WALTER PHILIPP

# An $n$-dimensional analogue of a theorem of H. Weyl

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# An $n$-dimensional analogue of a theorem of H. Weyl *[1]

by

Walter Philipp

It is wellknown that for any fixed basis $a > 1$ almost all real numbers $x$ are normal with respect to $a$. An equivalent statement is the following: For any fixed integer $a > 1$ the sequence $\{a^n x\}$ is uniformly distributed mod 1 for almost all $x$. This is a consequence of a theorem due to *H. Weyl* [4]: If $\{l_n\}$ is an increasing sequence of real numbers which does not increase too slowly in a sense to be determined later then $\{l_n x\}$ is uniformly distributed mod 1 for almost all $x$.

In another lecture contained in this volume *Cigler* (see also [1]) states the following

THEOREM 1: Let $A$ be a nonsingular $m \times m$-matrix with integral entries such that no eigenvalue of $A$ is a root of unity then the sequence of $m$-dimensional vectors $\{A^n \mathfrak{x}\}$ is uniformly distributed mod 1 for almost all vectors $\mathfrak{x} \in R^m$.

This is a consequence of a result of *Rochlin* [3] who proved that the transformation $A\mathfrak{x} - [A\mathfrak{x}]$ is ergodic and measure preserving with respect to Lebesgue measure if $A$ is a matrix with the above properties. But theorem 1 also follows from the following theorem which can be deduced from Weyl's criterion.

THEOREM 2: Let $\{A_n\}$ be a sequence of nonsingular $m \times m$-matrices with integral entries and for fixed $n$ and $k = 1, \ldots, n$ let $h_k^{(n)}$ be the number of integers $j$ $(1 \leqq j \leqq n)$ such that $\det(A_j - A_k) = 0$. If there are two positive constants $\varepsilon$ and $c$ such that

$$\max h_k^{(n)} = h^{(n)} \leqq \frac{c \cdot n}{(\log n)^{1+\varepsilon}}$$

then $\{A_n \mathfrak{x}\}$ is uniformly distributed mod 1 for almost all $\mathfrak{x}$.

Taking $A_n = A^n$ Theorem 1 follows immediately.

* Nijenrode lecture.

[1] The results of this lecture are published in [2].

Replacing $\mathfrak{x}$ by $1/N\,\mathfrak{x}$ where $N$ is a positive integer we see that the conclusion of the theorem holds also if $A_n = N^{-1}B_n$ with an arbitrary integral $B_n$ with the above properties. We now can prove the following:

THEOREM 3: Let $A$ be a real symmetric matrix with $m$ rows whose eigenvalues $\lambda_i$ $(1 \leqq i \leqq m)$ are all $> 1$ and let further $\{l_n\}$ be a sequence of real numbers increasing not too slowly, more precisely: Let there be two positive constants $\varepsilon$ and $c$ with the property that $l$ – considered as a function of the index – increases at least by $c$ as the index increases from $n$ to $n+(n/(\log n)^{1+\varepsilon})$; under these conditions the sequence $\{A^{l_n}\mathfrak{x}\}$ is uniformly distributed mod 1 for almost all $\mathfrak{x}$ .Moreover if $A$ is an arbitrary real square-matrix whose eigenvalues $\lambda_i$ satisfy $|\lambda_i| > 1$ the same conclusion is true if one supposes that the $l_n$ all are integral.

The example $A = \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ shows that the assumption $|\lambda_i| > 1$ cannot be replaced by a weaker one.

For the proof of the theorem we write $A = U^{-1}\Delta U$ where $\det(U) = \pm 1$ and $\Delta = [I_{\rho_1}(\lambda_1), \ldots, I_{\rho_r}(\lambda_r)]$ is a Jordan quasi-diagonal matrix (in the case where $A$ is symmetric we have even a diagonal matrix). Thus in each case $A^{l_n}$ is defined in an obvious way.

If $X = (x_{ij})$ is a square-matrix with $m$ rows we write $||X|| = m \cdot \max|x_{ij}|$. For two matrices $X$ and $Y$ we have $||X+Y|| \leqq ||X||+||Y||$ and $||XY|| \leqq ||X||\,||Y||$ and for any vector $\mathfrak{x}$ we have $|X\mathfrak{x}| \leqq m^{\frac{1}{2}}||X||\,|\mathfrak{x}|$ with $|\mathfrak{x}| = (\sum x_i^2)^{\frac{1}{2}}$.

Now let $N$ be a positive integer. We take now matrices $A_n$ whose elements are rational numbers, all with the same denominator $N$ such that

$$||A_n - \Delta^{l_n}|| \leqq \frac{m}{2N} \quad \text{or} \quad ||A_n - U^{-1}\Delta^{l_n}U|| \leqq \frac{m}{2N}.$$

For $l_k \neq l_j (1 \leqq j, k \leqq m)$ and for fixed $n$ we now put

$$\Omega_{kj} = (\Delta^{l_j} - \Delta^{l_k})^{-1}(UA_jU^{-1} - UA_kU^{-1}).$$

If $l_j - l_k \geqq c$ we have $||\Delta^{l_j} - \Delta^{l_k}|| = 0(1)$.

So we have $||\Omega_{kj} - E|| = 0(N^{-1})$ ($E$ . . . unit matrix). Therefore $\det(\Omega_{kj}) = 1 + 0(N^{-1})$. We have $|\det(\Delta^{l_j} - \Delta^{l_k}) - \det(A_j - A_k)| = |\det(\Delta^{l_j} - \Delta^{l_k})|\,|\det(\Omega_{kj}) - 1|$. So we can choose $N$ large enough to yield $|\det(A_j - A_k)| \geqq \frac{1}{2}|\det(\Delta^{l_j} - \Delta^{l_k})| \neq 0$ for those values $j$ which satisfy $l_j - l_k \geqq c$. But there are at most

$$\frac{2k}{(\log k)^{1+\varepsilon}} \leqq \frac{2n}{(\log n)^{1+\varepsilon}}$$

such numbers $j$ such that $l_j - l_k < c$. Therefore

$$h_k^{(n)} \leqq \frac{2n}{(\log n)^{1+\varepsilon}} \qquad k = 1, \ldots, n.$$

Because one can show in the same manner that $\det(A_n) \neq 0$ for all $n$ Theorem 2 applies. So we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e(\mathfrak{f}^* A_k \mathfrak{x}) = 0 \tag{1}$$

for almost all $\mathfrak{x}$ where $e(x) = e^{2\pi i x}$ and $\mathfrak{f}^*$ is the transposed vector of an arbitrary integral vector $\mathfrak{f} \neq 0$. For real $r_j, s_j, x_j$ $(1 \leqq j \leqq m)$ the following inequality holds

$$|e(\sum r_j x_j) - e(\sum s_j x_j)| \leqq 2\pi \sum |r_j - s_j||x_j|.$$

From this it follows easily that

$$\frac{1}{n} \sum e(\mathfrak{f}^* A_k \mathfrak{x}) - \frac{1}{n} \sum e(\mathfrak{f}^* A^{l_k} \mathfrak{x}) = 0(N^{-1}). \tag{2}$$

We now denote by $\mathfrak{A}_N$ the set of those $\mathfrak{x}$ for which (1) does not hold. The measure $m(\mathfrak{A}_N) = 0$. Let $\mathfrak{A} = \bigcup_N \mathfrak{A}_N \Rightarrow m(\mathfrak{A}) = 0$. From (2) we conclude that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e(\mathfrak{f}^* A^{l_k} \mathfrak{x}) = 0$$

for at least all $\mathfrak{x} \notin \mathfrak{A}$. This proves the theorem.

## LITERATURE

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