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by

Roop Narain Kesarwani

1

The object of this paper is to establish the following two Fourier series expansions for the Meijer’s $G$-functions.

\[
\sum_{r=0}^{\infty} G_{p+1, q+1}^{m+1, n+1} \left( z; a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q \right) \sin (2r+1) \theta = \sqrt{\pi}/2 \sin \theta \frac{a_1 \cdots a_p}{b_1 \cdots b_q} \times \sin \left( \frac{m+n-p/2-q/2}{2} \theta \right),
\]

where $0 \leq \theta \leq \pi$, $|\arg z| < (m+n-p/2-q/2)\pi$.

\[
G_{p+1, q+1}^{m+1, n+1} \left( z; a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q \right) + 2 \sum_{r=0}^{\infty} G_{p+1, q+1}^{m+1, n+1} \left( z; a_1, a_2, \ldots, a_p, 1+r \right) \cos r \theta = \sqrt{\pi} \frac{a_1 \cdots a_p}{b_1 \cdots b_q} \times \sin \left( \frac{m+n-p/2-q/2}{2} \theta \right),
\]

where $0 < \theta \leq \pi$, $|\arg z| < (m+n-p/2-q/2)\pi$.

The Meijer’s $G$-function is sum of hypergeometric functions each of which is usually an entire function. It is defined [1, p. 207] by a Mellin-Barnes type integral

\[
G_{p, q}^{m, n} (z; a_1, \ldots, a_p, b_1, \ldots, b_q) = \frac{1}{2\pi i} \int_{L} \frac{\prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\prod_{j=m+1}^{p} \Gamma(1-b_j+s) \prod_{j=n+1}^{q} \Gamma(a_j-s)} z^s ds,
\]

where $m, n, p, q$ are integers with $q \geq 1$, $0 \leq n \leq p$, $0 \leq m \leq q$. The parameters $a_j$ and $b_j$ are such that no pole of $\Gamma(b_j-s)$, $j = 1, \ldots, m$ coincides with any pole of $\Gamma(1-a_j+s)$, $j = 1, \ldots, n$.

The poles of the integrand must be simple and those of $\Gamma(b_j-s)$, $j = 1, \ldots, m$ lie on one side of the contour $L$ and those of $\Gamma(1-a_j+s)$, $j = 1, \ldots, n$ must lie on the other side.

To prove (1.1) and (1.2) whose conditions of validity are given in section 2, we require the following Fourier series established by McRobert [2, p. 79 and 3, p. 143].

\[
\frac{\sqrt{\pi} \Gamma(2-s)}{2 \Gamma(\frac{3}{2}-s)} (\sin \theta)^{1-2s} \sum_{r=0}^{\infty} \frac{(s; r)(2r+1)\theta}{(2-s; r)} \sin (2r+1) \theta,
\]
where \(0 \leq \theta \leq \pi\), \(R(s) \leq \frac{1}{2}\). Here \((s; 0) = 1\), \((s; r) = s(s+1) \ldots (s+r-1)\), \(r = 1, 2, 3, \ldots\)

\[
\frac{\sqrt{\pi} \Gamma(1-s)}{\Gamma(\frac{1}{2}-s)} (\sin (\theta/2))^{-2s} = 1 + 2 \sum_{r=0}^{\infty} \frac{(s; r)}{(1-s; r)} \cos r\theta
\]

where \(0 < \theta \leq \pi\).

2

From (1.3), the expression on the left of (1.1) can be written as

\[
\sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{3}{2}-s) \prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s) \Gamma(r+s)}{\Gamma(s) \prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s) \Gamma(2+r-s)} z^s ds \sin (2r+1)\theta.
\]

Here the path \(L\) of integration runs from \(c-i\infty\) to \(c+i\infty\). The conditions

\[
0 < c < \frac{3}{2}, \quad Rl(b_j) > c, \quad j = 1, \ldots, m
\]
\[
Rl(a_j) < c+1, \quad j = 1, \ldots, n
\]

ensure that all the poles of \(\Gamma(\frac{3}{2}-s)\) and \(\Gamma(b_j-s)\), \(j = 1, \ldots, m\) lie to the right of \(L\) and those of \(\Gamma(r+s)\) and \(\Gamma(1-a_j+s)\), \(j = 1, \ldots, n\) lie to the left of \(L\), as required for the definition of the \(G\)-function on the left side of (1.1). The integral converges if \(p+q < 2(m+n)\) and \(|\arg z| < (m+n-p/2-q/2)\pi\). On changing the order of integration and summation, which is easily seen to be justified, the above expression becomes

\[
\frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{3}{2}-s) \prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\Gamma(2-s) \prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s)} \cdot \left\{ \sum_{r=0}^{\infty} \frac{(s; r)}{(2-s; r)} \sin (2r+1)\theta \right\} z^s ds
\]

and on using (1.4), it takes the form

\[
\sqrt{\pi/2} \sin \theta \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s)} \left( \frac{z}{\sin^2 \theta} \right)^s ds
\]
which is just the expression on the right of (1.1).

(1.1) is the Fourier sine series for the $G$-functions. The Fourier cosine series (1.2) is proved in an analogous manner by using (1.3) and (1.5). The conditions of validity of (1.2) are

$$0 < c < \frac{1}{2}$$

$$\text{Re}(b_j) > c, \quad j = 1, \ldots, m$$

$$\text{Re}(a_j) < c+1, \quad j = 1, \ldots, n.$$ 

If we make use of the relation [1, p. 215]

$$G_{p+1,q}^{m,n} (\frac{a_0}{b_1}, \ldots, \frac{a_p}{b_q}) = E_{d_1, \ldots, d_q}^{c_1, \ldots, c_q},$$

where $E(\cdot)$ denotes McRobert's $E$-function [1, p. 208], the formulae (1.1) and (1.2) reduce to the Fourier series for $E$-functions obtained by McRobert [2, p. 79, eqns (1) and (2)].

$$3$$

From (1.1) and (1.2), we easily deduce the integrals

$$\int_0^\pi \sin (2r+1) \theta \sin \theta \ G_{p,q}^{m,n} (\frac{z}{\sin^2 \theta} | \frac{a_0}{b_1}, \ldots, \frac{a_p}{b_q}) d\theta$$

$$= \sqrt{\pi} \ G_{p+2,q+2}^{m+1,n+1} (\frac{z}{1-r} | \frac{a_0}{b_1}, \ldots, \frac{a_p}{b_q}^{r+1}),$$

and

$$\int_0^\pi \cos r \theta \ G_{p,q}^{m,n} (\frac{z}{\sin^2 (\theta/2)} | \frac{a_0}{b_1}, \ldots, \frac{a_p}{b_q}) d\theta$$

$$= \sqrt{\pi} \ G_{p+2,q+2}^{m+1,n+1} (\frac{z^\frac{1-r}{2}}{1} | \frac{a_0}{b_1}, \ldots, \frac{a_p}{b_q}^{r+1}),$$

where $p+q < 2(m+n)$, $|\arg z| < (m+n-p/2-q/2)\pi$ and $r = 0, 1, 2, \ldots.$

REFERENCES

Bateman Manuscript Project


McRobert, T. M.


McRobert, T. M.