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by

I. J. Schoenberg

Introduction

This is a brief report on a paper with the same title written in collaboration with Professor Ch. Pisot and concerning some modifications of Cauchy's equation $f(x+y) = f(x)+f(y)$ (See [4]). The background of the problem is a result of Erdös on additive arithmetic functions. An arithmetic function $F(n)$ $(n = 1, 2, \ldots)$ is said to be *additive* provided that $F(mn) = F(m)+F(n)$ whenever $(m, n) = 1$. In [2] Erdös found that if the additive function $F(n)$ is non-decreasing, i.e. $F(n) \leqq F(n+1)$ for all n, then it must be of the form $F(n) = C \log n$. This result was rediscovered by Moser and Lambek [3] and recently further proofs were given by Schoenberg [5] and Besicovitch [1].

Erdös remarkable characterization of the function $\log n$ raises the following question: Let $p_1, p_2, \ldots, p_k$ be a given set of k distinct prime numbers $(k \geqq 2)$. Let $F(n)$ be defined on the set A of integers n which allow no prime divisors except those among $p_1, \ldots, p_k$ and let $F(n)$ be additive, i.e.

$$(1) \qquad F(p_1^{u_1}p_2^{u_2}\ldots p_k^{u_k}) = F(p_1^{u_1})+F(p_2^{u_2})+ \ldots +F(p_k^{u_k}).$$

If we assume $F(n)$ to be non-decreasing over the set A, is it still true that $F(n) = C \log n$?

Communicating this problem to Erdös, I received from him in reply a letter dated February 13, 1961, in which Erdös states, with brief indications of proofs, that the answer to the above question is *affirmative* if $k \geqq 3$ and *negative* if $k = 2$. When Professor Pisot came to the University of Pennsylvania during the academic year 1961—62 as member of an Institute of Number Theory, I had forgotten about Erdös' letter and we investigated these questions as if they were still open problems. In a way my lapse of memory was fortunate for we would otherwise never have studied these

* Nijenrode lecture.

problems of which the case when $k = 2$ turned out to be particularly rewarding.

Let us change our notations. Setting $F(e^x) = f(x)$, $\alpha_i = \log p_i$ we find

$$F(\prod p_i^{u_i}) = F(e^{\sum u_i \log p_i}) = f(\sum u_i \log p_i) = f(\sum u_i \alpha_i)$$

and (1) becomes

$$(2) \qquad f(u_1\alpha_1 + \ldots + u_k\alpha_k) = f(u_1\alpha_1) + \ldots + f(u_k\alpha_k), \qquad (u_i \geqq 0).$$

The object of our study are the solutions, in particular monotone solutions, of this functional equation under various assumptions concerning the number k and the components α_i, which are assumed to be given positive numbers. The simplest case is obtained if the α_i have a common measure and may therefore be taken as natural integers. For a discussion of the solutions of (2) under this assumption we refer to [4, § 1]. Here we restrict ourselves to the cases when $k = 3$ and $k = 2$.

1. The 3-dimensional module

Assuming that $k = 3$ we may rewrite (2) as

$$(1.1) \qquad f(u\alpha+v\beta+w\gamma) = f(u\alpha)+f(v\beta)+f(w\gamma), \qquad (u, v, w \geqq 0),$$

where α, β, γ are given positive numbers such that the ratios α/β, α/γ and β/γ are irrational. Solutions $f(x)$ of (1.1) are defined in the set

$$S = \{x = u\alpha+v\beta+w\gamma \mid u, v, w \text{ integers} \geqq 0\}.$$

The main result is

Theorem 1. *If $f(x)$ is a solution of (1.1) which is non-decreasing in the set S then $f(x) = \lambda x$ for $x \in S$ (λ constant $\geqq 0$).*

Here is a sketch of the proof: $f(x)$ being a non-decreasing solution of (1.1), we show first that

$$(1.2) \qquad \lim \frac{f(x)}{x} = \lambda, \qquad (x \to \infty, x \in S),$$

exists. Next we define by

$$(1.3) \qquad f(x) = \lambda x + \omega(x)$$

the function $\omega(x)$ which evidently enjoys the properties

$$(1.4) \qquad \omega(u\alpha+v\beta+w\gamma) = \omega(u\alpha)+\omega(v\beta)+\omega(w\gamma)$$

$$(1.5)\qquad \lim \frac{\omega(x)}{x} = 0 \qquad (x \to \infty,\ x \in S).$$

Moreover, (1.3) being non-decreasing we also have

$$(1.6)\qquad \frac{\omega(y)-\omega(x)}{y-x} \geqq -\lambda, \qquad (x, y \in S,\ x < y).$$

Now (1.4) and (1.5) allow to derive from (1.6) by a process which may roughly be described as "amplification" the following fundamental inequality: If u, u' are given integers $\geqq 0$ and h, k are arbitrary integers, then

$$(1.7)\qquad \frac{\omega(u\alpha)-\omega(u'\alpha)}{(u-u')\alpha+h\beta+k\gamma} \geqq -\lambda,$$

provided that the denominator of the fraction does not vanish.

All of our results are essentially based on this inequality and its 2-dimensional analogue (2.10). To complete our proof: Given $u \geqq 0$, we select $u' = 0$ and (1.7) becomes

$$(1.8)\qquad \frac{\omega(u\alpha)}{u\alpha+h\beta+k\gamma} \geqq -\lambda.$$

Given $\varepsilon > 0$ we can find integers h and k such that $0 < u\alpha+h\beta+k\gamma < \varepsilon$ because β/γ is assumed to be irrational. Now (1.8) shows that $\omega(u\alpha) \geqq -\lambda\varepsilon$. Since ε is arbitrary we conclude that $\omega(u\alpha) \geqq 0$. Similarly we can select h, k such that $0 > u\alpha+h\beta+k\gamma > -\varepsilon$ and then (1.8) gives $\omega(u\alpha) \leqq \lambda\varepsilon$ and finally $\omega(u\alpha) \leqq 0$. Thus $\omega(u\alpha) = 0$ and similarly, because of the symmetry in α, β, γ, we can show that $\omega(v\beta) = 0$, $\omega(w\gamma) = 0$. Finally (1.4) shows that $\omega(x) = 0$ and (1.3) implies Theorem 1. This also implies Erdös' result on additive functions for $k = 3$.

2. The 2-dimensional module

For $k = 2$ we write (2) as

$$(2.1)\qquad f(u\alpha+v\beta) = f(u\alpha)+f(v\beta),\ (u, v \text{ integers} \geqq 0),$$

where α, β are given positive numbers such that α/β is irrational. Solutions $f(x)$ of (2.1) are defined on the set

$$(2.2)\qquad S = \{x = u\alpha+v\beta \mid u, v \text{ integers} \geqq 0\}$$

and we wish to study those solutions $f(x)$ which are non-decreasing on S.

We commence by constructing such solutions as follows: Taking the numbers $\{v\beta\}$ modulo α we obtain the set

$$S_\alpha = \{x = m\alpha + v\beta \mid m \text{ arbitrary}, v \geqq 0\} \tag{2.3}$$

which is everywhere dense and has the period α. On it we define an arbitrary function $\varphi(x)$, of period α, such that $\varphi(0) = 0$, and having all its difference quotients bounded below, i.e.

$$\inf_{x, y \in S_\alpha} \frac{\varphi(y) - \varphi(x)}{y - x} = -\mu \text{ is finite, } \mu \geqq 0. \tag{2.4}$$

Likewise we consider the set

$$S_\beta = \{x = u\alpha + n\beta \mid u \geqq 0, n \text{ arbitrary}\}, \tag{2.5}$$

having the period β and on it we define a function $\psi(x)$, of period β, such that $\psi(0) = 0$, and such that

$$\inf_{s, t \in S_\beta} \frac{\psi(t) - \psi(s)}{t - s} = -\nu \text{ is finite, } \nu \geqq 0. \tag{2.6}$$

Observe that $\varphi(x)$ and $\psi(x)$ are both defined on $S = S_\alpha \cap S_\beta$ and are solutions of (2.1). Indeed

$$\varphi(u\alpha + v\beta) = \varphi(v\beta) = \varphi(u\alpha) + \varphi(v\beta)$$

and similarly for $\psi(x)$. If λ is constant it is clear that also

$$f(x) = \lambda x + \varphi(x) + \psi(x), \qquad (x \in S), \tag{2.7}$$

is a solution of (2.1). If we now select λ such that

$$\lambda \geqq \mu + \nu \tag{2.8}$$

then (2.7) defines a *non-decreasing* solution of (2.1). Indeed, by (2.7), (2.4), (2.6) and (2.8) we find, if $x, y \in S$,

$$\frac{f(y) - f(x)}{y - x} = \lambda + \frac{\varphi(y) - \varphi(x)}{y - x} + \frac{\psi(y) - \psi(x)}{y - x} \geqq \lambda - \mu - \nu \geqq 0.$$

We finally observe that $\varphi(x)$ is bounded, because (2.4) and $\varphi(m\alpha) = 0$ imply that $|\varphi(x)| < \mu\alpha$ $(x \in S_\alpha)$, hence $\varphi(x) = o(x)$ as $x \to \infty$ $(x \in S)$. Similarly $\psi(x) = o(x)$ and finally (2.7) shows that

$$\lim \frac{f(x)}{x} = \lambda \qquad (x \to \infty, x \in S). \tag{2.9}$$

THEOREM 2. *The above construction gives all non-decreasing solutions of* (2.1) *in the following sense: If $f(x)$ is such a solution then λ,* *defined by* (2.9), *exists, and also two uniquely defined functions* $\varphi(x)$ *and* $\psi(x)$ *exist, enjoying all the properties described above, in particular* (2.4), (2.6) *and* (2.8), *such that the representation* (2.7) *holds.*

The uniqueness of both $\varphi(x)$ and $\psi(x)$ might at first glance seem puzzling and for this reason I add the following remarks: First (2.9) is established and then the "reduced" solution $\omega(x)$ is defined by $f(x) = \lambda x+\omega(x)$. This then allows to define

$$\varphi(m\alpha+v\beta) = \omega(v\beta),\ \psi(u\alpha+n\beta) = \omega(u\alpha).$$

Now the fundamental inequality (1.7) comes in, which in our case reduces to

$$\frac{\omega(u\alpha)-\omega(u'\alpha)}{(u-u')\alpha+h\beta} \geqq -\lambda, \qquad (h \text{ arbitrary integer}). \tag{2.10}$$

If $t = u\alpha+n\beta$, $s = u'\alpha+n'\beta$ are two distinct numbers in S_β and if we set $h = n-n'$ then $\psi(t) = \omega(u\alpha)$, $\psi(s) = \omega(u'\alpha)$ and (2.10) shows that

$$\inf_{s,t\in S_\beta} \frac{\psi(t)-\psi(s)}{t-s} \geqq -\lambda.$$

But then the infinum defined by (2.6) is surely finite and a similar argument shows that μ, defined by (2.4), is also finite. The proof of the inequality (2.8) is somewhat deeper and for this we refer to [4, § 8].

3. Extending the solutions

A study of the functional equation (2.1) suggests a similar discussion of the *unrestricted* functional equation

$$F(m\alpha+n\beta) = F(m\alpha)+F(n\beta), \quad (m, n \text{ arbitrary integers}), \tag{3.1}$$

whose solutions $F(x)$ are defined on the module

$$\Sigma = \{x = m\alpha+n\beta|\ m, n \text{ arbitrary}\}.$$

In particular the following question arises: Let $f(x)$ be a non-decreasing solution of (2.1); can $f(x)$ be extended to a function $F(x)$, defined on the module Σ, satisfying (3.1) and such that $F(x)$ is non-decreasing on Σ?

Let $f(x)$ be a non-decreasing solution of (2.1) and let (2.7) be its representation as furnished by Theorem 2. Observe that $\varphi(x)+\mu x$ is non-decreasing in the dense set S_α. But then $\varphi(x-0)$ and $\varphi(x+0)$ exist for all real x and $\varphi(x-0) \leqq \varphi(x+0)$. Similarly

$\psi(x-0) \leqq \psi(x+0)$ for all real x. Now we can easily solve the extension problem by the following

Construction: Define $\Phi(x)$ on Σ by the following three rules

1. $\Phi(x) = \varphi(x)$ if $x \in S_\alpha$.

2. If $0 < x < \alpha$, $x \in \Sigma - S_\alpha$, we select the value of $\Phi(x)$ at will such that $\varphi(x-0) \leqq \Phi(x) \leqq \varphi(x+0)$.

3. Extend $\Phi(x)$ to all of Σ so as to have the period α.

Similarly we define $\Psi(x)$ by

1'. $\Psi(x) = \psi(x)$ if $x \in S_\beta$;

2'. If $0 < x < \beta$, $x \in \Sigma - S_\beta$, we select the value of $\Psi(x)$ at will such that $\psi(x-0) \leqq \Psi(x) \leqq \psi(x+0)$.

3'. Extend $\Psi(x)$ to all of Σ so as to have the period β.

It follows from this construction that $\Phi(x)$ and $\Psi(x)$ share with $\varphi(x)$ and $\psi(x)$, respectively, all the properties of the latter *throughout the module* Σ, for instance $\Phi(x)+\mu x$ and $\Psi(x)+\nu x$ are non-decreasing and so forth. But then it is easily seen that

$$F(x) = \lambda x + \Phi(x) + \Psi(x), \qquad (x \in \Sigma),$$

is a non-decreasing solution of (3.1) such that $F(x) = f(x)$ if $x \in S$.

We can therefore always perform the required extension. A direct study of the monotone solutions of the unrestricted equation (3.1) allows to prove the converse

THEOREM 3. *The above construction gives all non-decreasing solutions $F(x)$ of* (3.1) *which are extensions of a given non-decreasing solution $f(x)$ of* (2.1).

In particular we have the

COROLLARY 1. *The above extension $F(x)$ of a given $f(x)$ is unique if and only if $\varphi(x)$ is continuous in $\Sigma - S_\alpha$ and $\psi(x)$ is continuous in $\Sigma - S_\beta$.*

Let us close with a few examples which illustrate these possibilities.

1. Let

$$f(x) = \left[\frac{x}{\alpha}\right] + \left[\frac{x}{\beta}\right] \qquad (\alpha, \beta > 0,\ \alpha/\beta \text{ irrational}), \tag{3.2}$$

which is non-decreasing in S, in fact for all x. The function $f(x)$ is a solution of (2.1) because (2.7) holds with

$$\lambda = \frac{1}{\alpha} + \frac{1}{\beta}, \quad \varphi(x) = \left[\frac{x}{\alpha}\right] - \frac{x}{\alpha}, \quad \psi(x) = \left[\frac{x}{\beta}\right] - \frac{x}{\beta},$$

where $\varphi(x)$, $\psi(x)$ have the periods α and β, respectively, $\varphi(0) =$

$\psi(0) = 0$, while $\mu = 1/\alpha$, $\nu = 1/\beta$, $\lambda = \mu + \nu$. Observe that $\varphi(x)$ is discontinuous at $x = m\alpha$ which points are all in S_α. Likewise $\psi(x)$ is discontinuous at $x = n\beta$ which are all in S_β. Thus $\varphi(x)$ and $\psi(x)$ are continuous in the sets $\Sigma - S_\alpha$ and $\Sigma - S_\beta$, respectively, and by Corollary 1 we conclude that there is a unique monotone extension $F(x)$, solution of (3.1), which is evidently also given by the formula (3.2).

2. Let

$$f(x) = \left[\frac{x}{\alpha}\right] + [x+\alpha], \qquad (0 < \alpha < 1,\ \alpha \text{ irrational},\ \beta = 1).$$

Again (2.7) holds with

$$\lambda = \frac{1}{\alpha} + 1, \qquad \varphi(x) = \left[\frac{x}{\alpha}\right] - \frac{x}{\alpha}, \qquad \psi(x) = [x+\alpha] - x,$$

where φ and ψ have the periods α and $\beta = 1$, respectively, $\varphi(0) = \psi(0) = 0$, $\mu = 1/\alpha$, $\nu = 1$, $\lambda = \mu + \nu$. However, $\psi(x)$ is discontinuous at $x = -\alpha \in \Sigma - S_\alpha$. We conclude by Corollary 1 that $f(x)$ $(x \in S)$ has infinitely many monotone extension $F(x)$, solutions of (3.1), which can all be easily described.

REFERENCES

A. S. Besicovitch

[1] a paper to appear in the G. Pólya Anniversary Volume, December 1962.

P. Erdös

[2] On the distribution function of additive functions, Ann. of Math. (2), vol. 47 (1946), pp. 1–20, in particular Theorem XI on p. 17.

L. Moser and J. Lambek

[3] On monotone multiplicative functions, Proc. Amer. Math. Soc., vol. 4 (1953), pp. 544–545.

Ch. Pisot and I. J. Schoenberg

[4] Arithmetic problems concerning Cauchy's functional equation, to appear in the Illinois J. of Math.

I. J. Schoenberg

[5] On two theorems of P. Erdös and A. Renyi, Illinois J. of Math., vol. 6 (1962), pp. 53–58.

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