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Compositio Mathematica, tome 17 (1965-1966), p. 146-148

<http://www.numdam.org/item?id=CM_1965-1966__17__146_0>
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by

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We establish a relation here between additive and multiplicative convolution averages of a bounded function. The real numbers are a locally compact Abelian group, under the operation of addition, with Haar measure \( dt \). The positive real numbers are a locally compact Abelian group, under the operation of multiplication, with Haar measure \( dt/t \). Given a bounded Lebesgue measurable function \( g \), with \( g(x) \) defined for all real numbers \( x \), we may study its behaviour for large values of \( x \) by forming certain averages. One kind is with respect to integrable functions on \( (-\infty, \infty) \), the other with respect to integrable functions on \( (0, \infty) \) where we use only the restriction of \( g \) to \( (0, \infty) \). In each case, integrability is with respect to the appropriate measure, and the average depends only on the behaviour of \( g \) at \( +\infty \). We call the first kind of average a Fourier average, the second kind a Mellin average, and we establish a connection between them. We shall assume that all our functions are Lebesgue measurable.

**Main Theorem.** If \( g \) is bounded, \( K \geq 0 \),

\[
\int_{-\infty}^{\infty} K(t)dt = 1, \quad H \geq 0, \quad \text{and} \quad \int_{0}^{\infty} H(t)dt/t = 1,
\]

then

\[
\limsup_{x \to \infty} \int_{0}^{\infty} H(x/t)g(t)dt/t \leq \limsup_{x \to \infty} \int_{-\infty}^{\infty} K(x-t)g(t)dt.
\]

The next result follows from the main theorem on normalizing \( K \) so that \( \int_{-\infty}^{\infty} K(t)dt = 1 \) (i.e. replacing \( K(t) \) by \( K(t)/\int_{-\infty}^{\infty} K(s)ds \)), and writing \( H = H^+ - H^- \), where \( H^+(t) = \max (H(t), 0) \) and \( H^-(t) = -\min (H(t), 0) \). Now considering the normalizations of \( H^+ \) and \( H^- \), the main theorem and the corresponding result for \( \lim \inf \) may be applied.

**Tauberian Theorem.** Suppose \( g \) is bounded,

\[
K \geq 0, \quad 0 < \int_{-\infty}^{\infty} K(t)dt < \infty,
\]

and

\[
\lim_{x \to \infty} \int_{-\infty}^{\infty} K(x-t)g(t)dt = 0.
\]
If
\[ \int_0^\infty |H(t)| \frac{dt}{t} < \infty, \]
then
\[ \lim_{x \to \infty} \int_0^\infty H(x/t) g(t) \frac{dt}{t} = 0. \]

**Proof.** We need the following result from [2, p. 1005].

**Lemma 1.** Given a bounded function \( g \) and \( 0 < \xi < 1 \), let

\[ L(\xi) = \lim sup_{x \to \infty} \frac{1}{x-\xi x} \int_{\xi x}^x g(t) dt, \]

and let \( L(1) = \sup_{0 < \xi < 1} L(\xi) \). There exists a bounded function \( g^* \) such that \( g^* \leq g \) and \( \lim_{x \to \infty} x^{-1} \int_0^x g^*(t) dt = L(1) \).

Let us write \( A = \lim sup \int_0^\infty H(x/t) g^*(t) \frac{dt}{t} \) and \( B = \lim sup \int_0^\infty K(x-t) g(t) dt \). We must prove \( A \leq B \), which obviously follows from the next two lemmas.

**Lemma 2.** \( A \leq L(1) \).

**Lemma 3.** \( L(1) \leq B \).

We prove Lemma 2 via Lemma 1. Since \( g \leq g^* \) and \( K \geq 0 \), we have \( A = \lim sup \int_0^\infty H(x/t) g^*(t) dt \). But since \( \lim x^{-1} \int_0^x g^*(t) dt = L(1) \) (i.e., the Cesaro limit of \( g^* \) is \( L(1) \)), we may apply the Mellin form of the Wiener Tauberian theorem [1, p. 296] to conclude that \( \lim \int_0^\infty H(x/t) g^*(t) \frac{dt}{t} = L(1) \), and hence \( A \leq L(1) \). In more detail, we have \( \lim x^{-1} \int_0^x g^*(t) dt = L(1) \), and we may write \( x^{-1} \int_1^x g^*(t) dt = \int_0^\infty g^*(t) C(x/t) \frac{dt}{t} \), where \( C(s) = 0 \) for \( 0 < s < 1 \), and \( C(s) = s^{-1} \) for \( s \geq 1 \). Denoting by \( C^\wedge \) the Mellin transform of \( C \), \( C^\wedge(r) = \int_0^\infty t^{i r} C(t) \frac{dt}{t} \), we have \( C^\wedge(r) = (1-i r)^{-1} \). Since \( C^\wedge(r) \neq 0 \) for real \( r \), we obtain the conclusion.

To prove Lemma 3, it is enough to do it under the special hypothesis that for some \( N \), \( K(x) = 0 \) for \( |x| \geq N \). The general case follows on letting

\[ K_N(x) = \begin{cases} K(x) / \int_{-N}^N K(t) dt & \text{for } |x| \leq N \\ 0 & \text{for } |x| > N, \end{cases} \]

and then letting \( N \to \infty \). Let us write

\( (K \ast g)(x) = \int_{-\infty}^x K(x-t) g(t) dt \).

We shall prove that for \( \xi < 1 \),

\[ \int_{\xi x}^x (K \ast g)(y) dy = \int_{\xi x}^x g(t) dt + o(x). \]

If this is done, we get
\[ L(\xi) \leq \limsup_{y \leq z \leq \xi} (K * g)(y) \]

from which Lemma 3 follows directly. To prove (1), write

\[ I(x) = \int_{\xi}^{x} (K * g)(y)dy = \int_{-\infty}^{\infty} g(t) \int_{\xi}^{x} K(y-t)dy dt. \]

But \( \int_{\xi}^{x} K(y-t)dy \) vanishes if \( t < \xi x - N \) or \( t > x + N \). And \( \int_{\xi}^{x} K(y-t)dy = \int_{\xi-x-N}^{\xi-x-N} K(y)dy \). Hence

\[ I(x) = \int_{\xi-x-N}^{\xi-x-N} g(t) \int_{\xi-x-N}^{x-t} K(y)dy dt. \]

We write \( \int_{\xi-x-N}^{x-N} = \int_{\xi-x-N}^{\xi-x-N} + \int_{\xi-x-N}^{\xi-x-N} + \int_{\xi-x-N}^{\xi-x-N} \). For \( \xi x + N < t < x - N \), \( \int_{\xi-x-N}^{\xi-x-N} K(y)dy = 1 \), and for any \( a \) and \( b \) with \( a < b \),

\[ 0 \leq \int_{a}^{b} K(y)dy \leq 1. \]

Hence

\[ I(x) = \int_{\xi-x-N}^{\xi-x-N} g(t)dt + 0(1) = \int_{\xi}^{\xi} g(t)dt + 0(1), \]

and the proof is complete.

REFERENCES

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