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Uniform distribution of sequences of integers *

by

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Let $A = \{a_i\}$ be an infinite sequence of integers. For any integers j and $m \geqq 2$ define $A(n, j, m)$ as the number of terms among $a_1, a_2, a_3, \ldots, a_n$ that satisfy $a_i \equiv j \pmod{m}$. We say that the sequence A is uniformly distributed modulo m in case

$$\lim_{n\to\infty} \frac{1}{n} A(n, j, m) = \frac{1}{m} \quad \text{for} \quad j = 1, 2, \ldots, m.$$

Further more we say that the sequence A is uniformly distributed in case A is uniformly distributed modulo m for every integer $m \geqq 2$. These definitions were introduced by I. Niven; see [1] in the bibilography at the end of this paper.

For example any arithmetic progression $\{ax+b;\, x = 1, 2, 3, \ldots\}$ is uniformly distributed modulo m if and only if g.c.d. $(a, m) = 1$. Such an arithmetic progression is uniformly distributed if and only if $a = 1$. The sequence of positive integers $1, 2, 3, \ldots$ is uniformly distributed, as is also the sequence of negative integers $-1, -2, -3, \ldots$. The sequence of primes is not uniformly distributed modulo m for any modulus m, whereas the sequence of composite integers is uniformly distributed.

For any irrational number θ the sequence obtained by taking the integer parts of the multiples of θ,

$$[\theta], [2\theta], [3\theta], \ldots$$

is uniformly distributed. This result is a consequence of the result of Weyl [2] that the sequence of fractional parts

$$\theta-[\theta], 2\theta-[2\theta], 3\theta-[3\theta], \ldots,$$

form a sequence that is uniformly distributed in the unit interval. (Alternative language for this is that the sequence is uniformly distributed modulo 1; note that in the definition of uniform distribution of a sequence of integers the modulus is greater than one.)

* Nijenrode lecture.

S. Uchiyama [3] extended a result of Niven and proved that a sequence $A = \{a_k\}$ is uniformly distributed modulo m if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \exp(2\pi i h a_k/m) = 0 \quad \text{for} \quad 1 \leqq h \leqq m-1, \tag{1}$$

and hence that A is uniformly distributed if and only if (1) holds for all pairs m, h of positive integers. This is analogous to the Weyl criterion that a sequence $\{\beta_i\}$ of real numbers is uniformly distributed modulo 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \exp(2\pi i \beta_k t) = 0$$

for all integers $t \neq 0$.

C. L. Van den Eynden [4] extended the work of Niven and proved that if $\{\beta_i\}$ is a sequence of real numbers such that the sequence $\{\beta_i/m\}$ is uniformly distributed modulo 1 for all integers $m \neq 0$ then the integer parts $\{[\beta_i]\}$ form a uniformly distributed sequence; also that a real sequence $\{\gamma_i\}$ is uniformly distributed modulo 1 if and only if the sequence of integer parts $\{[m\gamma_i]\}$ is uniformly distributed modulo m for all integers $m \geqq 2$. These results enable one to take many propositions in the theory of uniform distribution modulo 1 and extend them to propositions about sequences of integers. For example, if $f(x)$ is a polynomial with some irrational coefficient (other than $f(0)$) then the sequence $\{[f(n)]; n = 1, 2, 3, \ldots\}$ is uniformly distributed. Again, if p_i denotes the ith prime, then the sequence $\{[\theta p_i]; i = 1, 2, 3, \ldots\}$ is uniformly distributed for any irrational θ. Another result is that if λ is a normal number to base r then the sequence $\{[\lambda r^n]; n = 1, 2, 3, \ldots\}$ is uniformly distributed. A corollary of this can be obtained from the paper of Champernowne [5] that the sequence of integers

$$1, 12, 123, 1234, 12345, \ldots$$

formed from the digits of Champernowne's number

$$0.123456789101112131415161718192021 \ldots$$

is uniformly distributed.

We conclude with two negative results from [1]. Whereas if a sequence A is uniformly distributed modulo m it must then be uniformly distributed modulo d where d is any divisor of m, it is not true that uniform distribution modulo m_1 and m_2 implies uniform distribution modulo the least common multiple of m_1 and m_2. Also, if $f(x)$ is any polynomial with integral coefficients of degree $\geqq 2$, the sequence $\{f(n); n = 1, 2, 3, \ldots\}$ is not uniformly distributed.

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