COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 15 (1962-1964), p. 64-69

<http://www.numdam.org/item?id=CM_1962-1964__15__64_0>

On the Number of Representations of an Integer as a Sum of Primes belonging to given Arithmetical Progressions

by

A. Zulauf

1. **Introduction:** Let $K_1, \ldots, K_s$ be s given positive integers, and K their least common multiple. Let further $a_1, \ldots, a_s$ be given integers, $(a_\sigma, K_\sigma) = 1$ $(\sigma = 1, \ldots, s)$, and denote by $\kappa(n)$ the number of sets of residues $x_1, \ldots, x_s$ (mod K) which (i) are relatively prime to K, and (ii) satisfy the following system of congruences.

$$\begin{cases} x_\sigma \equiv a_\sigma \pmod{K_\sigma} \quad (\sigma = 1, \ldots, s), \\ \sum_{\sigma=1}^{s} x_\sigma \equiv n \pmod{K}. \end{cases}$$

Finally let $N(n)$ denote the number of representations of the positive integer n in the form

(1) $\quad n = p_1 + p_2 + \ldots + p_s, \quad p_\sigma \equiv a_\sigma \pmod{K_\sigma} \quad (\sigma = 1, \ldots, s)$

where the p_σ are odd prime numbers.

I have proved †) that if $n \equiv s \pmod 2$, and $s > 2$‡), then

(2) $$N(n) = \kappa(n) \frac{1}{\varphi^s(K)(s-1)!} \frac{n^{s-1}}{\log^s n} \mathfrak{S}^*(n) + O\left\{\frac{n^{s-1} \log\log n}{\log^{s+1} n}\right\},$$

where φ denotes Euler's function,

$$\mathfrak{S}^*(n) = \Omega K \prod_{p \nmid nK} \left\{1 - \left[\frac{-1}{p-1}\right]^s\right\} \times \prod_{\substack{p \mid n \\ p \nmid K}} \left\{1 - \left[\frac{-1}{p-1}\right]^{s-1}\right\},$$

where p runs through the odd prime numbers, and where $\Omega = 1$, or $= 2$, according as K is even, or odd.

In this paper we shall prove an explicit formula for $\kappa(n)$.

2. **Lemma:** Let p be a prime number $\geqq 2$, let $s \geqq 1$, $1 \leqq l \leqq L$, and let $\bar{\kappa}_p(s, l, L; n)$ denote the number of sets of residues $x_1, \ldots, x_s$ (mod p^L) which satisfy simultaneously

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†) see [1], p. 228.

‡) If $s = 2$ this results holds for "almost all" n. See [3] and [4].

$$\begin{cases} (x_\sigma, p) = 1 & (\sigma = 1, \ldots, s), \\ \sum_{\sigma=1}^{s} x_\sigma \equiv n \pmod{p^l}. \end{cases}$$

Then

$$\bar{\kappa}_p(s, l, L; n) = \begin{cases} p^{s(L-1)-l}\,(p-1)\,\{(p-1)^{s-1}-(-1)^{s-1}\} \text{ if } p|n \\ p^{s(L-1)-l}\,\{(p-1)^{s}-(-1)^{s}\} \text{ if } p \nmid n. \end{cases}$$

PROOF. The lemma is evidently true for $s = 1$, since the system

$$\{(x_1, p) = 1;\ x_1 \equiv n \pmod{p^l}\}$$

has p^{L-l}, or no, solutions $x_1 \pmod{p^L}$ according as $p \nmid n$ or $p|n$. Next, assuming the lemma true for $s = t$, we shall prove that it is also true for $s = t+1$.

The system

$$\{(x_{t+1}, p) = 1;\ x_{t+1} \equiv n - \sum_{\sigma=1}^{t} x_\sigma \pmod{p^l}\}$$

has p^{L-l}, or no, solutions $x_{t+1} \pmod{p^L}$ according as

$$\sum_{\sigma=1}^{t} x_\sigma \not\equiv n \pmod{p},$$

or not. Thus

$$\bar{\kappa}_p(t+1, l, L; n) = p^{L-l} \sum_{\substack{m=1 \\ m \not\equiv n(p)}}^{p} \bar{\kappa}_p(t, 1, L; m). \tag{3}$$

If $p|n$ then $p \nmid m$ for every $m \not\equiv n \pmod{p}$, and we obtain, by assumption, from (3)

$$\begin{aligned} \bar{\kappa}_p(t+1, l, L; n) &= p^{L-l}(p-1)p^{t(L-1)-1}\{(p-1)^t-(-1)^t\} \\ &= p^{(t+1)(L-1)-l}(p-1)\{(p-1)^t-(-1)^t\}. \end{aligned}$$

If $p \nmid n$ then $p \nmid m$ for $(p-2)$ residues $m \not\equiv n \pmod{p}$, and $p|m$ for one residue $m \not\equiv n \pmod{p}$. In this case we obtain, therefore, from (3)

$$\begin{aligned} \bar{\kappa}_p(t+1, l, L; n) &= p^{L-l+t(L-1)-1}[(p-2)\{(p-1)^t-(-1)^t\} \\ &\qquad + (p-1)\{(p-1)^{t-1}-(-1)^{t-1}\}] \\ &= p^{(t+1)(L-1)-l}\{(p-1)^{t+1}-(-1)^{t+1}\}. \end{aligned}$$

It follows, by induction, that our lemma is true for all $s \geqq 1$.

3. **Theorem.** Let $\kappa(n)$ be defined as in the introduction. Let $k = (K_1, \ldots, K_s)$, let q run through all prime factors of K, and put

$$m_q = n - \sum_{\substack{\sigma=1 \\ q|K_\sigma}}^{s} a_\sigma, \qquad s_q = \sum_{\substack{\sigma=1 \\ q \nmid K_\sigma}}^{s} 1.$$

Then

$$\kappa(n) = K^{s-1}k \times \prod_{\sigma=1}^{s} K_\sigma^{-1} \times \prod_{q \nmid km_q} q^{-s_q}\{(q-1)^{s_q}-(-1)^{s_q}\}$$
$$\times \prod_{\substack{q \nmid k \\ q | m_q}} q^{-s_q}(q-1)\{(q-1)^{s_q-1}-(-1)^{s_q-1}\},$$

or $\kappa(n) = 0$, according as

$$n \equiv \sum_{\sigma=1}^{s} a_\sigma \pmod{k}, \tag{4}$$

or not.

PROOF. Since the congruences

$$\{x_\sigma \equiv a_\sigma \pmod{K_\sigma} \quad (\sigma = 1, \ldots, s), \sum_{\sigma=1}^{s} x_\sigma \equiv n \pmod{K}\}$$

imply

$$n \equiv \sum_{\sigma=1}^{s} x_\sigma \equiv \sum_{\sigma=1}^{s} a_\sigma \pmod{k},$$

it is trivial that $\kappa(n) = 0$ if n does not satisfy the congruence (4).

Suppose now that n does satisfy the congruence (4). Let

$$K = \prod_q q^{L_q}, \ k = \prod_q q^{l_q}, \quad K_\sigma = \prod_q q^{\lambda_{q\sigma}}, \prod_{\sigma=1} K_\sigma = \prod_q q^{\Sigma_q},$$

so that

$$0 \leqq l_q \leqq \lambda_{q\sigma} \leqq L_q \quad (\sigma = 1, \ldots, s), \ L_q \geqq 1, \ \sum_{\sigma=1}^{s} \lambda_{q\sigma} = \Sigma_q.$$

If $\kappa_q(n)$ denotes the number of sets of residues $x_1, \ldots, x_s$ (mod q^{L_q}) which satisfy

$$\begin{cases} x_\sigma \equiv a_\sigma \pmod{q^{\lambda_{q\sigma}}}, \quad (x_\sigma, q) = 1 \quad (\sigma = 1, \ldots, s), \\ \sum_{\sigma=1}^{s} x_\sigma \equiv n \pmod{q^{L_q}}, \end{cases}$$

then, obviously,

$$\kappa(n) = \prod_q \kappa_q(n). \tag{5}$$

For finding the value of $\kappa_q(n)$, there is no loss of generality in assuming that

$$l_q = \lambda_{q1} \leqq \lambda_{q2} \leqq \cdots \leqq \lambda_{qs} = L_q.$$

Consider first the case $q|k$. If $q|k$ we have $s_q = 0$ and

$$(6) \qquad 1 \leqq l_q = \lambda_{q1} \leqq \lambda_{q\sigma} \leqq L_q \qquad (\sigma = 1, \ldots, s).$$

Since $\lambda_{q1} = l_q$, and hence

$$\sum_{\sigma=2}^{s} (L_q - \lambda_{q\sigma}) = (s-1)L_q + l_q - \Sigma_q,$$

the number of different sets of residues $x_2, \ldots, x_s \pmod{q^{L_q}}$ satisfying

$$(7) \qquad x_\sigma \equiv a_\sigma \pmod{q^{\lambda_{q\sigma}}} \qquad (\sigma = 2, \ldots, s)$$

is evidently given by

$$q^{(s-1)L_q + l_q - \Sigma_q}.$$

Now the congruence

$$x_1 \equiv n - \sum_{\sigma=2}^{s} x_\sigma \pmod{q^{L_q}}$$

uniquely determines a residue $x_1 \pmod{q^{L_q}}$, and if x_1 is so determined, then, by (6), (7) and (4), also

$$(8) \qquad x_1 \equiv n - \sum_{\sigma=2}^{s} x_\sigma \equiv n - \sum_{\sigma=2}^{s} a_\sigma \equiv a_1 \pmod{q^{\lambda_{q1}}}.$$

Since $(a_\sigma, K_\sigma) = 1$ and $\lambda_{q\sigma} \geqq 1$, the congruences (7) and (8) imply $(x_\sigma, q) = 1$ $(\sigma = 1, \ldots, s)$, and hence we conclude that

$$(9) \qquad \kappa_q(n) = q^{(s-1)L_q + l_q - \Sigma_q}$$

if $q|k$, and n satisfies (4).

Now consider the case $q \nmid k$. If $q \nmid k$ we have $1 \leqq s_q \leqq s-1$, and

$$(10) \quad 0 = l_q = \lambda_{q1} = \cdots = \lambda_{qs_q} < \lambda_{q(s_q+1)} \leqq \cdots \leqq \lambda_{qs} = L_q.$$

Since $0 = l_q = \lambda_{q1} = \cdots = \lambda_{qs_q}$, and hence

$$\sum_{\sigma=s_q+1}^{s} (L_q - \lambda_{q\sigma}) = (s-s_q)L_q + l_q - \Sigma_q,$$

the number of different sets of residues $x_{s_q+1}, \cdots, x_s \pmod{q^{L_q}}$ which satisfy

$$(11) \qquad x_\sigma \equiv a_\sigma \pmod{q^{\lambda_{q\sigma}}} \quad (s_q < \sigma \leqq s)$$

is evidently given by

$$q^{(s-s_q)L_q + l_q - \Sigma_q}.$$

It follows that there are

$$(12)\qquad q^{(s-s_q)L_q+l_q-\Sigma_q}\bar{\kappa}_q(s_q, L_q, L_q; n-\sum_{\sigma=s_q+1}^{s} x_\sigma)$$

different sets of residues $x_1, \ldots, x_s$ (mod q^{L_q}) which satisfy the congruences (11) and

$$\begin{cases} (x_\sigma, q) = 1 & (\sigma = 1, \ldots, s_q), \\ \sum_{\sigma=1}^{s} x_\sigma \equiv n \pmod{q^{L_q}}. \end{cases}$$

But $\lambda_{q\sigma} = 0$ and, consequently,

$$x_\sigma \equiv a_\sigma \pmod{q^{\lambda_{q\sigma}}} \quad \text{for } \sigma = 1, \ldots, \sigma_q.$$

Further, the congruences (11) and the conditions $(a_\sigma, K_\sigma) = 1$ imply

$$(x_\sigma, q) = 1 \quad \text{for } s_q < \sigma \leqq s,$$

since then $\lambda_{q\sigma} \geqq 1$. Hence, if $q \nmid k$, then $\kappa_q(n)$ is given by the expression (12). By (10) and (11), the condition

$$q | \{n - \sum_{\sigma=s_q+1}^{s} x_\sigma\}$$

is equivalent to $q|m_q$. We deduce, therefore, from (12) and the lemma that, in the case $q \nmid k$,

$$(13)\quad \kappa_q(n) = \begin{cases} q^{(s-1)L_q+l_q-\Sigma_q-s_q}(q-1)\{(q-1)^{s_q-1}-(-1)^{s_q-1}\} & \text{if } q|m_q \\ q^{(s-1)L_q+l_q-\Sigma_q-s_q}\{(q-1)^{s_q}-(-1)^{s_q}\} & \text{if } q \nmid m_q. \end{cases}$$

The truth of our theorem, when n satisfies (4), is thus established by (5), (9) and (13).

4. **Conclusion.** We have $\kappa(n) > 0$ if simultaneously

$$(14a)\qquad n \equiv s \pmod 2,$$

$$(14b)\qquad n \equiv \sum_{\sigma=1}^{s} a_\sigma \pmod k,$$

$$(14c)\qquad n \not\equiv \sum_{\substack{\sigma=1\\ \sigma\neq\sigma^*}}^{s} a_\sigma \pmod q \quad \begin{cases} \text{for every odd prime number } q \text{ which} \\ \text{divides all } K_\sigma \text{ except one, } K_{\sigma^*} \text{ say.} \end{cases}$$

(Condition (14c) may be stated as $q \nmid m_q$ for every odd prime number $q|K$ for which $s_q = 1$.)

It follows that all sufficiently large integers n satisfying the conditions (14) can be represented as a sum of primes in the form (1), and (2) will be an asymptotic formula for the number of such representations. [1])

To prove the above statement about $\kappa(n) > 0$, we observe that, since $(a_\sigma, K_\sigma) = 1$,

$$m_2 = n - \sum_{\substack{\sigma=1 \\ 2|K_\sigma}}^{s} a_\sigma \equiv n-(s-s_2) \equiv s_2 \pmod 2$$

provided that n satisfies (14a). Hence s_2 is odd if $2 \nmid m_2$, and (s_2-1) is odd if $2|m_2$. It follows that, if (14a) and (14b) are satisfied, then $\kappa(n)$ vanishes only if there is an odd prime number q for which $s_q = 1$ and $q|m_q$.

[1]) The above conclusions could also be drawn from general results proved in my paper [2].

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March 19th, 1959.

(Oblatum 30-5-1959).