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GILBERT HELMBERG

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A class of criteria concerning uniform distribution in compact groups *

by

Gilbert Helmberg

1.

Let X be a compact topological group satisfying the second axiom of countability. Let $\mathfrak{C}$ be the Banach space (with respect to uniform norm) of continuous complex-valued functions on X and let $\mathfrak{C}^*$ be the conjugate space of $\mathfrak{C}$. We identify bounded linear functionals on $\mathfrak{C}$ and regular Borel measures of finite total variation on X as elements of $\mathfrak{C}^*$ by means of the equation $\nu(f) = \int_X f(x) d\nu(x)$ for all $f \in \mathfrak{C}$. Let $\mathfrak{V} \in \mathfrak{C}^*$ be the set of non-negative normed measures and let μ be Haar measure on X.

A sequence (x_n) in X is called *summable*, if $\lim_{N\to\infty} 1/N \sum_{n=1}^{N} f(x_n)$ exists for all $f \in \mathfrak{C}$. Given a summable sequence (x_n), this limit defines a bounded linear functional $\nu \in \mathfrak{V}$. Referring to this particular functional ν we shall also call the sequence (x_n) *ν-summable*. A *μ-summable* sequence (x_n) will be called *uniformly distributed* (in X).

As usual, we define convolution of two measures $\nu_1, \nu_2 \in \mathfrak{C}^*$ by $\nu_1\nu_2(f) = \int_X \int_X f(xy) d\nu_1(x) d\nu_2(y)$ for all $f \in \mathfrak{C}$. It is well known that with convolution as multiplication $\mathfrak{V}$ becomes a compact semigroup in the weak topology. Haar measure μ may be characterized as the (unique) zero element of this semigroup, i.e., a measure $\lambda \in \mathfrak{V}$ coincides with μ if and only if $\lambda\nu = \lambda$ for all $\nu \in \mathfrak{V}$ [6].

The intention of this note is, in the first part, to characterize in a similar way among all sequences in X those that are uniformly distributed and, in the second part, to extend these results to functions of a real parameter s with values in X. The proofs of the theorems will only be sketched as the results of the first part are essentially contained in a paper already in print [3] and as the results of the second part are proved in a similar way.

In order to achieve this goal, we want to associate with every ordered pair of sequences (x_n) and (y_n) in X a new sequence (z_n)

* Nijenrode lecture.

in X such that, if (x_n) and (y_n) are ν_1- and ν_2-summable respectively, then (z_n) is $\nu_1\nu_2$-summable. If the sequences (x_n) and (y_n) are (statistically) independent, i.e. if the sequence (x_n, y_n) is summable with respect to product measure $\nu_1 \times \nu_2$ in $X \times X$, then ordinary multiplication $z_n = x_n y_n$ would meet these requirements. Since, however, we do not want to restrict our attention to pairs of independent summable sequences, we define the sequence, $(z_n) = (x_n) \times (y_n)$ by

$$z_1 = x_1 y_1$$
$$z_{(k-1)^2+2i-1} = x_k y_i \text{ for } 1 \leqq i \leqq k \text{ and } k > 1$$
$$z_{(k-1)^2+2i} = x_i y_k \text{ for } 1 \leqq i \leqq k-1 \text{ and } k > 1.$$

An ordering of this type has already been used in connection with uniform distribution in [5] and [2] (cf. also B. Volkmann "*An application of uniform distribution to additive number theory*") [1].

The following theorem states the desired relation between composition of sequences and convolution of measures:

THEOREM 1: *Let (x_n) and (y_n) be ν_1- and ν_2-summable sequences in X respectively. Then $(z_n) = (x_n) \times (y_n)$ is a $\nu_1\nu_2$-summable sequence.*

PROOF: Let $\{D^{(\kappa)} : \kappa \geqq 0\}$ be a complete system of inequivalent continuous irreducible unitary representations of X. By $D^{(0)}$ we denote the trivial representation of X. By Weyl's criterion we have only to show

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} D^{(\kappa)}(z_n) = \nu_1(D^{(\kappa)})\nu_2(D^{(\kappa)}) \text{ for all } \kappa \neq 0.$$

This is done by straightforward computation, using the well known matrix norm $||A|| = (\sum_{i,k} |a_{ik}|^2)^{\frac{1}{2}}$ for $A = (a_{ik})$ (cf. [4]).

In particular, if $\nu_1\nu_2 = \mu$, then (z_n) is uniformly distributed. This is the case, for instance, if X is the direct product of closed subgroups X_1 and X_2, and if ν_i is obtained from Haar measure μ_i on X_i by means of the equation $\nu_i(E) = \mu_i(E \cap X_i)$ for every Borel set $E \subset X$ $(i = 1, 2)$. A slightly less trivial example is provided by the dihedral group of the circle group X_1 (or, more general, of any compact abelian group). Let $x \to x'$ be a homeomorphism of X_1 onto a set X_1' and let multiplication in $X = X_1 \cup X_1'$ be defined by the additional relations

$$x'y = (xy)', \; xy' = (x^{-1}y)', \; x'y' = x^{-1}y$$

[1] This is the title of a Nijenrode lecture not submitted for this Volume.

(in particular we have $x'^2 = e =$ identity for all $x' \in X_1'$). Then X is a non-discrete non-commutative compact group with two connected components X_1 and X_1'. Let $X_2 = \{e, e'\}$. For any function $f \in \mathfrak{C}$ we have $\mu(f) = \frac{1}{2}(\int_{X_1} f(x)d\mu_1(x) + \int_{X_1} f(x')d\mu_1(x)) = \nu_1\nu_2(f)$ where ν_1 and ν_2 are obtained from Haar measure on X_1 and X_2 as indicated above. If (x_n) is a uniformly distributed sequence in X_1 (e.g. the multiples of an irrational number mod 1) and if $y_n = e'^n$ for all $n \geqq 1$, then (x_n) and (y_n) are ν_1- and ν_2-summable respectively in X and therefore $(z_n) = (x_n)\times(y_n)$ is uniformly distributed in X. In this case, composition of the two sequences essentially amounts to alternatingly taking elements of two sequences, uniformly distributed in the obvious sense in X_1 and X_1'.

Reformulating the characterization of Haar measure as zero element in the compact semigroup $\mathfrak{W}$ we obtain the following proposition: a summable sequence (x_n) in X is uniformly distributed if for every summable sequence (y_n) the sequence $(x_n)\times(y_n)$ is summable with respect to the same measure as (x_n). Of course, this statement does not contain any new information. Also, attention is again restricted to summable sequences in this result. The next theorem is both sharper and more general:

Theorem 2: *The following statements are equivalent:*

a) The sequence (x_n) is uniformly distributed.

b) The sequence $(x_n)\times(y_n)$ is summable for every sequence (y_n).

c) The sequence $(x_n)\times(y_n)$ is uniformly distributed for every sequence (y_n).

d) The sequence $(x_n)\times(x_n^{-1})$ is uniformly distributed.

e) The subsequence of $(x_n)\times(x_n^{-1})$ consisting of all products $x_i x_j^{-1}$ $(i > j)$ is uniformly distributed.

Proof: $a) \Rightarrow c)$ Direct computation as in the proof of theorem 1.

$c) \Rightarrow b)$ Trivial.

$b) \Rightarrow a)$ If (x_n) were not uniformly distributed, then either (x_n) is not summable at all or (x_n) is ν-summable and $\nu \neq \mu$. In the first case choose $y_n = e$ for all $n \geqq 1$ and let $(z_n) = (x_n)\times(e)$. It is easy to see that, for $N \to \infty$, $1/N \sum_{n=1}^{N} D^{(\kappa)}(z_n)$ diverges for at least one index κ. In the second case, combining the sequence (e) as used above and a uniformly distributed sequence (y_n), one may construct a sequence (y_n') such that, for $(z_n) = (x_n)\times(y_n')$ and as $N \to \infty$, $1/N \sum_{n=1}^{N} D^{(\kappa)}(z_n)$ oscillates between 0 and $\nu(D^{(\kappa)})$ for every $\kappa \neq 0$. Thus, (z_n) cannot be summable.

$a) \Rightarrow e)$ Direct computation, using the fact that the sequences

(xx_n^{-1}) $(x \in X)$ are equi-uniformly distributed.

$e) \Rightarrow d)$ If the subsequence in question is uniformly distributed, then so is the subsequence formed by the products $x_j x_i^{-1} = (x_i x_j^{-1})^{-1}$ $(i > j)$. Combining these two sequences and inserting a sequence of asymptotic density 0 we obtain the sequence $(x_n) \times (x_n^{-1})$.

$d) \Rightarrow a)$ If $A_N^{(\kappa)} = 1/N \sum_{n=1}^{N} D^{(\kappa)}(x_n)$ and if $A_N^{(\kappa)*}$ is the corresponding adjoint matrix, then the hypothesis $d)$ implies $\lim_{N\to\infty} A_N^{(\kappa)} A_N^{(\kappa)*} = 0$ and therefore $\lim_{N\to\infty} \|A_N^{(\kappa)}\| = 0$ for all $\kappa \neq 0$.

We add a few remarks:

ad b), c): An inspection of the proofs shows that the quantifier "for all sequences (y_n)" may be replaced by "for all not summable sequences (y_n)".

ad d): An estimate due in its sharpest formulation to Cassels [1] shows that, in the case of reals mod 1, if the discrepancy of the n^2 differences $x_i - x_j$ $(1 \leqq i, j \leqq n)$ tends to zero as $n \to \infty$, then so does the discrepancy of $x_1, \ldots, x_n$, i.e. the sequence (x_n) is uniformly distributed. In the case of a compact group X in general, there is no handy concept of discrepancy. Still, by $d)$, if the sequence $(x_n) \times (x_n^{-1})$ formed by the "differences" $x_i x_j^{-1}$ is uniformly distributed, then so is the sequence (x_n).

ad e): By van der Corput's fundamental theorem as transferred by Hlawka [4] to the compact group case, if every row of the infinite matrix

$$\begin{matrix} x_2 x_1^{-1} & x_3 x_2^{-1} & x_4 x_3^{-1} & \cdots \\ x_3 x_1^{-1} & x_4 x_2^{-1} & \cdots & \\ x_4 x_1^{-1} & \cdots & & \end{matrix}$$

is uniformly distributed, then so is the first (and therefore every) column and the sequence (x_n). By $e)$, uniform distribution of (x_n), i.e. of every column, is equivalent to uniform distribution of the sequence obtained by joining the (finite) secondary diagonals.

The second axiom of countability accounts for the countability of the complete system of inequivalent irreducible continuous representations $D^{(\kappa)}$ and for the existence of at least one uniformly distributed sequence. This fact, however, has only been used in the proof of the implication $b) \Rightarrow a)$ and is not essential even there. As a consequence, all results remain valid if this axiom is omitted. The same is true if X is taken to be any group (possibly without any topological structure) and if $\mathfrak{C}$ is replaced by any full module of almost periodic functions on X. Convolution of measures, then, has to be replaced by convolution of bounded linear functionals on this module and Haar measure by mean value. An application of this fact has been given by Hartman (cf. S. Hartman "*Remarks on equidistribution on non-compact groups*". This Vol. p. 66).

2.

It seems worth while to ask whether similar criteria for uniform distribution may be obtained for functions of a real parameter s as studied extensively (in the case of reals mod 1) by Kuipers, Meulenbeld, Hlawka, and others. We denote by R and by R' the real line and the non-negative real half-line respectively. Let $x(s)$ be a Borel-measurable function on R' with values in a compact topological group X (no countability axiom is assumed). The function $x(s)$ is called (ν-)*summable* if $\lim_{S\to\infty} 1/S \int_0^S f(x(s))ds$ exists (and equals $\nu(f)$) for all $f \in \mathfrak{C}$. A μ-summable function $x(s)$ will again be called *uniformly distributed.*

If we want to apply without substantial changes the method used in part 1 it seems appropriate to consider not just single functions $x(s)$ but pairs $\{x(s), (s_n)\}$ where

1) $x(s)$ is a measurable function on R' with values in X;

2) (s_n) is an increasing sequence in R' such that the ratio s_n/n has a positive limit for $n \to \infty$;

3) $$\lim_{N\to\infty} \left| \frac{1}{s_N} \int_0^{s_N} f(x(s))ds - \frac{1}{N} \sum_{n=1}^{N} f(x(s_n)) \right| = 0 \text{ for all } f \in \mathfrak{C}.$$

Any pair $\{x(s), (s_n)\}$ satisfying these three conditions will be called admissible. By 3), if the function $x(s)$ is ν-summable, then so is the sequence $x(s_n)$. We shall call an admissible pair of this type (ν-)-summable.

Let $\{x(s), (s_n)\}$ and $\{y(t), (t_n)\}$ be admissible pairs. Without loss of generality we may assume $s_1 = t_1 = 0$. Let us consider the graph consisting of countably many copies of R' used as non-negative s- and t-axes and as abscissas and ordinates with endpoints s_n and t_n on the s- and t-axis respectively (cf. fig. 1). If we omit the open intervals $]s_{2n}, s_{2n+1}[$ and $]t_{2n-1}, t_{2n}[$ on the s- and t-axis respectively ($n \geqq 1$), then the remaining graph may be traced as shown in fig. 1, starting in (0, 0) and using length u of the path $\mathfrak{Z}$ as parameter. Let u_n be the values of this parameter corresponding to consecutive corners of $\mathfrak{Z}$. If the point of $\mathfrak{Z}$ corre-

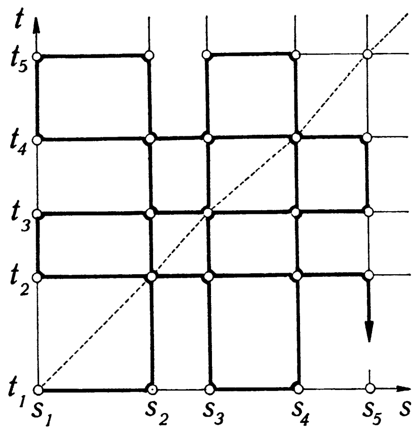

Figure 1

sponding to the parameter value u has abscissa s and ordinate t we define $z(u) = x(s)y(t)$. Thus,

$$\begin{aligned} z(u) &= x(u)y(0) && \text{for } 0 = u_1 \leqq u \leqq s_2 = u_2 \\ &= x(s_2)y(u) && \text{for } s_2 = u_2 \leqq u \leqq s_2+t_2 = u_3 \\ &= x(2s_2+t_2-u)y(t_2) && \text{for } s_2+t_2 = u_3 \leqq u \leqq 2s_2+t_2 = u_4 \\ &\ldots \end{aligned}$$

It is easy to verify that $\{z(u), (u_n)\}$ is again an admissible pair. We write $\{z(u), (u_n)\} = \{x(s), (s_n)\} \times \{y(t), (t_n)\}$. Furthermore, we denote by $\{x(s), (s_n)\} \overline{\times} \{y(t), (t_n)\}$ the admissible pair $\{\bar{z}(u), (\bar{u}_n)\}$ obtained in the following way: we omit the part of $\mathfrak{Z}$ lying above the "diagonal" joining the points (s_n, t_n) $(n \geqq 1)$ (dotted line in fig. 1). The remaining pieces may be joined in the points (s_{2n}, t_{2n}) so as to form a single path $\overline{\mathfrak{Z}}$. Let us denote length of this path by u and let $\bar{u}_n$ be the values of the parameter u corresponding to the corners of $\overline{\mathfrak{Z}}$. The element $\bar{z}(u)$ is defined on $\overline{\mathfrak{Z}}$ exactly as $z(u)$ has been defined on $\mathfrak{Z}$. If $x(s)$ and $y(t)$ are continuous functions of the parameters s and t respectively, then $z(u)$ and $\bar{z}(u)$ are continuous functions of the parameter u.

The proofs of the theorems given below follow the same lines as the proofs of theorems 1 and 2.

THEOREM 1′: *If $\{x(s), (s_n)\}$ and $\{y(t), (t_n)\}$ are ν_1- and ν_2-summable pairs respectively, then $\{z(u), (u_n)\} = \{x(s), (s_n)\} \times \{y(t), (t_n)\}$ is $\nu_1\nu_2$-summable.*

THEOREM 2′: *Let $\{x(s), (s_n)\}$ be an admissible pair. The following statements are equivalent:*

a′) The pair $\{x(s), (s_n)\}$ *is uniformly distributed.*

b′) The pair $\{x(s), (s_n)\} \times \{y(t), (t_n)\}$ *is summable for every admissible pair* $\{y(t), (t_n)\}$.

c′) The pair $\{x(s), (s_n)\} \times \{y(t), (t_n)\}$ *is uniformly distributed for every admissible pair* $\{y(t), (t_n)\}$.

d′) The pair $\{x(s), (s_n)\} \times \{x^{-1}(s), (s_n)\}$ *is uniformly distributed.*

e′) The pair $\{x(s), (s_n)\} \overline{\times} \{x^{-1}(s), (s_n)\}$ *is uniformly distributed.*

So far nothing has been said about the existence of admissible pairs. Let (x_n) $(n \geqq 0)$ be a summable sequence in X and let $[s]$ denote the largest integer in s. Then obviously $\{x_{[s]}, (n)\}$ is a summable pair (this fact has been used in the proof of $b') \Rightarrow a')$). We close by exhibiting a class of continuous summable functions $x(s)$ on R' with values in X for which admissible pairs may be constructed. The size of this class depends on the size of the class of continuous functions on R into X. Let $g(s)$ be a continuous real-valued almost periodic function on R in the sense of Bohr. Let $h(t)$ be a continuous function on R with values in X. Then $x(s) = h \circ g(s)$ is a continuous function on R into X. For every $f \in \mathfrak{C}$, the function $f(x(s))$ is continuous and almost periodic and therefore has a mean value $M(f) = \lim_{S\to\infty} 1/S \int_0^S f(x(s))\,ds$. As a consequence, $x(s)$ is summable with respect to a measure ν on X. If $g(s) \sim \sum_{m=1}^{\infty} \gamma_m \exp(2\pi i \lambda_m s)$ is the Fourier expansion of $g(s)$, let α be any real number that is rationally independent of the set of Fourier exponents λ_m $(m \geqq 1)$ and let $\mathfrak{A}$ be the full module of almost periodic functions on R generated by $g(s)$. The sequence (n/α) is dense and therefore uniformly distributed in the almost periodic compactification of R induced by $\mathfrak{A}$. Also, we have $f(x(s)) = (f \circ h) \circ g(s) \in \mathfrak{A}$ for every $f \in \mathfrak{C}$. Combining these facts, we get

$$\nu(f) = \lim_{S\to\infty} \frac{1}{S} \int_0^S f(x(s))\,ds = M(f \circ h \circ g) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\left(x\left(\frac{n}{\alpha}\right)\right)$$

for all $f \in \mathfrak{C}$. Therefore $\{x(s), (n/\alpha)\}$ is a ν-summable admissible pair.

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