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On the distribution of a sequence in a compact group *

by

J. H. B. Kemperman

1. Introduction

If $\{x_k\}$ is a sequence of real numbers such that each derived sequence $\{x_{k+h}-x_k\}$, h a positive integer, is uniformly distributed modulo 1 then so is the original sequence $\{x_k\}$. This important result due to van der Corput [6] can be sharpened and generalized in many different directions, cf. the survey [5] by Cigler and Helmberg.

For instance, even under weaker assumptions, one can assert that each arithmetic progression $\{x_{kL-j}\}$ (L and j fixed) is uniformly distributed modulo 1, see [4], [7] and [12].

Further, the group K of real numbers modulo 1 can be replaced by a connected compact abelian group G. After all, a sequence $\{x_k\}$ in G has asymptotically a uniform distribution if and only if $\varphi(x_k)$ is uniformly distributed in K for each non-trivial character $\varphi(x)$ of G, (= continuous homomorphism of G into K).

As was shown by Hlawka [7], the above difference theorem carries over to any additively written compact group, commutative or not. Subsequently, Cigler [4] (see also Tsuji [14]) generalized this result by replacing the ordinary Cesaro sum by a very general regular summation method. In [3] Cigler replaced points in G by probability measures on G. For the special case of a Cesaro type summation, Hlawka [9] considered in stead a function $x(k)$ of the real variable k taking values in a compact group G.

In the present paper (which is independent of Cigler's work, see [10]) the van der Corput difference theorem is carried over to a very general situation which covers all of the above cases. Our method has the advantage that it is based upon an estimate (Lemma 5.3) of the same type as van der Corput's fundamental inequality, which should also be useful in deriving quantitative

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results on the speed of convergence towards a uniform distribution.

In order to bring out more clearly the essential ideas of the proof, I have first taken up (sections 2 and 3) the special case of a sequence $\{x(k)\}$ of real numbers modulo 1. The general setting is given in section 4, the corresponding difference theorem in section 5. As will be seen, the method of section 5 could be regarded as a straightforward generalization of the one in section 2. The sections 6 and 7 are devoted to certain interesting corollaries of the results in section 5.

2. Auxiliary results

In the sections 2 and 3, $A = (a_{nk})$ denotes a fixed nonnegative regular summation method; $n, k = 1, 2, \ldots$. Thus,

$$a_{nk} \geqq 0, \quad \lim_{n\to\infty} a_{nk} = 0,$$

while

$$\lim_{n\to\infty} ||\alpha_n|| = 1, \text{ where } ||\alpha_n|| = \sum_{k=1}^{\infty} a_{nk}. \tag{2.1}$$

For each sequence $z = \{z(k)\}$ of complex numbers with

$$||z|| = \sup_k |z(k)| < \infty,$$

put

$$\alpha_n\{z(k)\} = \sum_{k=1}^{\infty} a_{nk} z(k). \tag{2.2}$$

It follows that

$$|\alpha_n\{z(k)-z(k+1)\}| \leqq ||\beta_n|| \, ||z||, \tag{2.3}$$

where

$$||\beta_n|| = \sum_{k=0}^{\infty} |a_{n,k}-a_{n,k+1}|, \tag{2.4}$$

$(a_{n,0} = 0)$. One has $||\beta_n|| = 2/n$ in the particular case $A = (C, 1)$, that is, $a_{nk} = 1/n$ if $1 \leqq k \leqq n$, $a_{nk} = 0$ if $k > n$.

The following result is analogous to the fundamental inequality of van der Corput [6], see [2, p. 71]. As will be seen, it can be generalized to much more general situations.

LEMMA 2.1. *Suppose that* $||z|| \leqq 1$. *Then, provided* $||\alpha_n|| > 0$, *we have for each positive integer* m *that*

$$||\alpha_n||^{-1} |\alpha_n\{z(k)\}|^2 \leqq \frac{4}{3} m||\beta_n|| + \frac{1}{m} ||\alpha_n|| + \frac{2}{m} \sum_{h=1}^{m-1} \left(1-\frac{h}{m}\right) \mathrm{Re}[\alpha_n\{z(k+h)\overline{z(k)}\}]. \tag{2.5}$$

Proof. Without loss of generality, we may assume that $||\alpha_n|| = 1$. Let n be fixed, and take

$$\gamma = \alpha_n\{z(k)\}, \quad \text{thus,} \quad |\gamma| \leqq 1. \tag{2.6}$$

Obviously,

$$\sum_{p=0}^{m-1}\sum_{q=0}^{m-1}(z(k+p)-\gamma)\overline{(z(k+q)-\gamma)} = \left|\sum_{p=0}^{m-1}(z(k+p)-\gamma)\right|^2 \geqq 0.$$

Multiplying by $a_{nk} \geqq 0$ and summing over k, this gives

$$\sum_{p=0}^{m-1}\alpha_n\{|z(k+p)|^2\}+\sum_{p>q}2\,\mathrm{Re}[\alpha_n\{z(k+p)\overline{z(k+q)}\}]$$
$$-m\sum_{p}2\mathrm{Re}[\bar{\gamma}\,\alpha_n\{z(k+p)\}]+m^2|\gamma|^2 \geqq 0.$$

By (2.3) and $||z|| \leqq 1$, replacing in the last sum $z(k+p)$ by $z(k)$ introduces an error

$$\leqq m\sum_{p=0}^{m-1}2p||\beta_n|| \leqq m^3||\beta_n||$$

in absolute value. Replacing in the second sum $z(k+p)\overline{z(k+q)}$ by $z(k+p-q)\overline{z(k)}$ introduces an error

$$\leqq \sum_{p>q}2q||\beta_n|| \leqq \tfrac{1}{3}m^3||\beta_n||$$

in absolute value. Finally, the first sum has its absolute value at most equal to m. Using (2.6), one obtains (2.5).

Next, let L be a given positive integer, and put

$$\rho_{nj} = \sum_{k=1}^{\infty}a_{n,kL-j}, \qquad (j = 0, 1, \ldots, L-1) \tag{2.7}$$

and

$$b_{n,L} = \sum_{k=-\infty}^{\infty}|a_{n,k}-a_{n,k+L}|; \qquad (a_{n,k} = 0 \text{ for } k \leqq 0).$$

Lemma 2.2. *If* $||z|| \leqq 1$ *then, for each positive integer* m,

$$\sum_{\substack{j=0\\ \rho_{nj}>0}}^{L-1}\rho_{nj}\left|\frac{1}{\rho_{nj}}\sum_{k=1}^{\infty}a_{n,kL-j}\,z(kL-j)\right|^2$$
$$\leqq \frac{1}{m}||\alpha_n||+\frac{4}{3}mb_{n,L}+\frac{2}{m}\sum_{h=1}^{m-1}\left(1-\frac{h}{m}\right)\mathrm{Re}[\alpha_n\{z(k+hL)\overline{z(k)}\}]. \tag{2.8}$$

Proof. Apply (2.5) with (a_{nk}) and $z(k)$ replaced by $(a_{n,kL-j})$ and $z(kL-j)$, respectively, $(j = 0, 1, \ldots, L-1)$. Adding the resulting inequalities, one obtains (2.8).

From now on, let us assume that

$$\lim_{n\to\infty} ||\beta_n|| = 0,$$

that is,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k} - a_{n,k+1}| = 0, \tag{2.9}$$

(as is true for most of the classical summation methods). It follows by (2.1) and (2.7) that

$$\lim_{n\to\infty} \rho_{n,j} = L^{-1}, \qquad (j = 0, 1, \ldots L-1).$$

Consequently, by Lemma 2.2,

LEMMA 2.3. *Suppose that* $||z|| < \infty$. *Let* L *be a positive integer such that*

$$\underline{\lim}_{m\to\infty} \overline{\lim}_{n\to\infty} \frac{2}{m} \sum_{h=1}^{m-1} \left(1-\frac{h}{m}\right) \mathrm{Re}[\alpha_n\{z(k+hL)\overline{z(k)}\}] = 0. \tag{2.10}$$

Then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,kL-j} z(kL-j) = 0, \qquad (j = 0, \ldots, L-1). \tag{2.11}$$

Note that, by (2.8), the left hand side of (2.10) is always nonnegative. By Hölder's inequality, a sufficient condition for (2.10) is that, for some $1 \leqq r < \infty$,

$$\underline{\lim}_{m\to\infty} \overline{\lim}_{n\to\infty} \frac{1}{m} \sum_{h=1}^{m-1} |\alpha_n\{z(k+hL)\overline{z(k)}\}|^r = 0.$$

Letting

$$\overline{\lim}_{n\to\infty} |\alpha_n\{z(k+h\}\overline{z(k)})\}| = \gamma_h,$$

a further sufficient condition for (2.10) is that γ_{hL} tends to zero when the positive integer h tends to infinity through some set S of integers of upper density 1.

3. Van der Corput's difference theorem

Let G denote the additive group of real numbers modulo 1. Let further $A = (a_{nk})$ be a given real and nonnegative regular summation matrix satisfying (2.9). A sequence of points $\{x(k)\}$ in G is said to have a given regular Borel measure μ as its *asymptotic A-distribution* if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} f(x(k)) = \int_G f(x)\mu(dx) \tag{3.1}$$

holds for each complex-valued continuous function f on G; such a function may be identified with a continuous function on the reals of period 1.

Clearly, μ must be a nonnegative measure on G of total mass 1, a so-called probability measure. If so then (3.1) is equivalent to the condition that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\, e^{2\pi i p x(k)} = \int_0^1 e^{2\pi i p x}\, \mu(dx), \tag{3.2}$$

for each integer $p = 1, 2, \ldots$. If μ is the Haar measure (= Lebesque measure) on G then (3.2) becomes

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\, e^{2\pi i p x(k)} = 0, \qquad (p = 1, 2, \ldots). \tag{3.3}$$

If (3.3) holds we say that the sequence $\{x(k)\}$ is uniformly distributed with respect to the summation method $A = (a_{nk})$.

Applying Lemma 2.3 with

$$z(k) = e^{2\pi i p x(k)},$$

(p a fixed positive integer), we obtain the following generalization of van der Corput's [6] difference theorem.

THEOREM 3.1. *Let L be a fixed positive integer, and suppose that, for each positive integer h, the sequence*

$$\{x(k+hL)-x(k);\quad k = 1, 2, \ldots\}$$

has an asymptotic A-distribution τ_h, say. Suppose further that, for each positive integer p, there exists a sequence $\{m_j\}$ of positive integers, $m_j \to \infty$, such that

$$\lim_{j\to\infty} \frac{1}{m_j} \sum_{h=1}^{m_j} \left(1 - \frac{h}{m_j}\right) \int_0^1 e^{2\pi i p x}\, \tau_h(dx) = 0. \tag{3.4}$$

Then, for each fixed $j = 0, 1, \ldots, L-1$, we have that the sequence $\{x(kL-j)\}$ is uniformly distributed with respect to the regular summation method $(La_{n,kL})$.

Essentially the same result was obtained independently by Cigler [4]; his method is related to a treatment by random variables in [11]. Tsuji [14] proved a certain special case of Theorem 3.1 with $L = 1$ and

$$a_{nk} = p_k(p_1 + \dots + p_n)^{-1} \text{ if } 1 \leqq k \leqq n,$$
$$= 0 \qquad \text{if } k > n,$$

($p_n > 0$, $\sum p_n = +\infty$). He assumed that both p_n and p_n/p_{n+h} ($h = 1, 2, \dots$) are decreasing in n. In view of (2.9), it is actually sufficient that p_n is monotone, $p_n = o(p_1 + \dots + p_n)$.

In the special case $A = (C, 1)$, Theorem 3.1 can also easily be deduced from the van der Corput fundamental inequality. Under the additional assumption that each sequence $\{x(k+h) - x(k)\}$ ($h = 1, 2, \dots$) is uniformly distributed modulo 1, this was already done by Korobov and Postnikov [12].

4. Generalities

In generalizing Theorem 3.1, we note that the sum in (3.1) can be written as

$$\int_Z f(x(z))\, \alpha_n(dz).$$

Here, Z denotes the collection of all positive integers. Further, α_n denotes the nonnegative measure on Z having a mass a_{nk} at the point, k, $k = 1, 2, \dots$; its total mass $||\alpha_n||$ tends to 1 as n tends to infinity. Moreover, we can write

$$x(k+hL) = x(T^h k),$$

where T denotes the transformation $z' = z+L$ of Z into itself. Finally, the quantity $b_{n,L}$ in (2.8) is precisely the total variation of the signed measure

$$\beta_n(A) = \alpha_n(T^{-1}A) - \alpha_n(A), \qquad (A \subset Z).$$

With this in mind, we now introduce the following entities.

(i) First, a fixed *measurable space* $Z = (Z, \mathfrak{A})$. That is, Z is a given abstract set, $\mathfrak{A}$ a σ-field of subsets of Z with $Z \in \mathfrak{A}$.

(ii) Second, a fixed *directed set* D. In other words, the relation $m \geqq n$ is defined for certain ordered pairs of elements in D such that $m \geqq m$, while $m \geqq n$, $n \geqq p$ imply $m \geqq p$; finally, given m, $n \in D$, there always exists a $q \in D$ such that both $q \geqq m$ and $q \geqq n$.

(iii) For each $n \in D$, let α_n be a given finite and nonnegative (σ-additive) measure on the measurable space Z. We shall assume that

(4.1) $$\lim_n \int_Z \alpha_n(dz) = 1,$$

the limit being taken in the sense of the directed set D.

(iv) Next, let T denote a fixed measurable transformation of Z into itself. By β_n we shall denote the finite signed measure on Z defined by

$$\beta_n(A) = \alpha_n(T^{-1}A) - \alpha_n(A), \tag{4.2}$$

for each $A \in \mathfrak{A}$. It folows that

$$\int [h(Tz) - h(z)]\alpha_n(dz) = \int h(z)\beta_n(dz), \tag{4.3}$$

for each bounded and measurable complex-valued function $h(z)$ on Z; (if no integration limits are specified the integral extends over all of Z). Later on, we shall assume that

$$\lim_n ||\beta_n|| = 0, \tag{4.4}$$

where

$$||\beta_n|| = \int |\beta_n(dz)|$$

denotes the total variation of the measure β_n.

(v) Let further G denote a given compact Hausdorff space. By $C(G)$ we shall denote the Banach space of all complex-valued continuous functions f on G with norm

$$||f|| = \sup_{x \in G} |f(x)|.$$

By the Riesz representation theorem, there is a 1 : 1 correspondence between the bounded linear functionals $\mu(f)$ on $C(G)$ on the one hand, and the regular complex-valued (finite) Borel measures μ on G on the other hand, namely by means of

$$\mu(f) = \int_G f(x)\mu(dx), \qquad (f \in C(G)). \tag{4.5}$$

The norm of the functional $\mu(f)$ is precisely the total variation $||\mu||$ of the corresponding measure μ.

Let $M(G)$ denote the linear space of all such bounded linear functionals on $C(G)$. The topology in $M(G)$ will be taken as the weak* topology. That is, a subset F of $M(G)$ is said to be open if, for each $\mu_0 \in F$, one can find a number $\varepsilon > 0$ and finitely many $f_1, \ldots, f_m$ in $C(G)$ such that

$$\{\mu : |\mu(f_i) - \mu_0(f_i)| < \varepsilon, i = 1, \ldots, m\} \subset F.$$

In other words, the net (also called generalized sequence) $\{\mu_m\}$

of elements in $M(G)$, (m running through some directed set E), converges to an element μ_0 in $M(G)$ if and only if $\{\mu_m(f)\}$ converges to $\mu_0(f)$ for each f in $C(G)$.

(vi) For each $n \in D$, let $x_n(z)$ be a given measurable function from Z to G, ("measurable" taken with respect to the σ-field of Borel subsets of G).

We shall be interested in the asymptotic distribution of the *values* $x_n(z)$ in G. More precisely, for each $n \in D$, the formula

$$\mu_n(f) = \int_Z f(x_n(z))\alpha_n(dz), \qquad (f \in C(G)), \tag{4.6}$$

defines a nonnegative regular Borel measure μ_n on G of norm $||\mu_n|| = ||\alpha_n||$. Equivalently,

$$\int_V \mu_n(dx) = \int_{x_n(z)\in V} \alpha_n(dz),$$

for each Baire measurable subset V of G. Thus, the measure α_n acts as a sort of weight function on Z.

Definition. We shall say that the net

$$\{x_n(z),\, n \in D\} \tag{4.7}$$

of measurable functions from Z to G has the measure μ_0 on G as its asymptotic distribution (with respect to the summation method $\{\alpha_n,\, n \in D\}$) if $\lim_n \mu_n = \mu_0$, that is, if

$$\lim_n \int_Z f(x_n(z))\alpha_n(dz) = \int_G f(x)\mu_0(dx) \tag{4.8}$$

holds for each $f \in C(G)$.

This in turn implies that (4.8) holds whenever f is a bounded function on G such that $\mu_0(\Delta_f) = 0$, where Δ_f denotes the set of points $x_0 \in G$ at which f is discontinuous; this generalizes a result of Hlawka [8]. His proof carries over immediately; namely, if f is bounded and $\mu_0(\Delta_f) = 0$ then ([1], p. 104, 109) for each number $\varepsilon > 0$ there exist real and continuous functions f_1 and f_2 on G such that $f_1(x) < \mathrm{Re}(f(x)) < f_2(x)$ and $\mu_0(f_2-f_1) < \varepsilon$.

5. A general difference theorem

From now on, we assume that (4.4) holds. Further, we shall take G as an additively written *compact group*, (not necessarily, commutative). By ν we denote the Haar measure on G, normalized in such a way that $||\nu|| = 1$. If (4.8) holds with μ_0 replaced by ν (all $f \in C(G)$) we say that the net (4.7) is (asymptotically) *uniformly distributed.*

THEOREM 5.1. *Suppose that, for h as a sufficiently large positive integer, the net*

$$\{x_n(T^h z) - x_n(z),\ n \in D\} \tag{5.1}$$

of functions from Z to G possesses an asymptotic distribution $\tau_h \in M(G)$ which tends to the uniform distribution ν as h tends to infinity. Then the net $\{x_n(z),\ n \in D\}$ is itself uniformly distributed.

This result corresponds to the special case $L = 1$ of Theorem 3.1. In generalizing the full Theorem 3.1, we shall assume that the measurable space $Z = (Z, \mathfrak{A})$ is the direct product of the pair of measurable spaces $Z' = (Z', \mathfrak{A}')$ and $Z'' = (Z'', \mathfrak{A}'')$, such that

$$(Tz)'' = z''. \tag{5.2}$$

Here, we think of a point $z \in Z$ as a pair of points $z' \in Z'$, $z'' \in Z''$, (the coordinates of z). Thus, (5.2) states that each section $z'' =$ const. is invariant under the transformation T.

Note that this assumption trivially holds on taking $Z' = Z$, (Z'' as a set consisting of one point only). Usually, also other decompositions are possible. For instance, if Z consists of the positive integers (or the positive real numbers) and $Tz = z+L$, this is true with $Z' = \{0, L, 2L, \ldots\}$ and Z'' as the interval $(0, L]$, (Z being a direct sum of Z' and Z'').

Let us denote by ρ_n the measure on Z'' which is the marginal of α_n, in other words, the projection of the measure α_n on $Z = Z' \times Z''$ onto the component Z''. In the present case we shall further assume, as we may in most applications, that there exists a function $\alpha_n(A \mid z'')$, $A \in \mathfrak{A}'$, $z'' \in Z''$, which is a *probability measure* in A for each fixed z'', and a measurable function in z'' for each fixed A, such that

$$\int_Z h(z)\alpha_n(dz) = \int_{Z''} \left[\int_{Z'} h(z', z'')\alpha_n(dz'|z'') \right] \rho_n(dz''), \tag{5.3}$$

for each bounded and measurable function $h(z) = h(z', z'')$ on Z.

THEOREM 5.2. *Suppose that, for h as a sufficiently large positive integer, $h \geqq h_0$, the net (5.1) has an asymptotic distribution τ_h. Let U be a given unitary representation of G and suppose that*

$$\lim_{m_j \to \infty} m_j^{-1} \sum_{h=h_0}^{m_j} \left(1 - \frac{h}{m_j}\right) \int_G U(x)\, \tau_h(dx) = 0, \tag{5.4}$$

for some sequence $\{m_j\}$ of integers tending to infinity. Then

$$\lim_{n \in D} \int_{Z''} \Phi_n(z'')\Phi_n(z'')^* \rho_n(dz'') = 0, \tag{5.5}$$

where

$$\Phi_n(z'') = \int_{Z'} U(x_n(z', z''))\alpha_n(dz'|z''). \tag{5.6}$$

Theorem 5.2 generalizes certain results of Cigler [3], [4] and Hlawka [7], [9], compare sections 6 and 7.

By a unitary representation $U = U(x)$ of degree r ($r = 1, 2, \ldots$) of the given compact group G we shall mean a *continuous* mapping of G into the group of all complex-valued unitary matrices of order r, such that

$$U(x+y) = U(x)U(y) \text{ for all } x, y \in G. \tag{5.7}$$

A matrix is said to be unitary if its inverse U^{-1} exists and is equal to its adjoint U^*, ($u^*_{ij} = \overline{u_{ji}}$). Thus, (5.7) implies

$$U(0) = I, \quad U(x)^* = U(-x), \tag{5.8}$$

(I denoting the identity matrix of order r).

The so-called trivial representation U_0 is the representation of degree 1 defined by $U_0(x) = 1$ for all $x \in G$. The unitary representation $U = U(x)$ of degree r is said to be irreducible if no non-trivial linear subspace of the r-dimensional Euclidean is invariant under all the transformations $U(x)$, $x \in G$. As is well-known, there exists a family

$$\{U_\gamma, \gamma \in \Gamma\} \tag{5.9}$$

of irreducible (mutually inequivalent) unitary representations of G (possibly of different degrees), such that all the matrix elements of all the $U_\gamma(x)$ together span a linear manifold which is dense in the Banach space $C(G)$. Finally, if U is any non-trivial irreducible unitary representation of G then (as can be seen from (5.7))

$$\int_G U(x)\nu(dx) = 0. \tag{5.10}$$

It follows from these remarks that the net $\{x_n(z)\}$ has (with respect to the summation method $\{\alpha_n\}$) asymptotically a *uniform distribution if and only if, for each non-trivial irreducible unitary representation $U = U(x)$, one has*

$$\lim_n \int U(x_n(z))\alpha_n(dz) = 0. \tag{5.11}$$

This shows that Theorem 5.1. is a special case of Theorem 5.2; (take $Z = Z' \times Z''$ with Z'' consisting of one point only).

That Theorem 3.1 is a special case of Theorem 5.2 can be seen from the remark following (5.2) and the well-known fact that the group K of real numbers modulo 1 has as its irreducible unitary representations the functions

$$U_p(x) = e^{2\pi i p x}, \qquad (p \text{ an integer}).$$

The proof of Theorem 5.2 is based on the following generalization of Lemma 2.2. Here, if V is a complex $r \times r$ matrix we denote by $V \gg 0$ the property that V is positive definite in the sense that $w.V.w^* \geqq 0$ for every complex $1 \times r$ vector $w = (w_1, \ldots w_r)$. One always has $V.V^* \gg 0$. We write $V \ll W$ or $W \gg V$ if and only if $W - V \gg 0$. This is a true partial ordering, for $0 \ll V \ll 0$ implies $V = 0$.

Lemma 5.3. *Let $U = U(x)$ denote a given unitary representation of G of degree r. Define $\Phi_n(z'')$ by* (5.6) *and $\Psi_{n,h}$ ($n \in D$, $h = 0, 1, \ldots$) by*

$$\Psi_{n,h} = \int U(x_n(T^h z) - x_n(z))\alpha_n(dz). \tag{5.12}$$

Then

$$\int_{Z''} \Phi_n(z'')\Phi_n(z'')^* \rho_n(dz'')$$

$$\ll \frac{1}{m}\,||\alpha_n||I + \frac{4}{3}\,m||\beta_n||I + \frac{1}{m}\sum_{h=1}^{m-1}\left(1 - \frac{h}{m}\right)(\Psi_{nh} + \Psi_{nh}^*). \tag{5.13}$$

Here, I denotes the identity matrix of order r.

Proof. Let $n \in D$ be fixed. The inequality being trivial when $||\alpha_n|| = 0$, we may assume that $||\alpha_n|| = 1$. Consider the $r \times r$ matrix

$$V(z) = \sum_{p=0}^{m-1} \left(U(x_n(T^p z)) - \Phi_n(z'')\right).$$

By $V(z)V(z)^* \gg 0$, (5.7) and (5.8), it follows that

$$0 \ll \sum_{p=0}^{m-1}\sum_{q=0}^{m-1} U(x_n(T^p z) - x_n(T^q z)) + m^2 \Phi_n(z'')\Phi_n(z'')^*$$
$$- m\sum_{p=0}^{m-1}[U(x_n(T^p z))\Phi_n(z'')^* + \Phi_n(z'')U(x_n(T^p z))^*].$$

We now integrate this relation over all of Z with respect to the nonnegative measure α_n. By (5.3), the second term yields

$$m^2 \int \Phi_n(z'')\Phi_n(z'')^* \rho_n(dz'').$$

The terms with $p = q$ in the double sum yield a contribution mI, while the terms with $p \neq q$ yield

$$\sum_{0 \leqq q < p \leqq m-1} [\Psi_{n,p-q} + \Psi^*_{n,p-q} + \Delta_{p-q,q} + \Delta^*_{p-q,q}].$$

Here,

$$\Delta_{h,q} = \int [D_h(T^q z) - D_h(z)]\alpha_n(dz), \tag{5.14}$$

where

$$D_h(z) = U(x_n(T^h z) - x_n(z)).$$

By (5.3) and (5.6), the last sum yields a contribution

$$-m \sum_{q=0}^{m-1} \left[2 \int \Phi_n(z'')\Phi_n(z'')^* \rho_n(dz'') + \nabla_q + \nabla^*_q \right],$$

where

$$\nabla_q = \int [U(x_n(T^q z) - U(x_n(z))]\Phi_n(z'')\alpha_n(dz).$$

Note that, by (5.2), we may write

$$\nabla_q = \int [E(T^q z) - E(z)]\alpha_n(dz), \tag{5.15}$$

where

$$E(z) = U(x_n(z))\Phi_n(z'').$$

Adding all these contributions, (and dividing by m^2), one obtains (5.13) provided it can be shown that

$$\Delta_{h,q} \ll q \, ||\beta_n|| \cdot I, \quad \nabla_q \ll q \, ||\beta_n|| \cdot I.$$

Let $w = (w_1, \ldots, w_r)$ be any complex $1 \times r$ vector of Euclidean length 1. It sufficies to show that

$$w\Delta_{h,q} w^* \leqq q \, ||\beta_n||, \quad w\nabla_q w^* \leqq q \, ||\beta_n||.$$

But $U(x)$ is unitary, thus, $|wD_h(z)w^*| \leqq 1$; similarly, by (5.6), $|wE(z)w^*| \leqq 1$. Hence, multiplying (5.14) and (5.15) on the left by w and on the right by w^*, the desired result follows by (4.3).

Proof of Theorem 5.2. It is given that, for $h \geqq h_0$,

$$\lim_{n} \Psi_{n,h} = \int U(x)\tau_h(dx) = V_h, \qquad \text{(say)}.$$

It follows by (4.4) and (5.13) that, given $\varepsilon > 0$ and the positive integer m, $m > h_0$, one has for n sufficiently large (in the sense of the directed set D) that

$$0 \ll \int_{Z''} \Phi_n(z'')\Phi_n(z'')^* \rho_n(dz'')$$
$$\ll \left(\varepsilon + \frac{1}{m}\,||\alpha_n||\right) I + \frac{1}{m} \sum_{h=h_0}^{m-1} \left(1 - \frac{h}{m}\right)(V_h + V_h^*).$$

Letting m tend to infinity and using (5.4), one obtains (5.5).

6. Random functions

Let again G be an arbitrary additively written compact group; by ν we denote the Haar measure on G, $||\nu|| = 1$. Let further $A = (a_{nk})$ $(n, k = 1, 2, \ldots)$ be a given nonnegative regular summation method satisfying (2.9).

Following Cigler [3], let us consider the asymptotic A-distribution of a sequence $\{\sigma_k\}$ of probability measures on G (= nonnegative regular Borel measures of total mass 1). Namely, we shall say that $\{\sigma_k\}$ has an asymptotic A-distribution σ if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\sigma_k = \sigma,$$

that is,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\, \sigma_k(f) = \sigma(f) \text{ for all } f \in C(G).$$

This generalizes (3.1); for, take σ_k as the measure having all its mass at the single point $x(k) \in G$.

Given the measures μ' and μ'' on G, let us define the measure $\mu' \ominus \mu''$ by

$$(6.1) \qquad (\mu' \ominus \mu'')(f) = \iint f(x-y)\mu'(dx)\mu''(dy),$$

$(f \in C(G))$. The following generalization of Theorem 3.1 is due to Cigler [3].

THEOREM 6.1. *Let L be a fixed positive integer. Suppose that, for $h = 1, 2, \ldots$, the sequence*

$$\{\sigma_{k+hL} \ominus \sigma_k\}$$

has an asymptotic A-distribution τ_h, *say. Suppose further that*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{h=1}^{m} \tau_h = \nu.$$

Then, for each $j = 0, 1, \ldots, L-1$,

$$\lim_{n\to\infty} L \sum_{k=1}^{\infty} a_{n,kL-j}\sigma_{kL-j} = \nu.$$

Theorem 6.1 in turn is a consequence of Theorem 5.2. Let

$$(\Omega, \mathscr{B}, P)$$

be a *fixed* measure space, where P is a probability measure on the σ-field $\mathscr{B}$. By a *random variable* taking values in G we shall mean a measurable function $X(\omega)$ from Ω to G. Its so-called distribution is the probability measure on G defined by

$$\mu(f) = \int_{\Omega} f(X(\omega))P(d\omega), \qquad (f \in C(G)). \tag{6.2}$$

The joint distribution of two random variables X' and X'' taking values in G is defined as the probability measure on $G\times G$ which is the distribution of the random variable $(X'(\omega), X''(\omega)) \in G\times G$. If this joint distribution is the direct product of the distributions μ' and μ'' of X' and X'', respectively, then the random variables X' and X'' are said to be independent; notice that in this case the random variable $X'-X''$ has its distribution precisely equal to $\mu' \ominus \mu''$.

Consider a sequence $\{\sigma_k\}$ of probability measures on G. Taking $(\Omega, \mathscr{B}, P)$ as the direct product of the measure spaces (G, σ_k) and letting $X_k(\omega)$ denote the k-th coordinate of a point ω in this product space, one obtains a sequence $\{X_k\}$ of independent random variables, such that the distribution of X_k is equal to σ_k.

Let Z, D, $\alpha_n (n \in D)$ and T be as in section 4; we assume that (4.4) holds. Suppose further that $Z = Z'\times Z''$, such that (5.2) and (5.3) hold. Next, for each $n \in D$ and $z \in Z$, let $X_n(z) = X_n(z, \omega)$ be a given random variable. We shall assume that $X_n(z, \omega)$ is *jointly measurable* in $z \in Z$, $\omega \in \Omega$.

Definition. The net

$$\{X_n(z), n \in D\}$$

of "random functions" on Z will be said to have the asymptotic distribution μ, with respect to the given summation method $\{\alpha_n, n \in D\}$, if

(6.3) $$\lim_n \int_Z \mu_{n,z}\, \alpha_n(dz) = \mu.$$

Here, $\mu_{n,z}$ denotes the distribution of the random variable $X_n(z)$, that is,

(6.4) $$\mu_{n,z}(f) = \int_\Omega f(X_n(z, \omega))P(d\omega), \quad \text{for all } f \in C(G).$$

Relation (6.3) generalizes (4.8) and denotes that

(6.5) $$\lim_n \int_Z \mu_{n,z}(f)\alpha_n(dz) = \mu(f) \quad \text{for all } f \in C(G).$$

If μ is the Haar measure on G then (6.5) is equivalent to

(6.6) $$\lim_n \int_Z \mu_{n,z}(U)\alpha_n(dz) = 0,$$

to hold for each non-trivial unitary representation U of G.

THEOREM 6.2. *Suppose that, for h as a sufficiently large positive integer, $h \geqq h_0$, the net of random variables*

(6.7) $$\{X_n(T^h z) - X_n(z), n \in D\}$$

has an asymptotic distribution τ_h, say. Let U be a given non-trivial unitary representation of G such that

(6.8) $$\lim_{j\to\infty} m_j^{-1} \sum_{h=h_0}^{m_j} \left(1 - \frac{h}{m_j}\right) \int U(x)\tau_h(dx) = 0,$$

for at least one sequence $\{m_j\}$ of positive integers tending to infinity. Then

(6.9) $$\lim_n \int_{Z''} \Phi_n(z'')\Phi_n(z'')^* \rho_n(dz'') = 0,$$

where

(6.10) $$\Phi_n(z'') = \int_{Z'} \mu_{n,z}(U)\alpha_n(dz' \mid z'').$$

Proof. In view of (6.4), this is an immediate consequence of Theorem 5.2, with Z', $\alpha_n(dz' \mid z'')$ and $x_n(z)$ replaced by

$$Z' \times \Omega, \quad \alpha_n(dz' \mid z'')P(d\omega) \text{ and } X_n(z, \omega),$$

respectively; (a slightly stronger conclusion would result if in stead we replace Z'' by $Z'' \times \Omega$).

Theorem 6.1 is obtained as the special case of Theorem 6.2, where Z is the set of positive integers, α_n the measure of mass a_{nk}

at $k \in Z$ ($k = 1, 2, \ldots$), T the translation $z' = z+L$; finally, the $X_n(z) = X(z)$ ($z = 1, 2, \ldots$) are taken as *independent* random variables, such that $X(k)$ has a preassigned distribution σ_k, compare the remarks following Theorem 6.1.

7. Further applications

(I) Suppose that in Lemma 5.3 the quantity $x_n(z)$ and/or the measure α_n depends on a hidden parameter $\lambda \in \Lambda$. If further $\lim_n \Psi_{n,h} = 0$ uniformly in λ then the left hand side of (5.13) tends to zero uniformly in λ; this remark implies a result of Hlawka [7, p. 10].

Actually, uniform convergence of a net $\{\psi_n(\lambda),\ n \in D\}$ is nothing but ordinary convergence of the net

$$\{\psi_n(\lambda), \quad (n, \lambda) \in D \times \Lambda\},$$

where $D \times \Lambda$ is the directed set defined by $(n, \lambda) \leqq (n', \lambda')$ if and only if $n \leqq n'$. In other words, uniform convergence with respect to a hidden parameter can simply be handled by merely replacing D by a new directed set.

(II) Suppose we take Z as the collection of all vectors $(y_1, \ldots, y_r)$ in r-dimensional Euclidean space having positive integral coordinates. Let D be any directed set. For each $n \in D$, let Q_n be a given r-dimensional interval in Z of the form

$$c_{nj} \leqq y_j \leqq d_{nj}, \qquad j = 1, \ldots, r,$$

($c_{nj} \leqq d_{nj}$ positive integers), and let α_n denote the measure on Z defined by

$$\alpha_n(A) = |A \cap Q_n|/|Q_n| \text{ for each set } A \subset Z,$$

($|B|$ denoting the number of elements in B). Then (4.6) reduces to

$$\mu_n(f) = |Q_n|^{-1} \sum_{z \in Q_n} f(x_n(z)).$$

Let us further choose T as the translation

$$T(y_1, y_2, \ldots, y_r) = (y_1+L, y_2, \ldots, y_r),$$

with L as a fixed positive integer. Finally, assume that (4.4) holds; this is equivalent to the assumption that

$$\lim_n (d_{n,1}-c_{n,1}) = +\infty.$$

The resulting special case of Theorem 5.1 generalizes a result of van der Corput [6, p. 411]. More can be said on applying Theorem 5.2.

(III) Let us finally consider the case where Z is chosen as the interval $(0, \infty)$ of all positive real numbers, together with the σ-field of all Lebesgue measurable subsets of Z; this case was first studied in detail by Kuipers [13].

Let D be any directed set. For each $n \in D$, let α_n denote the measure on Z defined by

$$\alpha_n(A) = \int_A a_n(z)dz. \tag{7.1}$$

Here, $a_n(z)$ denotes a given Lebesgue measurable function on $(0, \infty)$, such that

$$a_n(z) \geqq 0, \qquad \int_0^\infty a_n(z)dz = 1. \tag{7.2}$$

Suppose further that, for each fixed $z_0 > 0$,

$$\lim_n \int_0^{z_0} a_n(z)dz = 0; \quad \lim_n \int_0^\infty |a_n(z+z_0)-a_n(z)|dz = 0. \tag{7.3}$$

As an important illustration, choose

$$\begin{aligned} a_n(z) &= 1/d_n \text{ if } 0 \leqq z \leqq d_n, \\ &= 0 \quad\ \text{ if } z > d_n, \end{aligned} \tag{7.4}$$

where d_n is any positive function on D such that $\lim d_n = \infty$. In any case, the assumption (7.3) implies that the basic condition (4.4) with respect to each transformation from Z into Z of the special form

$$Tz = z+L, \tag{7.5}$$

where L is a fixed positive real number.

Let $\{x_n(z), n \in D\}$ be a *given* net of measurable functions from Z to a fixed compact group G.

Definition. By H we shall denote the collection of all positive numbers h, such that the net

$$\{x_n(z+h)-x_n(z), n \in D\}$$

is asymptotically uniformly distributed with respect to the summation method $\{\alpha_n, n \in D\}$, that is

$$\lim_{n \in D} \int_Z U(x_n(z+h)-x_n(z))a_n(z)dz = 0,$$

for each non-trivial irreducible unitary representation $U = U(x)$ of G.

For $L > 0$, m as a positive integer, put

$$H_L(m) = [\text{number of } k = 1, \ldots, m \text{ with } kL \notin H]. \tag{7.6}$$

It follows from Lemma 5.3 (with $Z' = Z$) that the net $\{x_n(z), n \in D\}$ is itself uniformly distributed (with respect to the summation method $\{\alpha_n, n \in D\}$), provided that

$$\varliminf_{m\to\infty} \frac{1}{m} \inf_{L>0} H_L(m) = 0. \tag{7.7}$$

Condition (7.7) holds, for instance, if H contains a measurable subset of density 1 at 0 (such as an interval $(0, c)$ with $c > 0$), and it holds also if, for some $L > 0$, we have $kL \in H$ for all positive integers k (outside some set of lower density zero). This generalizes a result of Hlawka [9] who took $a_n(z)$ of the form (7.4), $x_n(z) = x(z)$ independent of n (and continuous in z for the case $(0, c) \subset H$).

By an obvious modification of Lemma 5.3, one easily obtains the more general result that $\{x_n(z), n \in D\}$ is uniformly distributed whenever one can find *arbitrarily large* sets $\{z_1, \ldots, z_m\}$ of positive numbers z_j such that, for all but $o(m^2)$ pairs (z_i, z_j) with $z_i < z_j$, one has $z_j - z_i \in H$. However, it is not sufficient that H contains an interval as can be seen from counter examples of the type $x_n(z) = x(z) = y([z/c])$, where $\{y(k), k = 1, 2, \ldots\}$ is a sequence having two successive differences uniformly distributed modulo 1.

Let $L > 0$ be fixed, and let us consider in a little more detail the case that

$$kL \in H \text{ for all } k = 1, 2, \ldots, \tag{7.8}$$

(except for the integers k in some set of lower density zero). One may regard the measurable space Z as the direct product of

$$Z' = \{0, L, 2L, \ldots\} \text{ and } Z'' = \{z'' : 0 \leqq z'' < L\}.$$

In the notation (5.3), one has

$$\rho_n(dz'') = R_n(z'')dz'', \text{ where } R_n(z) = \sum_{k=0}^{\infty} a_n(z+kL),$$

($R_n(z) < \infty$ for almost all z). By (7.8) and Theorem 5.2, we have, for each non-trivial irreducible unitary representation U of G, that

$$\lim_n \int_0^L \Phi_n(z|U)\Phi_n(z|U)^* \, R_n(z)dz = 0, \tag{7.9}$$

where

$$\Phi_n(z|U) = R_n(z)^{-1} \sum_{k=0}^{\infty} a_n(z+kL)U(x_n(z+kL))$$

if $R_n(z) > 0$; (actually, (7.9) holds under much weaker conditions than (7.8), see Lemma 5.3.)

Now, suppose that G is second countable (so that we need to consider only denumerable many representations U) and further that D contains a countable cofinal subset.

It follows from (7.9) that each cofinal subset of D contains a cofinal *subsequence* $\{n_j\}$ such that for almost all $0 \leqq z < L$ one has

$$\lim_{j\to\infty} \Phi_{n_j}(z|U)R_{n_j}(z)^{1/2} = 0,$$

for *each* non-trivial irreducible unitary representation U of G.

Let us finally assume that $a_n(z)$ is of the special form (7.4) with $d_n \to \infty$. Then $R_n(z) \to 1/L$, consequently, the above $\{n_j\}$ is such that, for *almost all* $0 \leqq z < L$ we have the sequence $\{x_{n_j}(z+kL)\}$ uniformly distributed in the sense that

$$\lim_{j\to\infty} (L/d_{n_j}) \sum_{k=1}^{[d_{n_j}/L]} f(x_{n_j}(z+kL)) = \int_G f(x)\nu(dx),$$

for each $f \in C(G)$, ν denoting the Haar measure on G. In particular, if D denotes the positive integers and $d_n = n$, and further $x_n(z) = x(z)$ is independent of n, it follows from (7.8) that, for almost all real numbers $z > 0$, the sequence $\{x(z+kL),\ k = 1, 2, \ldots\}$ either has no distribution at all or it has the uniform distribution, (both with respect to the ordinary Cesaro summation).

REFERENCES

BOURBAKI, N.

[1] Éléments de Mathématique, Livre VI, Intégration, Hermann et Cie, Act. Sc. et Ind. no. 1175, Paris, 1952.

CASSELS, J. W. S.

[2] An Introduction to Diophantine Approximation, Cambridge University Press, Cambridge Tracts no. 345, Cambridge, 1957.

CIGLER, J.

[3] Folgen normierter Maße auf kompakten Gruppen, Z. Wahrscheinlichkeitstheorie vol. 1 (1962) pp. 3—13.

CIGLER, J.

[4] Über eine Verallgemeinerung des Hauptsatzes der Theorie der Gleichverteilung, J. für d. reine angew. Math. vol. 210 (1962) pp. 141—147.

CIGLER J. und HELMBERG, G.
[5] Neuere Entwicklungen der Theorie der Gleichverteilung, Jahresber. Deutsch. Math. Ver. vol. 64 (1961) pp. 1—50.

CORPUT, J. G. VAN DER
[6] Diophantische Ungleichungen I, Zur Gleichverteilung Modulo Eins, Acta Math. vol. 56 (1931) pp. 373—456.

HLAWKA, E.
[7] Zur formalen Theorie der Gleichverteilung in kompakten Gruppen, Rend. Circ. Mat. Palermo Ser. II vol. 4 (1955) pp. 1—15.

HLAWKA, E.
[8] Folgen auf kompakten Räumen, Abh. math. Sem. Univ. Hamburg vol. 20 (1956) pp. 223—241.

HLAWKA, E.
[9] Über C-Gleichverteilung, Ann. Mat. pura appl. Ser. IV vol. 49 (1960) pp. 311—326.

KEMPERMAN, J. H. B.
[10] Generalized distributions modulo 1, Int. Congress of Math., Abstracts p. 36, Stockholm, 1962.

KEMPERMAN, J. H. B.
[11] Probability methods in the theory of distributions modulo 1, this Vol., p. 106.

KOROBOV, N. M. and POSTNIKOV, A. G.
[12] Some general theorems on the uniform distribution of fractional parts, Dokl. Akad. Nauk SSSR vol. 84 (1952) pp. 217—220.

KUIPERS, L.
[13] De asymptotische verdeling modulo 1 van de waarden van meetbare functies, Thesis, Amsterdam, 1947.

TSUJI, M.
[14] On the uniform distribution of numbers mod. 1, J. Math. Soc. Japan vol. 4 (1952) pp. 313—322.

(Oblatum 29-5-63). University of Rochester.