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Uniform distribution of sequences of integers

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Uniform distribution of sequences of integers*

by

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Let \( A = \{a_i\} \) be an infinite sequence of integers. For any integers \( j \) and \( m \geq 2 \) define \( A(n, j, m) \) as the number of terms among \( a_1, a_2, a_3, \ldots, a_n \) that satisfy \( a_i \equiv j \pmod{m} \). We say that the sequence \( A \) is uniformly distributed modulo \( m \) in case

\[
\lim_{n \to \infty} \frac{1}{n} A(n, j, m) = \frac{1}{m} \quad \text{for} \quad j = 1, 2, \ldots, m.
\]

Furthermore we say that the sequence \( A \) is uniformly distributed in case \( A \) is uniformly distributed modulo \( m \) for every integer \( m \geq 2 \). These definitions were introduced by I. Niven; see [1] in the bibliography at the end of this paper.

For example any arithmetic progression \( \{ax + b; x = 1, 2, 3, \ldots\} \) is uniformly distributed modulo \( m \) if and only if \( \gcd(a, m) = 1 \). Such an arithmetic progression is uniformly distributed if and only if \( a = 1 \). The sequence of positive integers \( 1, 2, 3, \ldots \) is uniformly distributed, as is also the sequence of negative integers \( -1, -2, -3, \ldots \). The sequence of primes is not uniformly distributed modulo \( m \) for any modulus \( m \), whereas the sequence of composite integers is uniformly distributed.

For any irrational number \( \theta \) the sequence obtained by taking the integer parts of the multiples of \( \theta \),

\[ [\theta], [2\theta], [3\theta], \ldots \]

is uniformly distributed. This result is a consequence of the result of Weyl [2] that the sequence of fractional parts

\[ \theta - [\theta], 2\theta - [2\theta], 3\theta - [3\theta], \ldots, \]

form a sequence that is uniformly distributed in the unit interval. (Alternative language for this is that the sequence is uniformly distributed modulo 1; note that in the definition of uniform distribution of a sequence of integers the modulus is greater than one.)

* Nijenrode lecture.
S. Uchiyama [3] extended a result of Niven and proved that a sequence \( A = \{a_k\} \) is uniformly distributed modulo \( m \) if and only if
\[
(1) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \exp \left( 2\pi i a_k/m \right) = 0 \quad \text{for} \quad 1 \leq h \leq m-1,
\]
and hence that \( A \) is uniformly distributed if and only if (1) holds for all pairs \( m, h \) of positive integers. This is analogous to the Weyl criterion that a sequence \( \{\beta_i\} \) of real numbers is uniformly distributed modulo 1 if and only if
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \exp \left( 2\pi i \beta_k t \right) = 0
\]
for all integers \( t \neq 0 \).

C. L. Van den Eynden [4] extended the work of Niven and proved that if \( \{\beta_i\} \) is a sequence of real numbers such that the sequence \( \{\beta_i/m\} \) is uniformly distributed modulo 1 for all integers \( m \neq 0 \) then the integer parts \( \{[\beta_i]\} \) form a uniformly distributed sequence; also that a real sequence \( \{y_i\} \) is uniformly distributed modulo 1 if and only if the sequence of integer parts \( \{[m\gamma_i]\} \) is uniformly distributed modulo \( m \) for all integers \( m \geq 2 \). These results enable one to take many propositions in the theory of uniform distribution modulo 1 and extend them to propositions about sequences of integers. For example, if \( f(x) \) is a polynomial with some irrational coefficient (other than \( f(0) \)) then the sequence \( \{[f(n)]; n = 1, 2, 3, \ldots\} \) is uniformly distributed. Again, if \( p_i \) denotes the \( i \)th prime, then the sequence \( \{[\theta p_i]; i = 1, 2, 3, \ldots\} \) is uniformly distributed for any irrational \( \theta \). Another result is that if \( \lambda \) is a normal number to base \( r \) then the sequence \( \{[\lambda r^n]; n = 1, 2, 3, \ldots\} \) is uniformly distributed. A corollary of this can be obtained from the paper of Champernowne [5] that the sequence of integers
\[
1, 12, 123, 1234, 12345, \ldots
\]
formed from the digits of Champernowne's number
\[
0.123456789101112131415161718192021\ldots
\]
is uniformly distributed.

We conclude with two negative results from [1]. Whereas if a sequence \( A \) is uniformly distributed modulo \( m \) it must then be uniformly distributed modulo \( d \) where \( d \) is any divisor of \( m \), it is not true that uniform distribution modulo \( m_1 \) and \( m_2 \) implies
uniform distribution modulo the least common multiple of $m_1$ and $m_2$. Also, if $f(x)$ is any polynomial with integral coefficients of degree $\geq 2$, the sequence $\{f(n); n = 1, 2, 3, \ldots\}$ is not uniformly distributed.

BIBLIOGRAPHY

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