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On the Convergence of a Lacunary Trigonometrical Series

by

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1. A lacunary trigonometrical series is a series for which the terms different from zero are very sparse. This kind of trigonometrical series may be put in the form:

\[ \sum_{r=1}^{\infty} (a_r \cos n_r x + b_r \sin n_r x). \]

We here assume, for simplicity, that the constant term of the series also vanishes. A series \( \Sigma c_i \) is said to possess a gap \((u, v)\) if \( c_i = 0 \) for \( u < i < v \). It is known that [3, p. 251, § 10.81] if a series \( \Sigma C_i \) possesses infinitely many gaps \((m_k, m_k')\) such that \( m_k'/m_k > \mu > 1 \) for some \( \mu \) and is summable \( (C, 1) \) to sum \( S \), then \( S_{m_k} \) and also \( S_{m_k'} \) converges to \( S \).

2. Suppose that the above trigonometrical series is the Fourier series of an integrable function \( f(x) \). This series possesses infinitely many gaps \((n_v, n_{v+1})\) such that \( n_{v+1}/n_v > \lambda > 1 \) for some \( \lambda \). Write, for a given point \( x_0 \),

\[ \varphi(t) = \varphi_{x_0}(t) = \frac{1}{2} \{ f(x_0 + t) + f(x_0 - t) - 2f(x_0) \}. \]

Since

\[ \varphi^*(t) = \frac{1}{t} \int_0^t |\varphi| \, du = 0(1) \quad (t \to +0) \]

holds for almost all \( x \), the Fourier series of an integrable function is therefore almost everywhere summable \((C, 1)\) by Fejér's theorem. From this fact, we draw immediately the following well-known

Kolmogoroff's Theorem [1, 2]. If the Fourier series of an integrable function \( f(x) \) possesses infinitely many gaps \((n_v, n_{v+1})\) such that \( n_{v+1}/n_v > \mu > 1 \), then the partial sum \( S_{n_v} \) converges almost everywhere to \( f(x) \).

We can also conclude that, at a given point \( x_0 \) at which \( \varphi^*(t) = 0(1) \) \((t \to +0)\), if \( n_{v+1}/n_v > \lambda > 1 \), then \( S_{n_v}(x_0) \to f(x_0) \) as \( v \to \infty \).
3. We write

$$\varphi_1(t) = \frac{1}{t} \int_0^t \varphi \, du,$$

$$\varphi_2(t) = \frac{1}{t} \int_0^t \varphi_1 \, du.$$

In this note, we replace the condition $\varphi^*(t) = O(1)$ by the weaker condition $\varphi_1(t) = O(1)$ ($t \to +0$). We develop Kolmogoroff's theorem into the following manner.

Theorem. If the lacunary Fourier series of an integrable function $f(x)$ possesses infinitely many gaps $(n_\nu, n_{\nu+1})$ such that $n_{\nu+1}/n_\nu > \lambda > 1$, and if, at a given point $x_0$, (i) $\varphi_1(t) = O(1)$ ($t \to +0$) and

(ii) \[ \int_0^t |d\varphi_2| = O(1) \]

when $0 < t \leq \eta$ for some $\eta$, then $S_\nu(x_0) \to f(x_0)$ as $\nu \to \infty$.

4. Now, we are in a position to prove the theorem. Take, for instance, $n_\nu = n$, $n_{\nu+1} = m$ and denote respectively by $D_n(t)$ and $K_n(t)$ Dirichlet’s and Fejér’s kernels. Then

$$D_n(t) = \frac{1}{2} + \sum_{\nu=1}^n \cos \nu t,$$

$$(n + 1)K_n(t) = \sum_{\nu=0}^n D_\nu(t) = \frac{\sin^2 (n + 1) t/2}{2 \sin^2 t/2}.$$

Then, from the identity

$$mK_{m-1}(t) - nK_{n-1}(t) = \sum_{\nu=n}^{m-1} D_\nu(t)$$

$$= (m-n)D_n(t) + \sum_{\nu=1}^{m-n-1} (m-n-\nu) \cos (n+\nu)t$$

and in virtue of the special property of the lacunary Fourier series, we obtain

$$S_n(x_0) - f(x_0) = \frac{1}{\pi} \int_0^\pi \varphi(t) D_n(t) \, dt$$

$$= \frac{1}{\pi(m-n)} \int_0^\pi \varphi(t) (mK_{m-1}(t) - nK_{n-1}(t)) \, dt.$$

Write

$$\Phi(t) = \int_0^t \varphi \, du.$$
Integration by parts gives

\[ S_n(x_0) - f(x_0) = \frac{1}{\pi(m-n)} \left[ \Phi(t) \left( mK_{m-1}(t) - nK_{n-1}(t) \right) \right]_0^\pi \]

\[ - \frac{1}{\pi(m-n)} \int_0^\pi \Phi(t) \frac{d}{dt} \left( mK_{m-1}(t) - nK_{n-1}(t) \right) dt \]

\[ = 0(1) - \frac{1}{\pi(m-n)} \int_0^\pi \Phi(t) \left( mK'_{m-1}(t) - nK'_{n-1}(t) \right) dt \]

\[ = 0(1) - \frac{1}{2\pi(m-n)} \left( \frac{1}{4} \int_0^\pi \Phi(t) \frac{m \sin nt - n \sin nt}{\sin^2 t/2} dt \right. \]

\[ \left. - \int_0^\pi \Phi(t) \frac{\sin^2 mt - \sin^2 nt}{\sin^2 t/2} \cos t/2 dt \right) \]

\[ = 0(1) - \frac{1}{2\pi(m-n)} (I_{12} - I_2), \]

say. We are going to estimate the orders of the integrals \( I_1 \) and \( I_2 \) respectively. We write

\[ I_1 = m \int_0^\pi \Phi \frac{\sin mt}{\sin^2 t/2} dt - n \int_0^\pi \Phi \frac{\sin nt}{\sin^2 t/2} dt \]

\[ = mI_3 - nI_4, \]

say. Since \((2 \sin t/2)^3 - t^3\) is bounded, we obtain, by Riemann-Lebesgue’s theorem,

\[ I_3 = 4 \int_0^\pi \Phi \frac{\sin mt}{t^2} dt + 0(1) \]

\[ = 4 \int_0^\pi \varphi_1 \frac{\sin mt}{t} dt + 0(1) \]

\[ = 0(1) \]

as \( m \to \infty \) by De la Vallée Poussin’s test [8, p. 88, § 2.8] for the convergence of Fourier series at a given point by the condition (ii) and \( \varphi_2(t) = 0(1) \) as \( t \to +0 \). Similarly,

\[ I_4 = 4 \int_0^\pi \varphi_1 \frac{\sin nt}{t} dt + 0(1) \]

\[ = 0(1) \]
as $n \to \infty$ by the same test. It follows that
\[
\frac{1}{m-n} \cdot I_1 = 0 \left( \frac{m}{m-n} \right) + 0 \left( \frac{n}{m-n} \right)
\]
\[
= 0 \left( \frac{1}{1-\lambda^{-1}} \right) + 0 \left( \frac{1}{\lambda-1} \right)
\]
\[
= 0(1)
\]
as $n \to \infty$. In estimating the order of $I_2$, let us write
\[
I_2 = \int_0^\pi \varphi_1 \frac{\sin^2 mt/2 - \sin^2 nt/2}{\sin^2 t/2} \cdot \frac{t}{\tan t/2} \, dt
\]
\[
= \left( \int_0^{\pi/(m-n)} + \int_{\pi/(m-n)}^\varphi + \int_\varphi^\pi \right)
\]
\[
= I_5 + I_6 + I_7,
\]
say. We have
\[
|I_6| \leq 4 \max_{0 \leq t \leq \delta} |\varphi_1(t)| \int_0^{\pi/2(m-n)} \frac{\sin^2 mt - \sin^2 nt}{\sin^2 t} \left| \frac{t}{\tan t} \right| \, dt
\]
\[
= 4 \max_{0 \leq t \leq \delta} |\varphi_1(t)| \, I_6',
\]
say. Now,
\[
I_6' \leq \int_0^{\pi/2(m-n)} \frac{\sin (m+n)t \sin (m-n)t}{\sin^2 t} \, dt.
\]
Considering that
\[
\left| \frac{\sin \alpha t}{t} \right| \leq \alpha
\]
for $0 \leq t \leq \pi/2$ and $\alpha \geq 0$, we get
\[
I_6' \leq (m^2-n^2) \int_0^{\pi/2(m-n)} \, dt
\]
\[
= \frac{\pi}{2} (m+n).
\]
Therefore,
\[
|I_6| \leq 2\pi \max_{0 \leq t \leq \delta} |\varphi_1(t)| (m+n).
\]
Moreover, we have

\[ |I_6| \leq 4 \max_{0 \leq t \leq \delta} |\varphi_1(t)| \int_{n/2(m-n)}^{t/2} \csc^2 t \, dt \]

\[ < 8 \max_{0 \leq t \leq \delta} |\varphi_1(t)| \cdot \frac{\pi}{2} \cdot \frac{2(m-n)}{\pi} \]

\[ = 8 \max_{0 \leq t \leq \delta} |\varphi_1(t)| (m-n). \]

Last, by the second mean value theorem, we obtain

\[ |I_7| \leq \frac{A}{\delta^3}, \]

where \( A \) is an absolute constant. From the above analysis, it follows that

\[ |S_n(x_0) - f(x_0)| < 0(1) + \frac{1}{2\pi(m-n)} (m |I_3| + n |I_4|) \]

\[ + 2\pi \max_{0 \leq t \leq \delta} |\varphi_1(t)| (m + n) \]

\[ + 8 \max_{0 \leq t \leq \delta} |\varphi_1(t)| (m-n) \]

\[ + A/\delta^3) \]

\[ = 0(1) + \frac{1}{2\pi} \left( \frac{m}{m-n} |I_3| + \frac{n}{m-n} |I_4| \right) \]

\[ + 2\pi \max_{0 \leq t \leq \delta} |\varphi_1(t)| \frac{m+n}{m-n} \]

\[ + 8 \max_{0 \leq t \leq \delta} |\varphi_1(t)| + \frac{A}{(m-n)\delta^3} \]

\[ < 0(1) + \frac{1}{2\pi} \left( \frac{1}{1-\lambda^{-1}} |I_3| + \frac{1}{\lambda-1} |I_4| \right) \]

\[ + 2\pi \max_{0 \leq t \leq \delta} |\varphi_1(t)| \frac{1+\lambda^{-1}}{1-\lambda^{-1}} \]

\[ + 8 \max_{0 \leq t \leq \delta} |\varphi_1(t)| + \frac{A}{(m-n)\delta^3} \].

For a given \( \varepsilon > 0 \), we can choose a \( \delta \) so small that

\[ \max_{0 \leq t \leq \delta} |\varphi_1(t)| < \varepsilon \]

by the condition (i). After fixing \( \delta \), we take a sufficiently large
integer \( n \) which makes \( |I_3|, |I_4| \) and \((m-n)-1\delta^{-3}\) all less than \( \varepsilon \). Thus, we obtain finally

\[
|S_n(x_0) - f(x_0)| < 0(1) + \frac{1}{2\pi} \left( \frac{1}{1 - \lambda^{-1}} + \frac{1}{\lambda - 1} \right) + 2\pi \frac{1 + \lambda^{-1}}{1 - \lambda^{-1}} + 8 + A)\varepsilon.
\]

Since \( \varepsilon \) is an arbitrary small quantity, letting it tend to zero, we get

\[
S_n(x_0) - f(x_0) \to 0,
\]

i.e.,

\[
\lim_{n \to \infty} S_{n\nu}(x_0) = f(x_0).
\]

This proves the theorem.

REFERENCES.

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