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# A theorem on the zeros of an entire function (II)

by

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## 1

Let $f(z)$ be an entire function of order $\rho$ and genus $p$. Suppose further $z_1, z_2, \ldots, z_n$ are the zeros of $f(z)$; then its Hadamard representation is:

$$f(z) = z^m e^{Q(z)} f(z),$$

where $Q(z)$ is a polynomial of degree $q \leqq \rho$ and $P(z)$ is the canonical product of genus $p$ formed with the zeros (other than $z = 0$) of $f(z)$. In a recent note the author has proved the following theorem [2]:

THEOREM $A$: If $P(z)$ be a canonical product of genus $p$ and order $\rho(\rho > p)$, defined by

$$P(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left\{z/z_n + \tfrac{1}{2}(z/z_n)^2 + \ldots + \frac{1}{p}(z/z_n)^p\right\},$$

where $z_1, z_2, \ldots$ etc. are the zeros of $P(z)$ whose modulii $r_1, r_2 \ldots$ form a non-decreasing sequence such that $r_n > 1$ for all $n$ and where $r_n \to \infty$, as $n \to \infty$ then for $z$ in a domain exterior to the circles $r_n^{-h}$ $(h > \rho)$ described about the zeros $z_n$ as centres, we have:

$$\left|\frac{P'(z)}{P(z)}\right| < K \int_0^\infty \frac{n(x) r^p}{x^p (x+r)^2}\, dx,$$

where $K$ is a constant dependent of $p$ and $P'(z)$ is the first derivative of $P(z)$ and $n(x)$ denotes the number of zeros within and on the dircle $|z| = x$.

Again, suppose, as we may without loss of generality, that $n(r) = 0$ for $r \leqq 1$. In the present note the aim of the author is to give a few important uses of the above theorem. Let

$$M(r) = \max_{|z|=r} |f(z)|; \quad M'(r) = \max_{|z|=r} |f'(z)|.$$

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## 2

We show:

**Theorem 1:** If $f(z)$ is an entire function of non-integral order $\rho$ and genus $p$, then for arbitrarily large $r$,

$$n(r) \neq 0 \left\{\frac{rM'(r)}{M(r)}\right\}.$$

**Proof:** We shall consider two cases: according as $f(z)$ is of convergence class or divergence class [1]. To whatever class $f(z)$ belongs, we always have $p < \rho < p+1$, and $\int^{\infty} n(t)t^{-m}dt$ diverges for $m < \rho+1$ and converges for $m > \rho+1$. Now as we are considering entire functions of non-integral order, it is sufficient to prove the theorem for the canonical product $P(z)$, of $f(z)$. Then we have from the above theorem:

$$\frac{M'(r)}{M(r)} < K \int_1^{\infty} \frac{n(x)r^p}{x^p(x+r)^2}\,dx$$

or,

$$\frac{rM'(r)}{M(r)} < K\left\{r^{p-1}\int_1^r \frac{n(x)}{x^p}\,dx + r^{p+1}\int_r^{\infty}\frac{n(x)}{x^{p+2}}\,dx\right\}$$
$$\equiv K\varphi(r),\ (\text{say}).$$

Suppose now our result is false. Then for arbitrarily small positive $\varepsilon$ and for almost all increasing $r > r_0$; $r_0 \in E$ [2]

$$n(r) \leqq \varepsilon\varphi(r), \tag{1}$$

where we have omitted $K$.

Take $m$ so that $\rho+1 \leqq m < p+2$, and so $\int^{\infty} t^{-m}n(t)dt$ converges (it converges for $m > \rho+1$ in both cases). Multiply (1) by $r^{-m}$, and integrate it over $(R, \infty)$, $R > r_0$ and belongs to $E$, and then change the order of integration in the resulting iterated integrals (this can easily be effected), we obtain:

$$\int_R^{\infty} t^{-m}n(t)dt \leqq \varepsilon\int_1^R \frac{n(u)}{u^p}\,du\int_R^{\infty} t^{p-1-m}dt + \varepsilon\int_R^{\infty}\frac{n(u)}{u^p}\,du\int_u^{\infty} t^{p-1-m}dt$$
$$+\varepsilon\int_R^{\infty}\frac{n(u)}{u^{p+2}}\,du\int_R^u t^{p-m+1}dt$$
$$\leqq \frac{\varepsilon R^{p-m}}{m-p}\int_1^R \frac{n(u)}{u^p}\,du + \frac{\varepsilon}{m-p}\int_R^{\infty}\frac{n(u)}{u^m}\,du + \frac{\varepsilon}{p-m+2}\int_R^{\infty}\frac{n(u)}{u^m}\,du$$
$$= \frac{\varepsilon R^{p-m}}{m-p}\int_1^R \frac{n(u)}{u^p}\,du + \frac{2\varepsilon}{(m-p)(p-m+2)}\int_R^{\infty}\frac{n(u)}{u^m}\,du.$$

Let

$$\varepsilon < \frac{(p-m+2)(\rho-p)}{4}; \quad \rho < m.$$

Then

$$\tfrac{1}{2}\int_R^\infty \frac{n(t)}{t^m}\,dt \leqq \frac{\varepsilon R^{p-m}}{m-p}\int_1^R \frac{n(u)}{u^p}\,du.$$

*Case (i) when $f(z)$ is of divergence class*: Then letting $m \to \rho+1$, the left-hand side (2) becomes infinite whilst the right-hand side tends to a finite quantity and hence (1) gives a contradiction.

*Case (ii) when $f(z)$ is of convergence class*: Then from (2), taking $m = \rho+1$ to begin with, we have, since $n(r)$ increases,

$$\tfrac{1}{2}n(R)\rho^{-1}R^{-\rho} \leqq \frac{\varepsilon R^{p-\rho-1}}{\rho+1-p}\int_1^R \frac{n(u)}{u^p}\,du,$$

and since this is true for almost all large $R$ and $\varepsilon > 0$, we have

$$n(r) = 0\left\{r^{p-1}\int_1^r \frac{n(u)}{u^p}\,du\right\}.$$

Now $\int^\infty t^{-p-\alpha}n(t)dt$ diverges if $1 < \alpha < \rho+1-p$, and so for such $\alpha$, as $R \to \infty$

$$\begin{aligned}\int_1^R \frac{n(r)}{r^{p+\alpha}}\,dr &= 0\left\{\int_1^R r^{-\alpha-1}dr\int_u^r \frac{n(u)}{u^p}\,du\right\}\\ &= 0\left\{\int_1^R \frac{n(u)}{u^p}\,dx\int_u^R r^{-\alpha-1}dr\right\}\\ &= 0\left\{\int_1^R \frac{n(u)}{u^{p+\alpha}}\,du\right\},\end{aligned}$$

and this again shows a contradiction. Therefore (1) fails to hold good in both the cases. This proves the theorem.

THEOREM 2: If $f(z)$ is of order $\rho$ and divergence class, the integral

$$(\mathrm{I}_1) \qquad \int^\infty r^{-1-\rho+\varepsilon}\frac{rM'(r)}{M(r)}\,dr = \int^\infty r^{-\rho+\varepsilon}\frac{M'(r)}{M(r)}\,dr$$

diverges; and if

$$(\mathrm{I}_2) \qquad \int^\infty r^{-1-\rho}\frac{rM'(r)}{M(r)}\,dr = \int^\infty r^{-\rho}\frac{M'(r)}{M(r)}\,dr.$$

[1] A function $f(z)$ is said to be of convergence or divergence class according as $\int^\infty n(x)x^{-\rho-1}dx$ converges or diverges respectively.

[2] $E$ is the set of points of $r$ for which the inequality in Theorem A holds.

diverges then $f(z)$ is of divergence class provided the order $\rho$ of $f(z)$ is not an integer.

Proof: The first of the theorem is obvious and so omitted (the first follows with the help of Vijayaraghvan's inequality [3]:

$$M'(r) > \frac{M(r)}{r}\frac{\log M(r)}{\log r}, \qquad r > r_0(f) = r_0,$$

and the result of Boas ([1], p. 32; first part of (2.11.1))).

To prove the second part of the theorem, suppose $\rho$ is not an integer, then $f(z)$ is dominated by its canonical product $P(z)$. Now we have from Theorem A

$$\frac{rM'(r)}{M(r)} < K\left\{r^{p-1}\int_1^r \frac{n(t)}{t^p}\,dt + r^{p+1}\int_r^\infty \frac{n(t)}{t^{p+2}}\,dt\right\}$$
$$= K(J_1(r)+J_2(r)), \quad \text{(say)},$$

where $p < \rho < p+1$. If we can prove that the convergence of $\int^\infty (n(t)/t^{\rho+1})\,dt$ implies the convergence of $\int^\infty (J_j(r)/r^{\rho+1})\,dr$ $(j=1,2)$, our second part of the theorem will be established. So

$$\int_R^\infty \frac{J_2(r)}{r^{\rho+1}}\,dr = \int_R^\infty r^{p-\rho}\,dr\int_r^\infty \frac{n(t)}{t^{p+2}}\,dt$$
$$= \int_R^\infty \frac{n(t)}{t^{p+2}}\,dt\int_R^t r^{p-\rho}\,dr$$
$$\leqq (p-\rho+1)^{-1}\int_R^\infty \frac{n(t)}{t^{\rho+1}}\,dt < \infty$$

and

$$\int_1^R \frac{J_1(r)}{r^{\rho+1}}\,dr = \int_1^R r^{p-\rho-2}\,dr\int_1^r \frac{n(t)}{t^p}\,dt$$
$$\leqq \int_1^R r^{p-\rho-1}\,dr\int_1^r \frac{n(t)}{t^{p+1}}\,dt$$
$$\leqq (\rho-p)^{-1}\int_1^R \frac{n(t)}{t^{\rho+1}}\,dt$$

and so

$$\int^\infty r^{-\rho-1}\frac{rM'(r)}{M(r)}\,dr < \infty;$$

and the second part follows.

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