.,

$$\sigma_n^\alpha = \frac{1}{(\alpha)_n} \sum_{k=0}^n (\alpha)_k a_{n-k}, \quad (\alpha)_k = \Gamma(k+\alpha+1)/\Gamma(k+1)\Gamma(\alpha+1).$$

If the series

$$\sum |\sigma_n^\alpha - \sigma_{n-1}^\alpha|$$

converges, then, we say that the series  $\sum a_n$  is absolutely summable  $(C, \alpha)$  or summable  $|C, \alpha|$ . It is known that [2] if a series is summable  $|C|$ , then it is also summable  $|A|$ .

## 2.

Suppose now that  $f(x)$  is a function integrable in the sense of Lebesgue and periodic with period  $2\pi$ . Let

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum A_n(x).$$

Whittaker [5] proved that the series

$$\sum_{n=1}^{\infty} A_n(x)/n^\alpha \quad (\alpha > 0)$$

is summable  $|A|$  almost everywhere. Prasad [5] improved this result by showing that the series  $\sum A_n(x)$  when multiplied by one of the factors:

$$1/(\log n)^{1+\varepsilon}, 1/\log n(\log^2 n)^{1+\varepsilon}, \dots, \\ 1/(\log n) \cdot (\log^2 n) \dots (\log^{k-1} n)(\log^k n)^{1+\varepsilon},$$

where  $\varepsilon$  is any positive quantity and  $\log^1 n = \log n$ ,  $\log^k n = \log(\log^{k-1} n)$ , is summable  $|A|$  at the point  $x$ .

Let  $(\lambda_n)$  be a convex and bounded sequence [3], Chow [1] has demonstrated that the series

$$\sum A_n(x)\lambda_n$$

is summable  $|C, 1|$  almost everywhere, if the series  $\sum n^{-1}\lambda_n$  converges.

In the present note, we are interested particularly in the case of  $|C, 1|$  summability. For a fixed point of  $x$ , we write

$$\varphi(t) = \varphi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

We are going to establish at first the following

**THEOREM 1.** *If*

$$\Phi(t) = \int_0^t |\varphi(u)| du = O\left(\frac{t}{\log 1/t}\right)$$

as  $t \rightarrow +0$ , then the series

$$\sum_{n=2}^{\infty} \frac{A_n(x)}{(\log n)^{1+\varepsilon}}$$

is summable  $|C, 1|$  for every  $\varepsilon > 0$ .

### 3.

In the proof of the theorem, we require the following lemmas.

**LEMMA 1** [4]. *Let  $\alpha > -1$  and let  $\tau_n^\alpha$  be the  $n$ -th Cesàro mean of order  $\alpha$  of the sequence  $(na_n)$ , then*

$$\tau_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha),$$

where  $\sigma_n^\alpha$  is the  $n$ -th Cesàro mean of order  $\alpha$  of the series  $\sum a_n$ .

**LEMMA 2.** *Write*

$$S_n(t) = \sum_{k=0}^n (n+2-k) \cos(n+2-k)t,$$

then

$$S_n(t) = O \begin{cases} nt^{-1} & (nt \geq 1), \\ n^2 & (\text{for all } t). \end{cases}$$

In fact, we have

$$\begin{aligned} S_n(t) &= \mathcal{J} \left\{ \frac{d}{dt} \left( \overline{e^{t(n+2)t} \sum_{k=0}^n e^{-kt}} \right) \right\} \\ &= \mathcal{J} \left\{ \frac{d}{dt} \left( \frac{e^{t(n+2)t}}{1-e^{-t}} - \frac{e^{tt}}{1-e^{-t}} \right) \right\} \\ &= O(nt^{-1}) + O(t^{-2}) \\ &= O(nt^{-1}), \end{aligned}$$

if  $nt \geq 1$ . This proves the first part of the lemma. And the second part of the lemma is evident.

#### 4.

Now, we are in a position to prove the theorem. We use

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \varphi(t) \cos nt \, dt.$$

Let  $\tau_n(x)$  be the  $n$ -th Cesàro mean of first order of the sequence  $\{nA_n(x)(\log n)^{-(1+\varepsilon)}\}$ , then

$$\frac{\pi}{2} \tau_n(x) = \int_0^\pi \varphi(t) \frac{1}{n+1} \sum_{\nu=0}^n \frac{(\nu+2) \cos(\nu+2)t}{(\log(\nu+2))^{1+\varepsilon}} \, dt.$$

Abel's transformation gives

$$\begin{aligned} \frac{\pi}{2} \tau_n(x) &= \int_0^\pi \varphi(t) \frac{1}{n+1} \left\{ \sum_{\nu=0}^n S_\nu(t) \Delta \frac{1}{(\log(\nu+2))^{1+\varepsilon}} \right\} dt \\ &\quad + \int_0^\pi \varphi(t) \frac{1}{n+1} \cdot \frac{S_n(t)}{(\log(n+3))^{1+\varepsilon}} \, dt \\ &= I_1 + I_2, \end{aligned}$$

say. By Lemma 2, we have

$$\frac{1}{n+1} \left\{ \sum_{\nu=0}^n S_\nu(t) \Delta \frac{1}{(\log(\nu+2))^{1+\varepsilon}} \right\} = O \left\{ \begin{array}{l} \frac{1}{t(\log n)^{2+\varepsilon}} \quad (nt \geq 1), \\ \frac{n}{(\log n)^{2+\varepsilon}} \quad (\text{for all } t). \end{array} \right.$$

Thus, on writing

$$I_1 = \int_0^{1/n} + \int_{1/n}^\pi = I_3 + I_4,$$

say, we see that

$$I_3 = O \left\{ \frac{n}{(\log n)^{2+\varepsilon}} \int_0^{1/n} |\varphi| dt \right\} = O \left\{ \frac{1}{(\log n)^{2+\varepsilon}} \right\},$$

$$I_4 = O \left\{ \frac{1}{(\log n)^{2+\varepsilon}} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt \right\}.$$

Now,

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = \left( \frac{\Phi}{t} \right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2} dt$$

$$= O(\log n).$$

It follows that  $I_4 = O\{(\log n)^{-(1+\varepsilon)}\}$ . As before, we write

$$I_2 = \int_0^{1/n} + \int_{1/n}^{\pi} = I_5 + I_6,$$

say. Then,

$$I_5 = O \left\{ \frac{n}{(\log n)^{1+\varepsilon}} \int_0^{1/n} |\varphi| dt \right\}$$

$$= O \left\{ \frac{1}{(\log n)^{1+\varepsilon}} \right\}.$$

And

$$I_6 = O \left\{ \frac{1}{(\log n)^{1+\varepsilon}} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt \right\}.$$

Now,

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = \left( \frac{\Phi}{t} \right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2} dt$$

$$= O(1) + O \left( \int_{1/n}^{\pi} \frac{dt}{1/n + \log 1/t} \right)$$

$$= O(\log^2 n).$$

It follows that

$$I_6 = O \left( \frac{\log^2 n}{(\log n)^{1+\varepsilon}} \right).$$

By Lemma 1, we need only to prove that  $\sum |\tau_n(x)|/n$  converges. And from the above analysis, it concludes that

$$\sum |\tau_n(x)|/n = O \left\{ \sum_{n=2}^{\infty} \frac{\log^2 n}{n(\log n)^{1+\varepsilon}} \right\}$$

$$= O(1).$$

This completes the proof of Theorem 1.

## 5.

For the conjugate series

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum B_n(x),$$

we can derive an analogous theorem. Write

$$\bar{\Psi}(t) = \int_0^t |\psi(u)| du \equiv \int_0^t |f(x+u) - f(x-u)| du.$$

Then, we get the following

**THEOREM 2.** *If*

$$\bar{\Psi}(t) = O\left(\frac{t}{\log 1/t}\right)$$

*as*  $t \rightarrow +0$ , *then the series*

$$\sum \frac{B_n(x)}{(\log n)^{1+\varepsilon}}$$

*is summable*  $|C, 1|$  *at*  $x$ .

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