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A theorem on the zeros of an entire function (II)

by

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Let $f(z)$ be an entire function of order $p$ and genus $p$. Suppose further $z_1, z_2, \ldots, z_n$ are the zeros of $f(z)$; then its Hadamard representation is:

$$f(z) = z^n e^{Q(z)} f(z),$$

where $Q(z)$ is a polynomial of degree $q \leq p$ and $P(z)$ is the canonical product of genus $p$ formed with the zeros (other than $z = 0$) of $f(z)$. In a recent note the author has proved the following theorem [2]:

**Theorem A**: If $P(z)$ be a canonical product of genus $p$ and order $p(p > p)$, defined by

$$P(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp \left\{ \frac{1}{2} \left( \frac{z}{z_n} + \frac{1}{2} \left( \frac{z}{z_n} \right)^2 + \ldots + \frac{1}{p} \left( \frac{z}{z_n} \right)^p \right) \right\},$$

where $z_1, z_2, \ldots$ etc. are the zeros of $P(z)$ whose modulii $r_1, r_2 \ldots$ form a non-decreasing sequence such that $r_n > 1$ for all $n$ and where $r_n \to \infty$, as $n \to \infty$ then for $z$ in a domain exterior to the circles $r_n^{-h}$ $(h > p)$ described about the zeros $z_n$ as centres, we have:

$$\left| \frac{P'(z)}{P(z)} \right| < K \int_0^{\infty} \frac{n(x)r^p}{x^p(x+r)^2} \, dx,$$

where $K$ is a constant dependent of $p$ and $P'(z)$ is the first derivative of $P(z)$ and $n(x)$ denotes the number of zeros within and on the circle $|z| = x$.

Again, suppose, as we may without loss of generality, that $n(r) = 0$ for $r \leq 1$. In the present note the aim of the author is to give a few important uses of the above theorem. Let

$$M(r) = \max_{|z|=r} |f(z)|; \quad M'(r) = \max_{|z|=r} |f'(z)|.$$

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We show:

**THEOREM 1:** If \( f(z) \) is an entire function of non-integral order \( p \) and genus \( p \), then for arbitrarily large \( r \),

\[
n(r) \neq 0 \left\{ \frac{rM'(r)}{M(r)} \right\}.
\]

**PROOF:** We shall consider two cases: according as \( f(z) \) is of convergence class or divergence class\(^1\). To whatever class \( f(z) \) belongs, we always have \( p < \rho < p+1 \), and \( \int \infty \! n(t)t^{-m}dt \) diverges for \( m < \rho + 1 \) and converges for \( m > \rho + 1 \). Now as we are considering entire functions of non-integral order, it is sufficient to prove the theorem for the canonical product \( P(z) \), of \( f(z) \). Then we have from the above theorem:

\[
\frac{M'(r)}{M(r)} < K \int_1^\infty \! \frac{n(x)x^p}{x^p(x+r)^2} \, dx
\]

or,

\[
\frac{rM'(r)}{M(r)} < K \left\{ r^{p-1} \int_1^r \! \frac{n(x)x^p}{x^p} \, dx + r^{p+1} \int_r^\infty \! \frac{n(x)x^p}{x^{p+2}} \, dx \right\} = K \varphi(r), \text{ (say)}.
\]

Suppose now our result is false. Then for arbitrarily small positive \( \varepsilon \) and for almost all increasing \( r > r_0 \); \( r_0 \in E \)

\[
(1) \quad n(r) \leq \varepsilon \varphi(r),
\]

where we have omitted \( K \).

Take \( m \) so that \( \rho + 1 \leq m < p + 2 \), and so \( \int \infty \! t^{-m}n(t)dt \) converges (it converges for \( m > \rho + 1 \) in both cases). Multiply \((1)\) by \( r^{-m} \), and integrate it over \((R, \infty)\), \( R > r_0 \) and belongs to \( E \), and then change the order of integration in the resulting iterated integrals (this can easily be effected), we obtain:

\[
\int_R^\infty \! t^{-m}n(t)dt \leq \varepsilon \int_1^R \! \frac{n(u)}{u^p} \, du \int_R^\infty \! t^{p-1-m}dt + \varepsilon \int_R^\infty \! \frac{n(u)}{u^p} \, du \int_u^\infty \! t^{p-1-m}dt + \varepsilon \int_R^\infty \! \frac{n(u)}{u^{p+2}} \, du \int_R^u \! t^{p-m+1}dt
\]

\[
\leq \frac{\epsilon R^{p-m}}{m-p} \int_1^R \! \frac{n(u)}{u^p} \, du + \frac{\epsilon}{m-p} \int_R^\infty \! \frac{n(u)}{u^m} \, du + \frac{\epsilon}{p-m+2} \int_R^\infty \! \frac{n(u)}{u^m} \, du
\]

\[
= \frac{\epsilon R^{p-m}}{m-p} \int_1^R \! \frac{n(u)}{u^p} \, du + \frac{2\epsilon}{(m-p)(p-m+2)} \int_R^\infty \! \frac{n(u)}{u^m} \, du.
\]
Let 
\[ \varepsilon < \frac{(p-m+2)(\rho-p)}{4}; \quad \rho < m. \]

Then
\[ \frac{1}{2} \int_{R}^{\infty} \frac{n(t)}{t^m} \, dt \leq \frac{\varepsilon R^{\rho-m}}{m-p} \int_{1}^{R} \frac{n(u)}{u^p} \, du. \]

*Case (i) when \( f(z) \) is of divergence class:* Then letting \( m \to \rho + 1 \), the left-hand side (2) becomes infinite whilst the right-hand side tends to a finite quantity and hence (1) gives a contradiction.

*Case (ii) when \( f(z) \) is of convergence class:* Then from (2), taking \( m = \rho + 1 \) to begin with, we have, since \( n(r) \) increases,
\[ \frac{1}{2} n(R) \rho^{-1} R^{-\rho} \leq \frac{\varepsilon R^{\rho-p-1}}{\rho+1-p} \int_{1}^{R} \frac{n(u)}{u^p} \, du, \]

and since this is true for almost all large \( R \) and \( \varepsilon > 0 \), we have
\[ n(r) = 0 \left\{ r^{p-1} \int_{1}^{r} \frac{n(u)}{u^p} \, du \right\}. \]

Now \( \int_{1}^{\infty} t^{-\alpha} n(t) \, dt \) diverges if \( 1 < \alpha < \rho + 1 - \rho \), and so for such \( \alpha \), as \( R \to \infty \)
\[ \int_{1}^{R} \frac{n(r)}{r^{p+\alpha}} \, dr = 0 \left\{ \int_{1}^{R} r^{-\alpha-1} \, dr \int_{1}^{r} \frac{n(u)}{u^p} \, du \right\} \]
\[ = 0 \left\{ \int_{1}^{R} \frac{n(u)}{u^p} \, dx \int_{1}^{R} r^{-\alpha-1} \, dr \right\} \]
\[ = 0 \left\{ \int_{1}^{R} \frac{n(u)}{u^p+\alpha} \, du \right\}, \]

and this again shows a contradiction. Therefore (1) fails to hold good in both the cases. This proves the theorem.

**Theorem 2:** If \( f(z) \) is of order \( \rho \) and divergence class, the integral
\[ (I_1) \quad \int_{r}^{\infty} r^{-1-\rho+\varepsilon} \frac{r M'(r)}{M(r)} \, dr = \int_{\infty}^{r} r^{-\rho+\varepsilon} \frac{M'(r)}{M(r)} \, dr \]
diverges; and if
\[ (I_2) \quad \int_{r}^{\infty} r^{-1-\rho} \frac{r M'(r)}{M(r)} \, dr = \int_{\infty}^{r} r^{-\rho} \frac{M'(r)}{M(r)} \, dr. \]

\(^1\) A function \( f(z) \) is said to be of convergence or divergence class according as \( \int_{1}^{\infty} n(z)x^{-\rho-1} \, dx \) converges or diverges respectively.

\(^2\) \( E \) is the set of points of \( r \) for which the inequality in Theorem A holds.
diverges then \( f(z) \) is of divergence class provided the order \( \rho \) of \( f(z) \) is not an integer.

**Proof:** The first of the theorem is obvious and so omitted (the first follows with the help of Vijayaraghvan’s inequality [3]:

\[
M'(r) > \frac{M(r) \log M(r)}{r \log r}, \quad r > r_0(f) = r_0,
\]

and the result of Boas ([1], p. 32; first part of (2.11.1)).

To prove the second part of the theorem, suppose \( \rho \) is not an integer, then \( f(z) \) is dominated by its canonical product \( P(z) \).

Now we have from Theorem A

\[
\frac{rM'(r)}{M(r)} < K \left\{ r^{p-1} \int_1^r \frac{n(t)}{t^\rho} \, dt + r^{p+1} \int_r^\infty \frac{n(t)}{t^{p+2}} \, dt \right\}
= K(J_1(r) + J_2(r)), \quad \text{(say),}
\]

where \( p < \rho < p+1 \). If we can prove that the convergence of \( \int_1^\infty (n(t)/t^{p+1}) \, dt \) implies the convergence of \( \int_1^\infty (J_j(r)/t^{p+1}) \, dr \) \((j = 1, 2)\), our second part of the theorem will be established. So

\[
\int_1^\infty \frac{J_2(r)}{r^{p+1}} \, dr = \int_1^\infty r^{p-\rho} \, dr \int_r^\infty \frac{n(t)}{t^{p+2}} \, dt
= \int_r^\infty \frac{n(t)}{t^{p+2}} \, dt \int_r^1 r^{p-\rho} \, dr
\leq (p-\rho+1)^{-1} \int_r^\infty \frac{n(t)}{t^{p+1}} \, dt < \infty
\]

and

\[
\int_1^R \frac{J_1(r)}{r^{p+1}} \, dr = \int_1^R r^{p-\rho-1} \, dr \int_1^r \frac{n(t)}{t^\rho} \, dt
\leq \int_1^R r^{p-\rho-1} \, dr \int_1^r \frac{n(t)}{t^{p+1}} \, dt
\leq (p-\rho)^{-1} \int_1^R \frac{n(t)}{t^{p+1}} \, dt
\]

and so

\[
\int_1^\infty r^{p-1} \frac{rM'(r)}{M(r)} \, dr < \infty;
\]

and the second part follows.

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