COMPOSITIO MATHEMATICA

FU CHENG HSIANG

On the convergence of a lacunary trigonometrical series

Compositio Mathematica, tome 15 (1962-1964), p. 28-33
<http://www.numdam.org/item?id=CM_1962-1964__15__28_0>

© Foundation Compositio Mathematica, 1962-1964, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (http://http://www.compositio.nl/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

On the Convergence of a Lacunary Trigonometrical Series

by

Fu Cheng Hsiang

1. A lacunary trigonometrical series is a series for which the terms different from zero are very sparse. This kind of trigonometrical series may be put in the form:

$$\sum_{\nu=1}^{\infty} (a_\nu \cos n_\nu x + b_\nu \sin n_\nu x).$$

We here assume, for simplicity, that the constant term of the series also vanishes. A series $\Sigma\, c_i$ is said to possess a gap (u, v) if $c_i = 0$ for $u < i < v$. It is known that [3, p. 251, § 10.31] *if a series* $\Sigma\, C_i$ *possesses infinitely many gaps* (m_k, m'_k) *such that* $m'_k/m_k > \mu > 1$ *for some* μ *and is summable* $(C, 1)$ *to sum* S, *then* S_{m_k} *and also* $S_{m'_k}$ *converges to* S.

2. Suppose that the above trigonometrical series is the Fourier series of an integrable function $f(x)$. This series possesses infinitely many gaps $(n_\nu, n_{\nu+1})$ such that $n_{\nu+1}/n_\nu > \lambda > 1$ for some λ. Write, for a given point x_0,

$$\varphi(t) = \varphi_{x_0}(t) = \tfrac{1}{2}\{f(x_0+t)+f(x_0-t)-2f(x_0)\}.$$

Since

$$\varphi^*(t) = \frac{1}{t}\int_0^t |\varphi|\, du = 0(1) \quad (t \to +0)$$

holds for almost all x, the Fourier series of an integrable function is therefore almost everywhere summable $(C, 1)$ by Fejér's theorem. From this fact, we draw immediately the following well-known

Kolmogoroff's Theorem [1, 2]. *If the Fourier series of an integrable function* $f(x)$ *possesses infinitely many gaps* $(n_\nu, n_{\nu+1})$ *such that* $n_{\nu+1}/n_\nu > \mu > 1$, *then the partial sum* S_{n_ν} *converges almost everywhere to* $f(x)$.

We can also conclude that, at a given point x_0 at which $\varphi^*(t) = 0(1)$ $(t \to +0)$, if $n_{\nu+1}/n_\nu > \lambda > 1$, then $S_{n_\nu}(x_0) \to f(x_0)$ as $\nu \to \infty$.

3. We write

$$\varphi_1(t) = \frac{1}{t}\int_0^t \varphi\, du,$$

$$\varphi_2(t) = \frac{1}{t}\int_0^t \varphi_1\, du.$$

In this note, we replace the condition $\varphi^*(t) = 0(1)$ by the weaker condition $\varphi_1(t) = 0(1)$ $(t \to +0)$. We develop Kolmogoroff's theorem into the following manner.

Theorem. *If the lacunary Fourier series of an integrable function $f(x)$ possesses infinitely many gaps $(n_\nu, n_{\nu+1})$ such that $n_{\nu+1}/n_\nu > \lambda > 1$, and if, at a given point x_0,* (i) $\varphi_1(t) = 0(1)$ $(t \to +0)$ *and*

(ii) $$\int_0^t |d\varphi_2| = O(1)$$

when $0 < t \leqq \eta$ for some η, then $S_{n_\nu}(x_0) \to f(x_0)$ as $\nu \to \infty$.

4. Now, we are in a position to prove the theorem. Take, for instance, $n_\nu = n$, $n_{\nu+1} = m$ and denote respectively by $D_n(t)$ and $K_n(t)$ Dirichlet's and Fejér's kernels. Then

$$D_n(t) = \tfrac{1}{2} + \sum_{\nu=1}^{n} \cos \nu t,$$

$$(n+1)K_n(t) = \sum_{\nu=0}^{n} D_\nu(t) = \frac{\sin^2 (n+1)t/2}{2\sin^2 t/2}.$$

Then, from the identity

$$\begin{aligned} mK_{m-1}(t) - nK_{n-1}(t) &= \sum_{\nu=n}^{m-1} D_\nu(t) \\ &= (m-n)D_n(t) + \sum_{\nu=1}^{m-n-1} (m-n-\nu)\cos(n+\nu)t \end{aligned}$$

and in virtue of the special property of the lacunary Fourier series, we obtain

$$\begin{aligned} S_n(x_0) - f(x_0) &= \frac{1}{\pi}\int_0^\pi \varphi(t)D_n(t)dt \\ &= \frac{1}{\pi(m-n)}\int_0^\pi \varphi(t)\left(mK_{m-1}(t) - nK_{n-1}(t)\right)dt. \end{aligned}$$

Write

$$\Phi(t) = \int_0^t \varphi\, du.$$

Integration by parts gives

$$S_n(x_0)-f(x_0)=\frac{1}{\pi(m-n)}[\Phi(t)\,(mK_{m-1}(t)-nK_{n-1}(t))]_0^\pi$$

$$-\frac{1}{\pi(m-n)}\int_0^\pi \Phi(t)\frac{d}{dt}(mK_{m-1}(t)-nK_{n-1}(t))\,dt$$

$$=0(1)-\frac{1}{\pi(m-n)}\int_0^\pi \Phi(t)\,(mK'_{m-1}(t)-nK'_{n-1}(t))\,dt$$

$$=0(1)-\frac{1}{2\pi(m-n)}\left(\tfrac{1}{2}\int_0^\pi \Phi(t)\frac{m\sin mt-n\sin nt}{\sin^2 t/2}dt\right.$$

$$\left.-\int_0^\pi \Phi(t)\frac{\sin^2 mt/2-\sin^2 nt/2}{\sin^3 t/2}\cos t/2\,dt\right)$$

$$=0(1)-\frac{1}{2\pi(m-n)}(I_{1/2}-I_2),$$

say. We are going to estimate the orders of the integrals I_1 and I_2 respectively. We write

$$I_1=m\int_0^\pi \Phi\frac{\sin mt}{\sin^2 t/2}dt-n\int_0^\pi \Phi\frac{\sin nt}{\sin^2 t/2}dt$$

$$=mI_3-nI_4,$$

say. Since $(2\sin t/2)^{-2}-t^{-2}$ is bounded, we obtain, by Riemann-Lebesgue's theorem,

$$I_3=4\int_0^\pi \Phi\frac{\sin mt}{t^2}dt+0(1)$$

$$=4\int_0^\pi \varphi_1\frac{\sin mt}{t}dt+0(1)$$

$$=0(1)$$

as $m\to\infty$ by De la Vallée Poussin's test [8, p. 38, § 2.8] for the convergence of Fourier series at a given point by the condition (ii) and $\varphi_2(t)=0(1)$ as $t\to+0$. Similarly,

$$I_4=4\int_0^\pi \varphi_1\frac{\sin nt}{t}dt+0(1)$$

$$=0(1)$$

as $n \to \infty$ by the same test. It follows that

$$\begin{aligned}\frac{1}{m-n}\cdot I_1 &= 0\left(\frac{m}{m-n}\right)+0\left(\frac{n}{m-n}\right)\\ &= 0\left(\frac{1}{1-\lambda^{-1}}\right)+0\left(\frac{1}{\lambda-1}\right)\\ &= 0\,(1)\end{aligned}$$

as $n \to \infty$. In estimating the order of I_2, let us write

$$\begin{aligned}I_2 &= \int_0^{\pi} \varphi_1 \frac{\sin^2 mt/2-\sin^2 nt/2}{\sin^2 t/2}\cdot\frac{t}{\tan t/2}\,dt\\ &= \left(\int_0^{\pi/(m-n)} + \int_{\pi/(m-n)}^{\delta} + \int_{\delta}^{\pi}\right)\\ &= I_5 + I_6 + I_7,\end{aligned}$$

say. We have

$$\begin{aligned}|I_5| &\leqq 4 \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)| \int_0^{\pi/2(m-n)} \frac{|\sin^2 mt-\sin^2 nt|}{\sin^2 t}\left|\frac{t}{\tan t}\right| dt\\ &= 4 \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)|\, I_5',\end{aligned}$$

say. Now,

$$I_5' \leqq \int_0^{\pi/2(m-n)} \frac{|\sin(m+n)t\sin(m-n)t|}{\sin^2 t}\,dt.$$

Considering that

$$\left|\frac{\sin \alpha t}{t}\right| \leqq \alpha$$

for $0 \leqq t \leqq \pi/2$ and $\alpha \geqq 0$, we get

$$\begin{aligned}I_5' &\leqq (m^2-n^2)\int_0^{\pi/2(m-n)} dt\\ &= \frac{\pi}{2}(m+n).\end{aligned}$$

Therefore,

$$|I_5| \leqq 2\pi \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)|\,(m+n).$$

Moreover, we have

$$|I_6| \leqq 4 \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)| \int_{\pi/2(m-n)}^{\delta/2} csc^2\, t\, dt$$
$$< 8 \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)| \cdot \frac{\pi}{2} \cdot \frac{2(m-n)}{\pi}$$
$$= 8 \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)|\,(m-n).$$

Last, by the second mean value theorem, we obtain

$$|I_7| \leqq \frac{A}{\delta^3},$$

where A is an absolute constant. From the above analysis, it follows that

$$|S_n(x_0)-f(x_0)| < 0(1) + \frac{1}{2\pi(m-n)}(m\,|I_3| + n\,|I_4|$$
$$+ 2\pi \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)|\,(m+n)$$
$$+ 8 \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)|\,(m-n)$$
$$+ A/\delta^3)$$
$$= 0(1) + \frac{1}{2\pi}\left(\frac{m}{m-n}|I_3| + \frac{n}{m-n}|I_4|\right.$$
$$+ 2\pi \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)| \frac{m+n}{m-n}$$
$$\left. + 8 \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)| + \frac{A}{(m-n)\delta^3}\right)$$
$$< 0(1) + \frac{1}{2\pi}\left(\frac{1}{1-\lambda^{-1}}|I_3| + \frac{1}{\lambda-1}|I_4|\right.$$
$$+ 2\pi \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)| \frac{1+\lambda^{-1}}{1-\lambda^{-1}}$$
$$\left. + 8 \operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)| + \frac{A}{(m-n)\delta^3}\right).$$

For a given $\varepsilon > 0$, we can choose a δ so small that

$$\operatorname*{Max}_{0\leqq t\leqq\delta} |\varphi_1(t)| < \varepsilon$$

by the condition (i). After fixing δ, we take a sufficiently large integer n which makes $|I_3|$, $|I_4|$ and $(m-n)^{-1}\delta^{-3}$ all less than ε. Thus, we obtain finally

$$|S_n(x_0) - f(x_0)| < o(1) + \frac{1}{2\pi}\left(\frac{1}{1-\lambda^{-1}} + \frac{1}{\lambda-1} + 2\pi\frac{1+\lambda^{-1}}{1-\lambda^{-1}} + 8 + A\right)\varepsilon.$$

Since ε is an arbitrary small quantity, letting it tend to zero, we get

$$S_n(x_0) - f(x_0) \to 0,$$

i.e.,

$$\lim_{\nu\to\infty} S_{n_\nu}(x_0) = f(x_0).$$

This proves the theorem.

REFERENCES.

A. Kolmogoroff

[1] Une contribution à l'étude de la convergence des séries de Fourier, Fund. Math., 5 (1924), 96—97.

J. Marcinkiewicz

[2] A new proof of a theorem of Fourier series, Journ. London Math. Soc., 8 (1933), 279.

A. Zygmund

[3] Trigonometrical Series, Warsaw (1935).

National Taiwan University, Taipei, China.