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BODO VOLKMANN

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On uniform distribution and the density of sets of lattice points *

by

Bodo Volkmann

In 1953 M. Kneser [1] proved a theorem on the asymptotic density of the sum set of two sets of non-negative integers which states that, in general, the analogue of Mann's inequality holds, and describes the sum set in the exceptional cases where this inequality is violated. So far, no generalization of Kneser's theorem to lattice point sets appears to be known, but it has been proved by the speaker [3] that the inequality under consideration is true, at least, for a certain class of pairs of lattice point sets which are defined by means of uniformly distributed sequences of real numbers.

The details are as follows: Let Λ_k be the set of all lattice points $\mathfrak{a} = (a_1, a_2, \ldots, a_k)$, $a_i \geqq 0$, in the euclidean space R_k, and define for any $\mathfrak{a}$,

$$||\mathfrak{a}|| = \max(a_1, \ldots, a_k).$$

For any set $A \subseteqq \Lambda_k$ and any $x \geqq 0$ let $A(x) = \sum_{\mathfrak{a} \in A, ||\mathfrak{a}|| \leqq x} 1$ and $D(A) = \lim_{x \to \infty} A(x)/x$ if this limit exists. Furthermore we consider fixed positive irrational numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$, and we map each lattice point $\mathfrak{a} = (a_1, \ldots, a_k)$ onto the point $\mathfrak{p}(\mathfrak{a})$ in the unit cube C_k whose coordinates are the fractional parts of $\lambda_\kappa a_\kappa$ $(\kappa = 1, \ldots, k)$. With any set $M \subseteqq C_k$ we associate the set A_M of those lattice point $\mathfrak{a} \in \Lambda_k$ for which $\mathfrak{p}(\mathfrak{a}) \in M$. If we define the sum $A+B$ of two sets in Λ_k by ordinary vector addition the following theorem is true:

THEOREM: For any two open sets $M_1, M_2 \subseteqq C_k$, the densities $D(A_{M_1})$, $D(A_{M_2})$, $D(A_{M_1}+A_{M_2})$ exist and satisfy the inequality

$$D(A_{M_1}+A_{M_2}) \geqq \min\big(1, D(A_{M_1})+D(A_{M_2})\big).$$

The proof consists in showing that, if the elements of a given set $A \subseteqq \Lambda_k$ are ordered in any way compatible with the partial order-

* Nijenrode lecture.

ing induced by $||\mathfrak{a}|| \leqq ||\mathfrak{b}||$, then the sequence $\mathfrak{p}(\mathfrak{a}_1), \mathfrak{p}(\mathfrak{a}_2), \ldots$ is uniformly distributed in C_k. This implies that, for any open set $M \subseteqq C_k$, the density $D(A_M)$ equals the Jordan content of M, as is easily demonstrated. Furthermore, it can be shown, denoting addition mod. 1 by $\oplus$, that always $A_{M_1}+A_{M_2} \doteq A_{M_1 \oplus M_2}$. Thus, the problem reduces to the corresponding question for sums of open sets in C_k, with content taking the place of density. The theorem then follows from a result by A. M. Macbeath [2].

REFERENCES

M. Kneser

[1] Abschätzung der asymptotischen Dichte von Summenmengen, Math. Z. vol. 58 (1953), pp. 459—484.

A. M. Macbeath

[2] On measures of sum sets. II. The sum theorem for the torus, Proc. Cambr. Phil. Soc. vol. 49 (1953), pp. 40—43.

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[3] On uniform distribution and the density of sum sets, Proc. Amer. Math. Soc. vol. 8 (1957), pp. 130—136.

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Universität Mainz.