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The fundamental theorem of van der Corput on uniform distribution and its generalizations [1*]

by

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The fundamental theorem of van der Corput on uniform distribution says the following: *Let* $\{x_n\}$ *be a sequence of real numbers. If for every* $h = 1, 2, 3, \ldots$ *the sequence* $\{x_{n+h}-x_n\}$ *is uniformly distributed modulo* 1, *then the same is true for the original sequence* $\{x_n\}$. For the proof of this theorem van der Corput used his so-called fundamental inequality. There have been many generalizations and improvements of this theorem both qualitative and quantitative. We shall be concerned exclusively with the qualitative aspect. M. Tsuji [9] replaced the arithmetic mean in the definition of uniform distribution by certain weighted means and proved an analogue of van der Corput's theorem for this case. E. Hlawka [5], [6] generalized van der Corput's theorem to sequences in compact groups and replaced the requirement of uniform distribution of the sequences $\{x_{n+h}-x_n\}$ by a weaker condition. All these generalizations used modifications of the fundamental inequality. A different method was used by J. Bass [1] to prove van der Corput's theorem. He used the theorem of Bochner-Herglotz on positive definite functions. This idea turned out to be very fruitful and we shall show in the sequel how it can be applied to prove a rather strong generalization of the fundamental theorem which includes all previously known special cases.

Let $A = (a_{nk})$ be a positive strongly regular matrix, i.e. $a_{nk} \geqq 0$, $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$, $\lim_{n\to\infty} a_{nk} = 0$, $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k+1}-a_{nk}| = 0$. Special cases are the arithmetic mean and the weighted means (M, λ_n) with $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n \geqq \ldots > 0$ and $\sum \lambda_n = \infty$. Here (M, λ_n) denotes the matrix (a_{nk}) defined by

[1]) The content of this paper was published in a slightly different form in [2]. The main results were also obtained independently by P. Bertrandias (cf. his paper in this volume). Analogous questions have also been considered by J. H. B. Kemperman (cf. his article in this volume).

*) Nijenrode lecture.

$$a_{nk} = \frac{\lambda_k}{\lambda_1+\lambda_2+\ldots+\lambda_n} \quad \text{for} \quad k \leqq n$$
$$= 0 \qquad\qquad \text{for} \quad k > n.$$

We say that a sequence $\{x_k\}$ has the A-distribution measure μ if for every continuous $f(x)$ with period 1

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} f(x_k) = \int_0^1 f(x) d\mu(x).$$

In the special case of weighted means this means

$$\lim_{n\to\infty} \frac{\lambda_1 f(x_1)+\ldots+\lambda_n f(x_n)}{\lambda_1+\lambda_2+\ldots+\lambda_n} = \int_0^1 f(x) d\mu(x).$$

Let now $f(k)$ be an arbitrary bounded function on the natural numbers. We say that m_f is an A-mean value of the sequence $f(k)$, if m_f is a cluster point of the sequence $\sum_k a_{nk} f(k)$, or to put it differently, if there exists a sequence n' such that $\lim_{n'} \sum_k a_{n'k} f(k) = m_f$. If even $\lim_{n\to\infty} \sum_k a_{nk} f(k)$ exists, we denote this limit by $M_k f(k)$. We now consider the sequence $\sum_k a_{nk}\, f(k+h)\overline{f(k)}$, denoting by $\overline{f(k)}$ the conjugate complex of $f(k)$. If for a monotone increasing sequence n' and every $h = 1, 2, 3, \ldots$ the limit

$$\gamma_f(h) = \lim_{n'} \sum_k a_{n'k}\, f(k+h)\, \overline{f(k)},$$

exists, then we say that $\gamma_f(h)$ is an A-correlation function of the sequence $f(k)$. Obviously always at least one A-correlation function exists. It is possible to define $\gamma_f(h)$ even for negative integers h. Then we have $\gamma_f(0) \geqq 0$ and $\gamma_f(-h) = \overline{\gamma_f(h)}$.

It follows from the properties of strongly regular matrices A that every A-correlation function is positive definite. This means that for every finite set of complex numbers λ_p

$$\sum_{p=1}^{n} \sum_{q=1}^{n} \lambda_p \bar{\lambda}_q \gamma_f(p-q) \geqq 0.$$

Therefore by the theorem of Bochner-Herglotz, for every A-correlation function $\gamma_f(h)$ there exists a positive normed Radon measure $\sigma_f = \sigma_{\gamma_f}$ on $[0, 1)$ with

$$\gamma_f(h) = \int_0^1 e^{2\pi i h x}\, d\sigma_f(x).$$

From this representation it follows easily that $M_k\, \gamma_f(k) = \sigma_f(0) \geqq 0$. We sketch the proof:

$$\underset{k}{M}\,\gamma_f(k) = \lim_n \sum_k a_{nk}\gamma_f(k) = \lim_n \sum_k a_{nk}\int_0^1 e^{2\pi ihx}\,d\sigma_f(x)$$
$$= \lim_n \int_0^1 \big(\sum_k a_{nk}e^{2\pi ikx}\big)\,d\sigma_f(x) = \sigma_f(0) \geqq 0.$$

For we have

$$\lim_n \sum_k a_{nk}\,e^{2\pi ikx} = \begin{matrix} 1 & (x \equiv 0 \bmod 1) \\ 0 & (x \not\equiv 0 \bmod 1). \end{matrix}$$

This follows from the strong regularity of the matrix A. The interchange of limit and integration may be justified by Lebesgue's theorem on bounded convergence. This proof also shows that the mean value of $\gamma(h)$ is independent of the special summation method considered, provided it is strongly regular.

We now denote by Γ_f the set of all A-correlation functions $\gamma_f(h)$ of f. Then for every A mean value m_f we have

$$|m_f|^2 \leqq \limsup_{\gamma_f\in\Gamma_f} \underset{k}{M}\,\gamma_f(k).$$

For let $\lim'_n \sum_k a_{n'k}\,f(k) = m_f$. If we set $f(k) = f'(k)+m_f$, then we obviously have $\lim_{n'} \sum_k a_{n'k}\,f'(k) = 0$. From the equation

$$\overline{f(k)}\,f(k+h) = |m_f|^2+m_f\,\overline{f'(k)}+\overline{m}_f\,f'(k+h)+\overline{f'(k)}\,f'(k+h)$$

we conclude therefore

$$\gamma_f(h) = |m_f|^2+\gamma_{f'}(h)$$

for every A'-correlation function, where A' denotes the matrix $A' = (a_{n'k})$. It follows that $M\gamma_f(h) = |m_f|^2+M\gamma_{f'}(h) \geqq |m_f|^2$, considering that $M\gamma_{f'}(h) \geqq 0$. From this equation our theorem follows.

An easy consequence is the corollary:

If for each $\gamma_f(h) \in \Gamma_f$ the mean value $M_h\gamma_f(h) = 0$, then $M_kf(k) = \lim_{n\to\infty} \sum_k a_{nk}\,f(k) = 0$. In other words: let σ_f be the Radon measure associated with γ_f, then $\sigma_f(0) = 0$ for all such σ_f is a sufficient condition to ensure that $M_k\,f(k) = 0$.

Let now $l \geqq 2$ and $j (0 \leqq j \leqq l-1)$ be integers. We denote by A_{lj} the matrix $A_{lj} = (b_{nk})$ with $b_{nk} = l\,a_{n,lk+j}$. Obviously the matrix A_{lj} is positive and strongly regular. If we denote by $M_{klj}f(k)$ the A_{lj}-mean value of the function $f(k)$, then it is easy to see that $M_h\gamma(lh) = 0$ implies $M_{lj}\gamma(lh) = 0$ and also $\underset{k}{M}_{lj}\gamma^{lj}(lh) = 0$ for every A_{lj}-correlation function γ^{lj} of the sequence $f(lk+j)$.

Therefore we get the following theorem:

If for every A-correlation function γ_f $M_h \gamma_f(lh) = 0$, i,e. if for the associated measure σ_f we have $\sigma_f(x) = 0$ for $x = 0, 1/l, \ldots, l-1/l$, then $M_{kj} f(lk+j) = 0$.

We now apply these theorems to obtain a generalization of the fundamental theorem of van der Corput.

Let $f_l(k) = e^{2\pi i l x_k}$ $(l = 0, 1, 2, \ldots)$ where $\{x_k\}$ is a sequence of real numbers. The sequence $\{x_k\}$ is A-uniformly distributed if and only if $M_k f_l(k) = M_k e^{2\pi i l x_k} = 0$ for $l = 1, 2, 3, \ldots$.

We now suppose that $M_k \overline{f_l(k)} f_l(k+h)$ exists for every $l, h = 1, 2, 3, \ldots$ and we denote this limit by $\gamma_{lh} = \gamma_l(h)$. Let h be fixed. Then we have for each $l = 1, 2, 3, \ldots$

$$\lim_n \sum_k a_{nk} e^{2\pi i l(x_{k+h}-x_k)} = \underset{k}{M} f_l\overline{(k)f_l}(k+h) = \gamma_{lh}.$$

This means by definition that for each $h = 1, 2, 3, \ldots$ the sequence $\{x_{k+h}-x_k\}$ has a certain distribution measure which we denote by τ_h. Therefore we get $\gamma_{lh} = \int_0^1 e^{2\pi i l x} d\tau_h(x)$.

On the other hand we have shown that for each fixed l there is a Radon measure σ_l such that $\gamma_{lh} = \int_0^1 e^{2\pi i h x} d\sigma_l(x)$. These measures τ_h and σ_l are related to each other by the equations

$$\gamma_{lh} = \int_0^1 e^{2\pi i l x} d\tau_h(x) = \int_0^1 e^{2\pi i h x} d\sigma_l(x).$$

From our results it follows easily that, if τ_h denotes the distribution measure of the sequence $\{x_{k+h}-x_k\}$ $(h = 1, 2, 3, \ldots)$ and if the associated measures σ_l satisfy $\sigma_l(0) = 0$ $(l = 1, 2, 3, \ldots)$ then the sequence $\{x_k\}$ is A-uniformly distributed mod 1.

An easy consequence is the following theorem:

Let τ_h be the distribution measure of the sequence $\{x_{k+h}-x_k\}$. If

$$(*) \qquad \lim_n \frac{1}{n} \sum_{k=1}^{n} \left| \int_0^1 e^{2\pi i l x} d\tau_k(x) \right|^2 = 0 \qquad (l = 1, 2, 3, \ldots)$$

then the sequence $\{x_k\}$ is A-uniformly distributed mod 1.

Proof: The condition (*) is equivalent to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \int_0^1 e^{2\pi i k x} d\sigma_l(x) \right|^2 = 0$$

and this means, by a well-known theorem of Schoenberg and Wiener (cf. [4], § 7), that each point has σ_l-measure zero. But we saw before that $\sigma_l(0) = 0$ implies the A-uniform distribution of the sequence $\{x_k\}$.

This condition is obviously fulfilled if for every $l = 1, 2, 3, \ldots$

$$\lim_{k\to\infty} \int_0^1 e^{2\pi i l x} d\tau_k(x) = 0.$$

This condition has been given in the special case of the arithmetic mean by E. Hlawka [6].

Another easy consequence is the following theorem: *Under the conditions of the previous theorem the sequence* $\{x_{lk+j}\}$ *is* A_{lj}*-uniformly distributed.* ($l \geqq 1$ *and* j *arbitrary integers*). In the special case where A is the matrix of the arithmetic mean we have $A_{lj} \equiv A$ and therefore not only the sequence $\{x_k\}$ but also the sequence $\{x_{lk+j}\}$ is uniformly distributed.

Special cases of this theorem were given by Korobov and Postnikov [8] and E. Hlawka [5], [6], [7].

The theorem of van der Corput itself is of course implied by our results. For in this case τ_h equals Lebesgue measure and therefore σ_l, too, equals Lebesgue measure.

Another special case is Tsuji's theorem mentioned in the beginning. The conditions imposed by Tsuji turn out to be unnecessarily strong. It suffices to take weighted means (M, λ_n) with $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n \geqq \ldots > 0$ and $\sum \lambda_n = \infty$. *Then the* (M, λ_n)*-uniform distribution of the sequences* $\{x_{n+h}-x_n\}$ *implies the* (M, λ_n)*-uniform distribution of the original sequences* $\{x_n\}$.

It is interesting that even the sequences $\{k\vartheta\}$ with ϑ irrational can be handled with our results. For $x_{k+h}-x_k = h\vartheta$, a constant. It has the distribution measure $\varepsilon_{h\vartheta}$, which is concentrated in the point $h\vartheta-[h\vartheta]$. Now it is easy to see that $\tau_l = \sigma_l = \varepsilon_{l\vartheta}$. Now $\varepsilon_{l\vartheta}(0) = 0$ because ϑ is irrational. Therefore our theorem implies that the sequence $\{k\vartheta\}$ is A-uniformly distributed mod 1.

There are generalizations of our theorem to compact groups, but we shall not deal with this aspect here.

Another useful corollary of our theorem is the following one: [2]) *If for each* $l = 1, 2, 3, \ldots$

$$(**) \qquad \lim_n \frac{1}{n} \sum_{k=1}^{n} \int_0^1 e^{2\pi i l x} d\tau_k(x) = 0,$$

then the sequence $\{x_k\}$ *is* A*-uniformly distributed.*

All these theorems can be generalized to sequences of normed measures. For this topic see [3].

[2]) (**) is of course a weaker condition than (*). But if the measures σ_l are known, (*) may be more convenient.

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