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Uniform distribution modulo 1 and numerical analysis *

by

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One of the oldest applications of uniform distribution modulo 1 seems to be a theorem of Fejer and Kalmar, going back to a theorem of C. Runge about Lagrange interpolation: Let C be a closed bounded pointset of the finite z-plane, consisting of a finite number of mutually exterior Jordan regions. Let the points $z_1^{(n)}, \ldots z_{n+1}^{(n)}$ $(n = 1, 2, \ldots)$ lie on the boundary B of C; a necessary and sufficient condition that the sequence of polynomials $p_n(z)$ of degree n found by interpolation to an arbitrary function $f(z)$ analytic on C in the points $z_1^{(n)}, \ldots, z_{n+1}^{(n)}$ converges uniformly to $f(z)$ is, that the transforms of the $z_1^{(n)}, \ldots z_{n+1}^{(n)}$, be uniformly distributed on the unit circle. The definition of the transform w of the point z is the following: Let K be the exterior of C and let $w = \varphi(z)$ map K on to the exterior of $|w| = 1$, so that the points at infinity in the two planes correspond to each other. I do not want to go further in the details of this theorem, because there is the excellent exposition in the wellknown book of Walsh [1]. I want to speak now about the calculation of multiple integrals.

Let us suppose, that a function f is defined on the unit cube $E = E^s$

$$0 \leqq p_1 < 1, \ldots 0 \leqq p_s < 1 \qquad (s \geqq 1) \tag{1}$$

periodic with period one and integrable in the Riemann-sense. This shall be at first the only condition about f. The classical formulas in numerical calculation of integrals are working with sequences

$$\left(\frac{k_1}{n}, \ldots, \frac{k_s}{n}\right) \qquad 0 \leqq k_j \leqq n-1 \qquad (n \geqq 1). \tag{2}$$

The number of the points is $T = n^s$ and if we consider a Riemann-sum

* Nijenrode lecture.

$$\sum_T = \frac{1}{n^s} \sum_{k_1,\ldots k_s} f\left(\frac{k_1}{n}, \ldots, \frac{k_s}{n}\right)$$

then we have [2] if f is of bounded variation,

$$|\Delta_T| = |\sum_T - \mu(f)| = o(T^{-1/s}), \qquad \mu(f) = \int_E f dx_1 \ldots dx_s$$

and we can easily find functions of bounded variation, for which

$$\Delta_T \geqq CT^{-1/s}.$$

If you want, that $\Delta_T < \varepsilon$ for a given ε, then you must take $T \sim (1/\varepsilon)^s$ that is for large s beyond the realm of computers. The situation is the same, if you work for instance with Simpson-formulas in more dimensions. Therefore is often used the Monte Carlo-method [3]. Here one uses a sequence of independent uniformly distributed random variables $x_1, \ldots, x_T$ in the unit cube and if you consider

$$F = \frac{1}{T} \sum_{j=1}^{T} f(x_j)$$

then the expected value $E(F)$ is $\mu(f)$ and for the variance σ_F^2 you have σ_f^2/T, therefore the probable error in mean is $T^{-\frac{1}{2}}$. This looks very nice, but you have to replace the random variables by a sample of pseudo random numbers and then the problem remains, if the error Δ_T is also of this order. This is not so, and this has been shown by numerical examples by Ph. Davis and Rabinovits [4] and also in a later paper by N. M. Korobow [9]. It seems, that there are many mathematicians, who where suspicious about the Monte Carlo-method, but it is not very easy to find this in the literature. I mention further a paper of J. Bass and J. Guilloud [5] and a lecture of J. M. Hammersley [6]. There seems to be also a paper of Richtmyer (1951) in the Los Alomos reports, in which is not worked with a Monte-Carlo-method, but with the theory of uniform distribution. Also a paper of G. Peck (1953) seems to belong to the ideas of the paper of Richtmyer. Further I want to mention a new paper of Richtmyer, Devany, Metropolis [7] where it is said, that it would be very important to find new methods for calculating multiple integrals. Perhaps it was unknown to these authors the work of N. M. Korobow [7] and mine[2].

Korobow always works with functions f of class E_s^α:

$$f = \sum_h C_h e^{2\pi i \langle hx \rangle}$$

where the Fourier coefficients

$$C_h = 0(R(h)^{-\alpha}),\ R(h) = R(h_1, \ldots h_s) = \prod_{j=1}^{s} \mathrm{Max}(|h_j|, 1)$$

with $\alpha > 1$, so that the Fourier series of f is absolute convergent. In a paper [8] of 1957 he showed, that for functions of the class E_s^2 and the sequence $(k/p^2, k^2/p^2, \ldots k^s/p^2)$ mod 1

$$\begin{gathered} p \text{ prime},\ k \leqq p^2-1,\ T = p^2 \\ \Delta_T = 0(p^{-\frac{1}{2}}), \end{gathered} \tag{3}$$

the error is the same as in the Monte Carlo-method. In lectures, given in Princeton at the Columbia University 1959 and at the II. Hungarian Mat. Congress in Budapest 1960, I have shown, [2] that the discrepancy of *this* sequence is

$$\leqq (70 \log p)^s/\sqrt{p} \tag{4}$$

and then you have for functions of bounded variation

$$\Delta_T \leqq V(f) \frac{(70 \log p)^s}{\sqrt{p}} \tag{5}$$

using[1] the formula I (6).

You get the same result with the sequence,

$$\omega_1(s, T) : \left(\frac{k}{p}, \frac{k^2}{p}, \ldots, \frac{k^s}{p}\right), \qquad k \leqq p-1, \quad T = p \tag{6}$$

applying the theorem of Hasse and A. Weil about the zeros of the Riemann congruence zeta-function. It can be shown, that in this formula the power of $\log p$ cannot be cancelled. But there remains the possibility that the formula is true with a power of $\log \log p$. Korobow and I independently found sequences of the form $(\beta > 0)$

$$\omega_2(s, T) : \left(\frac{kg_1}{p}, \ldots, \frac{kg_s}{p}\right) \bmod 1, \qquad k \leqq p-1, \quad T = p \tag{7}$$

for which the error is of the order $0\left((\log p)^{s\beta}/p\right)$ therefore better than the Monte-Carlo-method. Korobow showed this only for the class E_s^α and his work was continued by I. N. Sobol, Solodov, Bachalov and I. Sarygin [9]. I showed it for functions of bounded variations and the results are more general and more precise [2]. More than $(\log p)^{(s-1)/2}/p$ is not possible after the theorem of

[1]) The lecture "Discrepancy and uniform distribution" is cited with I. (This Vol. p. 83).

K. Roth. Another sequence with the same properties as the sequence of Korobow and me, was found by Halton, [10] generalizing a sequence of van der Corput on the suggestion of Hammersley, but only calculating the discrepancy of the sequence. Sometimes people are astonished to hear, that it is possible to find sequences which give a smaller error than the Monte-Carlo-method for large T. But they forget the following thing: It follows from the theorem of large numbers, that for almost all sequences the error term is of the order $0(p^{-\frac{1}{2}} \log \log p)$. This means for all sequences except a set of measure zero, depending on f. But in this set of measure zero, there may exist sequences with larger errors, but also with smaller errors and my theorem and also the theorem of Korobow show, that there exist sequences, that for large classes of f the error terms constructed with these sequences are smaller than in the Monte-Carlo-method [11]. Further it must be said, that it is impossible, that an error term is independent of s for all functions of bounded variation.

This follows from the theorem of Roth.

Now I want to go into details. My method is very simple and was inspired by some methods in geometry of numbers. I call a lattice point $g = (g_1, \ldots, g_s)$ a good lattice point modulo p (p prime) if for all lattice points $h = (h_1, \ldots, h_s)$ with $0 < \|h\| \leqq (p-1)/2$ and

$$\langle hg \rangle \equiv h_1 g_1 + \ldots + h_s g_s = 0 \pmod{p}$$
$$R(h) \geqq p\,(8 \log p)^{-s}.$$

Such good lattice points exist. I will show this for $s = 2$. We can use $g_1 = 1$ and consider all solutions g of

$$h_1 + h_2 g \equiv 0 \pmod{p}, \quad 0 < \operatorname{Max}(|h_1|, |h_2|) \leqq \frac{p-1}{2},$$
$$|h_1 h_2| \leqq p/(8 \log p)$$

For fixed h_1 we have $p/(8h_1 \log p)$ possibilities for g, therefore for all h_1, h_2 the total number of g is

$$\frac{p}{8 \log p} \sum \frac{1}{h_1} < \frac{p}{4}$$

therefore exists a g_2 such that for all h_1, h_2 with $h_1 + h_2 g_2 \equiv 0 \pmod{p}$, we have

$$|h_1 \cdot h_2| \geqq \frac{p}{8 \log p}.$$

If we have already found $g_1 = 1, g_2, \ldots g_{s-1}$ we consider again the congruence

$$h_1+h_2g_2+\ldots+h_{s-1}g_{s-1}+h_sg \equiv 0 \pmod p,\ R(h) < \frac{p}{(8\log p)^{s-1}}$$

we see that the total number of all this g with the same method as before is

$$\frac{p}{(8\log p)^{s-1}} \sum_h \frac{1}{R(h)} < \frac{p}{4}$$

and so it is shown, how we can find successively $g_1, g_2, \ldots g_s$.

The g_j exists also, if p is not a prime, but instead of the constant 8 you have to take another constant. Especially important for the numerical applications is the case that p is a power of two. Now it can be shown, that the discrepancy of the sequence (7) with the numbers g_i is smaller than

$$< \frac{(1000\log p)^{2s}}{p} \tag{8}$$

and it can be shown, that we can choose these numbers g so, that the discrepancy is smaller than

$$< \frac{(1000\log p)^{s}}{p}. \tag{8'}$$

Now I want to say some words about the Corput-Halton sequence. Let R be a natural number, then we have for any natural number n

$$n = n_0+n_1R+\ldots.$$

We define

$$\varphi_R(n) = \frac{n_0}{R} + \frac{n_1}{R} + \ldots,$$

then the sequence is

$$\omega_3(s, T) : (\varphi_{R_1}(n), \ldots \varphi_{R_s}(n)) \qquad n = 1, 2, \ldots T \tag{9}$$

where $R_1, \ldots R_s$ have the greatest common divisor 1. For example $R_1, \ldots R_s$ can be the first s primes. It can be shown, that

$$D(\omega_3(s, T)) < \frac{(\log T)^s}{T} \prod_{i=1}^{s} \left(\frac{3R_i-2}{\log R_i}\right),\ T > \text{Max}\,(R_1, \ldots R_s).$$

This was shown by van der Corput for $s = 2$. You can get even a better sequence

$$\left(\frac{n}{T}, \varphi_{R_1}(n), \ldots \varphi_{R_{s-1}}(n)\right) \tag{9'}$$

with

$$D < \frac{(\log T)^{s-1}}{T} C_s \qquad (C_s \text{ constant}).$$

It is not necessary to work with prime numbers, because for large s there would be necessary a large table of prime numbers [12]. I would propose to work with the Fermat-numbers, because after a well known theorem, the Fermat-numbers are relative prime.

Now suppose, that we know more about f.

Let us suppose, that all derivatives for fixed m ($m \geqq 2$)

$$\frac{\partial^{mr} f}{\partial x_{l_1}^m \ldots \partial x_{l_r}^m} \qquad (1 \leqq r \leqq s, l_1 < l_2 < \ldots < l_r)$$

exist are periodic and continuous in the unit cube. Then we can show [2] [9] that for $\omega_2(s, T)$

$$|\lambda_T(f) - \mu(f)| < \frac{C\,(30 \log p)^{(m+1)s-1}}{p^m}. \tag{10}$$

There are two possibilities to show this [13]. The first method is, to develop f in a Fourier series. But it seems better to work with a generalization of the Euler summation formula to get smaller constancy in the error. We have

$$f(x) = \mu(f) + \sum_{r=1}^{m} \frac{P_r(x)}{r!}\left(f^{(r-1)}(1) - f^{(r-1)}(0)\right) - \int_0^1 \frac{P_m(x-t)}{m!} f^{(m)}(t)dt \tag{11}$$

in one variable, therefore for periodic function with continuous derivatives till the order m

$$f(x) = \mu(f) - \int_0^1 \frac{P_m(x-t)}{m!} f^{(m)}(t)dt \tag{12}$$

and for more dimension

$$f(x) - \mu(f) = \sum_l \int_{E_{l_1,\ldots l_r}^l} \frac{\partial^{mr} f(t)}{\partial x_{l_1}^m \ldots \partial x_{l_r}^m} P_{m_1,\ldots m_r}(x-t)dt_{l_1} \ldots dt_{l_r}$$

$$P_m(x) = P_{m_1,\ldots m_r}(x) = P_m(x_{l_1}) \ldots P_m(x_{l_r})$$

and therefore

$$|\lambda_T(f) - \mu(f)| \leqq \sum_l \int_{E_l} \left|\frac{\partial^{mr} f}{\partial x_{l_1}^m \ldots dx_{l_r}^m}\right| \left|\frac{1}{T}\sum_{n=1}^{T} P_m(x_n)\right|$$

and now we have

$$\frac{1}{T}\sum_{n=1}^{T} P_m(x_n) = \sum \frac{1}{R(h)^m}\frac{1}{T}\sum_{n=1}^{T} e^{2\pi i\langle hx_n\rangle}$$

and estimating this sum we get the above result.

Now let us suppose that the derivatives are continuous in the unit cube, but not periodic with period one. To fix the ideas I suppose that $m = 2l$ and $s = 1$. We form $\varphi(x) = \frac{1}{2}(f(x)+f(1-x))$ then we have, because $P_\nu(x)+P_\nu(1-x) = 0$ for ν odd, $(\Delta^k f = f^{(k)}(1)-f^{(k)}(0))$

$$\varphi(x) = \mu(f)+\sum_{\nu=1}^{l}\frac{P_{2\nu}}{(2\nu)!}\Delta^{2\nu-1}f-\int_0^1\frac{P_{2l}(x-t)}{(2l)!}\varphi^{(2l)}(t)dt$$

then I apply the operator

$$U^k\varphi = \left(\varphi\left(\frac{x}{2}\right)+\varphi(\tfrac{1}{2}+x)-\frac{1}{2^{k-1}}\varphi(x)\right)\Big/\left(1-\frac{1}{2^{k-1}}\right)$$

and form

$$\varphi_2 = U^2\varphi,\ \varphi_4 = U^4\varphi_2,\ \ldots\ \varphi_{2l} = U^{2l}\varphi_2(l-1)$$

and we have

$$\varphi_{2l}(x) = \mu(f)-C_{2l}\int_0^1 P_{2l}(x-t)\varphi_{2l}^{(2l)}(t)dt \tag{14}$$

more general $(0 \leqq k \leqq l,\ \varphi_0 = \varphi)$

$$\varphi_{2k}(x) = \mu(f)-\sum_{\nu=k+1}^{l} C_k\frac{P_{2k}}{(2k)!}\Delta^{(2k-1)}(f)-\int_0^1\frac{P_{2l}(x-t)}{(2l)!}\varphi^{(2l)}(t)dt.$$

This can be easily shown by complete induction in k. For $k = 0$ it is clear and we have

$$\varphi_{2k+2} = U^{2k+2}\varphi_{2k} = \mu(f)+\sum_{\nu=k+1}^{l} U^{2k+2}P_{2\nu}(x)\Delta^{2\nu-1}f-\int_0^1 U^{2k+2}P_{2l}\varphi^{(2l)}$$

now

$$U^{2k+2}P_{2k}(x) = P_{2k+2}\left(\frac{x}{2}\right)+P_{2k+2}(\tfrac{1}{2}+x)-\frac{1}{2^{2k+1}}P_{2k+2}(x) = 0.$$

We can easily generalize it to more variables, if we apply the operators to all variables and then we came back to formula (15) where we have now instead of f the function φ as (6) shows, with $\mu(\varphi) = \mu(f)$.

We can also work with the operators

$$U_p^k\varphi = \left(\sum_{h=0}^{p-1}\varphi\left(\frac{x}{p}+\frac{h}{p}\right)-\frac{1}{p^{k-1}}\varphi(x)\right)\Big/\left(1-\frac{1}{p^{k-1}}\right)$$

or better

$$S_p^k \varphi = \left(\sum_{h=0}^{k-1} \varphi\left(x + \frac{h}{p}\right) - \frac{1}{p^{k-1}} \varphi(\{px\}) \right) \Big/ \left(1 - \frac{1}{p^{k-1}}\right)$$

if we work with the sequence (7).

Now let us suppose we want to calculate a multiple integral about a simplex [14]. Now we can apply the formula (9) of I. We get a much better result if we apply the second formula I (8) and we take for ω a good sequence (7) or (9) and we have for the simplex $\tilde{S}: 0 \leqq p_1 \leqq p_2 \ldots \leqq p_s < 1$ if $f(x)$ is of bounded variation on $\tilde{S}$

$$\left| \lambda_T(f, \tilde{\omega}) - \frac{\tilde{\mu}(f)}{\tilde{\mu}(\tilde{S})} \right| \leqq C_s \frac{(30 \log T)^s}{T} V_{\tilde{S}}(f) \tag{15}$$

where $\tilde{\mu}(f) = \int_{\tilde{S}} f dx$ and $\mu(\tilde{S}) = 1/s!$ is the volume of $\tilde{S}$.

With the help of one affine mapping we can therefore construct to any simplex S a good sequence ω (T, S), so that

$$\left| \lambda_T(f, \omega) - \frac{\mu(f)}{\mu(S)} \right| < C^s \frac{(\log T)^s}{T} V_s(f) \tag{15'}$$

where $\mu(f)$ is the integral of f about S, $V_s(f)$ the variation of f on S, $\mu(S)$ the volume of S. It follows immediately that there must exist a sequence ω defined on any complex S with the same formula (15') and it is then easy, to see, how to construct such ω, when S is a differentiable compact manifold, because it is a wellknown theorem in topology, that S can be covered by a finite complex. There exist many good sequences ω on S, but it is essentially for the numerical work to find sequences, which can easily be computed, e.g. when S is a sphere, but about these things I will speak on another occasion, but compare the sequence at the end of I.

Now we will consider some applications.

We consider an integral equation [16]

$$\varphi(x) - \lambda \int_0^1 K(x, y) \varphi(y) dy = f(x). \tag{16}$$

We suppose that $K(x, y)$ is defined on the unit interval $E = E^2$: $0 \leqq x < 1, 0 \leqq y < 1$ and that it is periodic with period one and that it has a finite variation $V(K)$ in E^2. Further we set $\sigma(K) = \sup_{E^2} |K(x, y)|$, $M = \operatorname{Max}(\sigma(K), V(K))$. We suppose further that $f(x)$ is defined on E^1: $0 \leqq x < 1$, periodic and of bounded Variation $V(f)$, $\sigma(f) = \sup|f|$, $A = \operatorname{Max}(\sigma(f), V(f))$. The solution of the equation is given by the Neumann-series

$$\varphi(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_0^1 K^{(n)}(x, y) f(y) dy \tag{17}$$

where

$$K^{(n)}(x, y) = \int_0^1 \dots \int_0^1 K(x, x_2) K(x_2, x_3) \dots K(x_n, y) dx_2 \dots dx_n. \tag{18}$$

The series is convergent, if $|\lambda| M < 1$.

We set $\Pi^{(n)}(x_1, \dots x_n) = K(x_1, x_2) K(x_2, x_3) \dots K(x_{n-1}, x_n)$ defined on the unit cube $E^n : 0 \leqq x_1 \leqq 1, 0 \leqq x_n \leqq 1$. It is easy to show, that for the variation we have $V(\Pi^{(n)}) \leqq 2(6M)^{n-1}$ and for the variation of $F(\xi, x_1, \dots x_n) = \Pi^{(n)}(\xi, x_1, \dots x_n) f(x_n)$, we have $V(F) \leqq 2A(6M)^n$.

With these definitions, the Neumann-series has the form

$$\varphi(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_{E^m} F(x, x_1, \dots x_m) dx_1 \dots dx_m$$

and if s is a natural number

$$\varphi(x) = f(x) + \sum_{m=1}^{s} \lambda^m \int_{E^m} F(x, x_1, \dots, x_m) dx_1 \dots dx_m + R_1,$$

$$|R_1| \leqq A \frac{(|\lambda| M)^{s+1}}{1 - |\lambda| M}.$$

Now we apply our quadrature formula for $\int_{E^m} F (m \leqq s)$ and choose a sequence $\omega(m, T) = (y_{k1}, \dots y_{km})$ $(k = 1, \dots T)$ with discrepancy $D(\omega(m, T))$ (m is the dimension), then we have

$$\varphi(x) = f(x) + \frac{1}{T} \sum_{k=1}^{T} \sum_{m=1}^{s} \lambda^m K(x, y_{k1}) K(y_{k1}, y_{k2}) \dots K(y_{k,m-1}, y_{km}) f(y_{km}) + R \tag{19}$$

where

$$|R| = |R(s, T, \omega)| < A \left(2 \sum_{m=1}^{s} (6M|\lambda|)^m D(\omega(m, T)) + (|\lambda| M)^{s+1} \Big/ (1 - |\lambda| M) \right)$$

If we consider a sequence $\omega(s, T)$ and we take for the sequence $\omega(m, T)$ $(m \leqq s)$ the sequence of points, whose coordinates are the first m coordinates of the points of $\omega(s, T)$, we have $D(\omega(m, T)) \leqq D(\omega(s, T))$ therefore we get the simpler form of the remainder term, $B = \text{Max}(M, 1)$.

$$|R| < A(12(6B)^s D(\omega(s, T)) + (|\lambda| M)^{s+1} / (1 - |\lambda| M)). \tag{20}$$

If we take one of the good sequences, discussed before, and we take

$$s = \left[\frac{1}{4}\frac{\log T}{\operatorname{loglog} T}\right],$$

we have for $\varepsilon > 0$, $u = 4\log(\varepsilon(1-|\lambda|M))/\log|\lambda|M$

$$|R| < 2A\varepsilon \text{ for } T > \operatorname{Max}(\exp(u\log^2 u), \varepsilon^{-2}, \exp(1000\,B)) = T_0(\varepsilon). \tag{21}$$

If we take the sequence $(k/p, \ldots k^s/p) \bmod 1$ $T = p-1$, p prime, we have

$$R < 2A\sqrt{\varepsilon} \tag{22}$$

with the same s and we have, when we go with T to infinity

$$\varphi(x) = f(x) + \lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}\sum_{m=1}^{s(T)} K(x, y_{k1}) K(y_{k1}, y_{k2}) \ldots K(y_{k,m-1}, y_{km}) f(y_{km}). \tag{23}$$

Now let us consider the homogeneous integral equation

$$\varphi(x) = \lambda\int_0^1 K(x, y)\varphi(y)dy \tag{24}$$

with $K(x, y) = K(y, x)$ and let us suppose, that $K(x, y)$ is always > 0, then we have after the wellknown theorem of Jentsch for the Eigenvalues $\lambda_1, \lambda_2, \ldots$ of this equation $0 < \lambda_1 < |\lambda_2| \leqq |\lambda_3| \leqq \ldots$. This means that the lowest Eigenvalue is simple and positive. We can always suppose, that $\lambda_1 \geqq 1$. The Eigenfunctions to these Eigenvalues let us denote $\varphi_1, \varphi_2 \ldots$. Now we want to calculate the lowest Eigenvalue λ_1. After Kellogg, we have $\lambda_1 = \lim_{s\to\infty} I_{s-1}/I_s$ where $I_s = \int_0^1\int_0^1 K_{(x,y)}^{(s-1)}$ and more precise:

$$\left|\lambda_1 - \frac{I_{s-1}}{I_s}\right| < c\left(\frac{\lambda_1}{\lambda_2}\right)^{s-1}.$$

we have

$$I_s = \int_{E^s} \Pi^{(s)}(x_1, \ldots x_s)dx_1 \ldots dx_s.$$

If we apply again our quadrature formula, we get a result of the Type

$$\lambda_1 = \frac{\sum_{k=1}^{T} K(y_{k1}, y_{k2}) \ldots K(y_{k,s-2}, y_{k,s-1})}{\sum_{k=1}^{T} K(y_{k1}, y_{k2}) \ldots K(y_{k,s-1}, y_{k,s})} + R \tag{25}$$

and we get

$$\lambda_1 = \lim_{T\to\infty} \text{————————}$$

Our method can also be applied for linear equations

$$\varphi(i) - \frac{\lambda}{U}\sum_{k=1}^{l} K(i,k)\varphi(k) = f(i) \qquad 1 \leqq i \leqq U \tag{28}$$

because it is only a special case of an integral equation, with the kernel of bounded variation. But it is of interest to write explicitly the formulas for this case. The Neumann series is in this case

$$\varphi(i) = f(i) + \sum_{m=1}^{\infty} \lambda^m \frac{1}{U}\sum_{i=1}^{U} K^{(n)}(i,k)f(k) \tag{17'}$$

where

$$K^{(n)}(i,k) = \frac{1}{U^n}\sum_{i_2,\dots i_n=1}^{U} K(i,i_2)K(i_2,i_3)\dots K(i_n,k)$$

$$V(K) = \sum_{i,k=1}^{U} |K(i,k) - K(i-1,k) - K(i,k-1) + K(i-1,k-1)| \tag{18'}$$
$$+ \sum_{i=1}^{U} |K(i,1) - K(i-1,1)| + \sum_{k=1}^{U} |K(1,k) - K(1,k-1)|, \quad \sigma(K)$$
$$= \sup |K(i,k)|, \quad V(f) = \sum_{i=1}^{U} |f(i) - f(i-1)|.$$

The Neumann series is convergent for $|\lambda|M < 1$. If we write the system of linear equations in the usual form

$$\varphi(i) - \mu\sum_{k=1}^{U} K(i,k)\varphi(k) = f(i) \text{ with } |\mu|M < U.$$

If we set

$$\tilde{\varphi}(x) = \varphi(i), \text{ if } \frac{i-1}{U} \leqq x < \frac{i}{U} \qquad (i = 1,\dots U)$$

$$\tilde{K}(x,y) = K(i,k) \text{ if } \frac{i-1}{U} \leqq x < \frac{i}{U}, \frac{k-1}{U} \leqq y < \frac{k}{U} \quad (i,k = 1,\dots U)$$

$$\tilde{f}(x) = f(i) \text{ if } \frac{i-1}{U} \leqq x < \frac{i}{U} \qquad (i = 1,\dots U)$$

then we have

$$\int_0^1 \tilde{K}(x,y)\tilde{\varphi}(y)dy = \sum_{k=1}^{U}\int_{k-1/U}^{k/U} \tilde{K}(x,y)\tilde{\varphi}(y)dy = \frac{1}{U}\sum_{k=1}^{U} K(i,k)\varphi(k)$$

and the system of linear equations has the form

$$\tilde{\varphi}(x)-\lambda\int_0^1 \tilde{K}(x,y)\tilde{\varphi}(y)dy=\tilde{f}(x).$$

It is $\tilde{K}(x,y)=K([xU]+1,[yU]+1)$ and we have

$$\varphi(i)=f(i)+\frac{1}{T}\sum_{k=1}^{T}\sum_{m=1}^{S}\lambda^m K(i,[y_{k1}U]+1)$$
$$K([y_{k1}U]+1,[y_{k2}U]+1)\ldots f([y_{km}U]+1)+R$$

where

$$|R|<2A\varepsilon$$

for $T>T_0$ after (21).

Now let us suppose we want to calculate $\varphi(i)$ with an error smaller than δ, then we must take

$$T>T_0(\delta/2A)=\rho,$$

for instance $T=[\rho]+1$. Now we calculate the number of multiplications that are necessary to calculate a fixed $\varphi(i)$ with this error term. The number of multiplications is

$$\sum_{k=1}^{T}\sum_{m=1}^{S}(m+1)<2Ts^2=0\left(T\left(\frac{\log T}{\log\log T}\right)^2\right)$$

independent of U! This is not so surprising as it looks and well-known in the Monte Carlo-method. The Gausz elimination method needs $\sim U^3$ multiplications, the usual iterations method KU^2 multiplications where K is depending on ρ. The above method is only useful when U is very large, so that $Ts^2(T)<KU^2$.

The method can also be applied to matrix inversion: If the problem is to solve $\mathfrak{AH}=\mathfrak{E}$, where $\mathfrak{A}$ is a given quadratic matrix of s^2 elements then we can get this equation in the form

$$\mathfrak{H}=\mathfrak{A}^{-1}=\mathfrak{B}+(\mathfrak{E}+\mathfrak{F}+\mathfrak{F}^2+\ldots)\mathfrak{F}\mathfrak{B},$$

where $\mathfrak{B}$ is an estimate of $\mathfrak{A}^{-1}$ and $\mathfrak{F}=\mathfrak{E}-\mathfrak{B}\mathfrak{A}$ and it can be assumed that $||\mathfrak{F}||<1$. Then we can work as before.

In general we have for any function $F(i_1,\ldots,i_m)$ defined for all m-tuples of integers $1\leqq i_1\leqq U,\ldots,1\leqq i_m\leqq U$ and for any sequences $\omega=(y_k)$

$$1\leqq k\leqq T,\ y_k=(y_{k1},\ldots y_{km})$$

after I 6

$$\left|\frac{1}{T}\sum_{k=1}^{T} F([Uy_{k1}], [Uy_{k2}], \ldots [Uy_{km}]) - \frac{1}{U^m}\sum_{i_1,\ldots i_m=1}^{U} F(i_1, \ldots i_m)\right| \leqq D(\omega)V(F).$$

$V(F)$ is analogously defined as $V(K)$ for $m = 2$.

This follows from I(6), if we define for $(x_1, \ldots x_m)$ in E^m

$$f(x_1, \ldots x_m) = F([x_1, U], \ldots [x_m U]).$$

Final remarks: The methods developed above can be applied for calculating the eigen values of matrices and differential equations. I have applied them also for interpolation formulas in more than one variables.

NOTES

[1] Compare J. L. WALSH, Interpolation and Approximation (1935) Amer. Math. Soc. Coll. Publ. 20. With the theory of discrepancy this theorem can be improved.

[2] Compare E. HLAWKA, Monatshefte für Mathematik 66, 140–151 (1962) Satz 2. If f has continuous derivatives there are results of L. C. Hsu and others, compare the paper of Hsu, Numerische Mathematik 3 (1961) 69–73.

[3] Compare for instance Symposium on Monte Carlo Methods, editor H. A. Meyer, John Wiley & Sons (1954).

[4] Math. Tables Aids Comput. 10 (1956) 1–8.

[5] CHIFFRES, Revue Assoc. France. Calcul 1, (1958) 149–155.

[6] Proc. Nat. Acad. Sci. 86 (1960) 844–874.

[7] Numer. Math. 4 (1961) 68–84. There are cited the papers of Richtmyer and G. Peck.

[8] Dokladi CCCR 115 (1937) 1062–65.

[9] Compare the paper of Korobow Trudi MHAH, LX 196–210 and the literature cited there.

[10[Numer. Math. 2 (1960) S. 84.

[11] There is an interesting note of I. M. Sobol Dokladi Acad. 114 (1957) 7067–69 but without details. It is unknown to me, whether there exists a further paper on this line of thoughts.

[12] Some numerical work has been done with this sequence for $s = 10$ at the Math. Inst. of the technical university of Stuttgart, by P. Ross & K. Arnold (Österr. Akad. Wiss.; in print).

[13] It is unknown if this is true for the Halton-sequence.
If f is analytic, then the method of Ph-Davis, Numer. Approx. Proc. Sympos. Math. Res-Center Madison April 21–23 (1958) 45–49 can be generalized to more dimension.

[14] E. STIEFEL und H. RUTISHAUSER, C. R. Acad. Sci. Paris 252 (1961) 1899–1900 ZAMM 41 (1961); E. Stiefel Einführung in die numerische Mathematik (1961), B. G. Teubner Stuttgart.

[15] There are some results of P. C. Hammer, O. J. Marlowe, A. H. Stroud, compare the report of J. Albrecht a. L. Collatz, ZAMM 38 (1958) 1–15.

[16] E. Hlawka, Sitzungsberichte Österr. Akad. Wissenschaften, Bd. 171 (1962), E. Hlawka & K. Kreiter, ebenda (1963) (in print).
N. M. Korobow has applied also his results for inhomogeneous integral equations, see [9].

[17] For numerical work it will be useful to combine it with other methods for instance with over relaxation methods.