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Autohomeomorphism Groups of 0-dimensional Spaces

by

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If T is a topological space, we denote by $A(T)$ the group of all homeomorphisms of T onto itself. In [2], it was shown that given an arbitrary group G, one can find a topological space T such that G and $A(T)$ are isomorphic; in fact, such a T can be found among the compact connected Hausdorff spaces. In general, no such T can be found among the spaces with a base of open – and – closed sets, i.e., the spaces T such that $\dim T = 0$. The present paper investigates the following question. What can be said, in general, about $A(T)$ if T is a completely regular Hausdorff space and $\dim T = 0$?

If α is any cardinal $\geqq 1$, we shall denote by S_α the restricted permutation group on α objects; that is, the group of all those permutations which involve only finitely many objects. We will find it convenient to let S_0 denote the group of one element. ΣC_2 will denote the direct sum of $\aleph_1$ groups of order two. Throughout this paper, "space" will be used to mean "completely regular Hausdorff space". For any 0-dimensional space T, we shall show that $A(T)$ must

(1) consist of a single element (in which case we say T is "rigid"),

(2) contain a subgroup S_α for some α,

or (3) contain a subgroup of the form $S_\alpha + \Sigma C_2$. This result is best possible, in the sense that for any cardinal α, we can construct spaces whose autohomeomorphism group is precisely S_α or $S_\alpha + \Sigma C_2$. We produce examples of arbitrarily high weight,[2]) but we leave open the problem of constructing *compact* rigid 0-dimensional spaces of arbitrarily high weight.

In particular, if T is dense in itself, $A(T)$ equals the unit

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[2]) The weight of a space is $\mathfrak{m}$ if there exists an open base of $\mathfrak{m}$ and not less than $\mathfrak{m}$ sets.

element or contains a subgroup ΣC_2. On the other hand, one can construct compact 0-dimensional Hausdorff spaces H, dense in itself, for which $A(H) = 1$ or $A(H)$ equals the direct sum of continuously many groups of order two (in the last case one takes the Čech-Stone compactification of [2; § 5, example I]).

Some of the results of this paper were announced in [3].

I. $A(T)$ for 0-dimensional Spaces

1.1. Lemma. Let $\{x_i\}$ and $\{y_i\}$, $i \in N$ (the natural numbers) be sets of distinct isolated points in the space T such that, for every $J \subset N$, $\{x_j\}$ and $\{y_j)$, $j \in J$, have identical boundaries in T; then T admits of uncountably many distinct autohomeomorphisms of order two.

Proof. It is easy to see that the map interchanging x_i and y_i for each i in N, and leaving all other points of T fixed, is an autohomeomorphism; the same is clearly true for every subset J of N, and there are uncountably many such subsets.

In what follows, we shall need the following well known (and easily proved) result from group theory.

1.2. Proposition. If G is a group in which all elements distinct from the identity have order two, then G can be represented as the direct sum of cyclic groups of order two.

1.3. Theorem. Let T be a 0-dimensional completely regular Hausdorff space, containing α isolated points (α may be 0). Then either $A(T) = S_\alpha$, or $A(T)$ contains a subgroup of the form $S_\alpha + \Sigma C_2$.

Proof. $A(T)$ clearly contains a subgroup isomorphic to S_α, since every one – one onto map moving a finite number of isolated points, and leaving all other points fixed, is a homeomorphism. Thus we need only show that if T admits any autohomeomorphism which does *more* than this, then T contains a subgroup isomorphic to $S_\alpha + \Sigma C_2$.

Note first of all that if $\alpha > \aleph_0$, there is no problem, since S_α itself contains such a subgroup. So we assume $\alpha \leqq \aleph_0$, and we distinguish two cases.

(1) There is an autohomeomorphism φ on T which moves a non-isolated point p. Then we can find an open – and – closed set U containing p such that $U \cap \varphi(U) = \phi$. If U has no countable base, we can find more than $\aleph_0$ distinct open- and -closed subsets $K \subset U$, and interchanging K and $\varphi(K)$ gives us an autohomeo-

morphism of order two. If U has a countable base, let $D = \{x_i\}$ be the set of all isolated points in U. If D is finite, then $M = (U \setminus D) \cup \varphi(U \setminus D)$ is open-and-closed, dense in itself, separable, metrizable and 0-dimensional, and is therefore homeomorphic to a dense-in-itself subset of the Cantor set. Since M is not rigid, $A(M)$ (and hence $A(T)$) contains a subgroup of the form ΣC_2, by [2; p. 90, (i)]. If D is infinite and closed, let $\{x_i\}$ be any enumeration of D; then $\{x_i\}$ and $\{\varphi(x_i)\}$ satisfy the hypotheses of Lemma 1.1; if D is not closed, it has a limit point q and a subsequence $\{y_i\}$ converging to q. In that case, $\{y_{2i-1}\}$ and $\{y_{2i}\}$ satisfy the hypotheses of 1.1.

(2) No autohomeomorphism moves a non-isolated point. Let φ be a homeomorphism moving an infinite set of isolated points $\{y_i\}$. If we can find a set of isolated points $\{x_i\}$ such that $\{x_i\} \cap \{\varphi x_i\} = \phi$, then $\{x_i\}$ and $\{\varphi x_i\}$ clearly satisfy the hypotheses of 1.1. But such a set $\{x_i\}$ is easily found, for if there is a $y \in \{y_i\}$ with infinite orbit, let $x_i = \varphi^{2i} y$; if each y_i has finite orbit, form $\{x_i\}$ by choosing one point from each of the orbits determined by y_i.

It follows that $A(T)$ contains a group isomorphic to ΣC_2; from the construction, it is easily seen that by dividing the isolated points into two disjoint infinite sets if necessary, one can find a subgroup isomorphic to $S_\alpha + \Sigma C_2$.

It should be pointed out that in only one case in the proof of 1.3 do we fail to find *continuously* many distinct autohomeomorphisms of order two. We could replace $\sum C_2$ in the statement of the theorem by the direct sum of continuously many groups of order two if we could prove the following: if U and V are 0-dimensional, disjoint homeomorphic spaces having no countable base, and $X = U \cup V$, then $A(X)$ contains $\mathbf{c}$ elements of order two.

II. Rigid Spaces

In this section, we extend the methods of [2] to produce rigid 0-dimensional spaces of arbitrary (infinite) weight. We shall require some ideas in the theory of uniform spaces; the reader is referred to [1] and [4] for a development of these ideas.

First, we extend a metric space theorem to uniform spaces in a routine manner.

2.1. Definition. An intersection of $\mathbf{m}$ open sets will be called a $G_{\mathbf{m}\delta}$-set; a $G_{\aleph_0\delta}$-set will be called, as usual, a G_δ-set.

2.2. Theorem. Let X be a completely regular Hausdorff space of weight $\mathbf{m}$, complete in a uniformity $\mathscr{D}$ generated by a set D of $\mathbf{m}$ pseudometrics. Then every continuous map f from a subset H of X into X can be extended continuously to a map $\bar{f}$ from a $G_{\mathbf{m}\delta}$-set $G \supset H$ into X.

PROOF. For each $d \in \mathscr{D}$, and each $x \in \bar{H}$, let $\omega_d(x)$ be the oscillation of f at x with respect to d. Let

$$G_d = \{x \in \bar{H} : \omega_d(x) = 0\}.$$

G_d is evidently a G_δ-set. Let

$$G = \bigcap_{d \in \mathscr{D}} G_d;$$

then $G \supset H$ is a $G_{\mathbf{m}\delta}$-set.

Now f can be extended continuously over G. For let $\{h_\alpha\}$ be any net in H converging to a point $x \in G$. Then, in the uniformity generated by $\mathscr{D}$, $\{f(h_\alpha)\}$ is a Cauchy net, by the definition of G. Hence $\{f(x_\alpha)\}$ converges to some point $p \in X$; set $\bar{f}(x) = p$. $\bar{f}$ is evidently continuous at x.

Now, using 2.2, we extend some of the results in [2].

2.3. DEFINITION. If X is a topological space, and f a map from a subset of X into X, then f is called a *continuous displacement of order* $\mathbf{M}$ if f is continuous, and is a displacement of order $\mathbf{M}$. A continuous displacement of order $\mathbf{c}$ will be called, as usual, a continuous displacement [2; § 2].

2.4. THEOREM. Let X be a completely regular Hausdorff space of weight $\mathbf{m}$, complete in a uniformity $\mathscr{D}$ generated by $\mathbf{m}$ pseudometrics, and let $|X| = 2^{\mathbf{m}} = \mathbf{M}$. Further, let $\{K_\beta\}$ be any family of $\mathbf{M}$ subsets of X, each of cardinal $\mathbf{M}$. Then there is a family $\{F_\gamma\}$ of $2^{\mathbf{M}}$ subsets of X such that

(1) For $\gamma \neq \gamma'$, $|F_\gamma \backslash F_{\gamma'}| = \mathbf{M}$.

(2) No F_γ admits of any continuous displacement of order $\mathbf{M}$ onto itself or any other $F_{\gamma'}$.

(3) For every β, γ, $|F_\gamma \cap K_\beta| = \mathbf{M}$, and $|(X \backslash F_\gamma) \cap K_\beta| = \mathbf{M}$.

PROOF. There exist only $\mathbf{M}$ $G_{\mathbf{m}\delta}$-sets in X, and a fixed subset of X admits at most $\mathbf{M}$ continuous maps into X, and therefore at most $\mathbf{M}$ continuous displacements of order $\mathbf{M}$. Let f_β be a continuous displacement of order $\mathbf{M}$ whose domain is a $G_{\mathbf{m}\delta}$-set. The family $\{f_\beta\}$ of all such mappings has cardinal at most $\mathbf{M}$. This family is non-empty (otherwise the theorem is trivial), so by counting a given displacement $\mathbf{M}$ times if necessary, we may assume that $|\{f_\beta\}| = \mathbf{M}$.

Now we apply [2; Lemma 1], with $X = N$, $\mathbf{M} = \mathbf{m}$, and $\{f_\beta\}$. We obtain a family $\{F_\gamma\}$ of $2^{\mathbf{M}}$ subsets of X satisfying (1) and (3). Suppose (2) is false, and there is a continuous displacement of order $\mathbf{M}$, φ, from F_γ onto $F_{\gamma'}$. This φ can be extended (Theorem 2.4) to a continuous map $\tilde{\varphi}$ of a $G_{\mathbf{m}\delta}$-set $G_\gamma \supset F$ into X, so $\tilde{\varphi} = f_\beta$ for some β. Hence, by [2; Lemma 1, (2.3)], for every pair γ, γ', $f_\beta F_\gamma \backslash F_{\gamma'} \neq \phi$, and so, since $\varphi = f_\beta$ on F_γ, $\varphi F_\gamma \backslash F_{\gamma'} \neq \phi$, i.e., φ maps F_γ onto no member of $\{F_\gamma\}$.

2.5. Lemma. Let P be a space in which every open set has cardinal at least $\mathbf{M}$. If $\varphi : P \to P$ is non-trivial, and is either locally topologically into P or continuous onto P, then φ is a displacement of order $\mathbf{M}$.

Proof. The proof is word for word the proof of [2; Lemma 2], with "$\aleph$" replaced by "$\mathbf{M}$", and "continuous displacement" replaced by "continuous displacement of order $\mathbf{M}$".

2.6. Theorem. Let X be a locally compact Hausdorff space of weight $\mathbf{m}$, complete in a uniformity generated by $\mathbf{m}$ pseudometrics, such that every open set in X has $2^{\mathbf{m}}$ points. Let K be the set of all compact subsets of X whose cardinal is $2^{\mathbf{m}}$. Then the sets $\{F_\gamma\}$ constructed in Theorem 2.4 are such that no $\{F_\gamma\}$ can be mapped topologically into or continuously onto itself or any other $F_{\gamma'}$.

Proof. Each open set in each F_γ will have $2^{\mathbf{m}}$ points. By Lemma 2.5 and (1), Theorem 2.4, any non-trivial φ satisfying either condition of the theorem is a continuous displacement of order $\mathbf{M}$. But this contradicts (2), Theorem 2.4.

2.7. Example. Theorem 2.6 enables us to construct many examples of rigid 0-dimensional spaces of arbitrary weight. For instance, let

$$X = \Pi_{\alpha \in A} X_\alpha,$$

where $|A| = \mathbf{m}$, and, for each α, X_α is a discrete space of cardinal two. Then X has weight $\mathbf{m}$, X is compact, and hence complete in any uniformity, so X is complete in a uniformity generated by $\mathbf{m}$ pseudometrics. Further, every open set in X contains $2^{\mathbf{m}}$ points. Now, applying Theorem 2.6, we get a collection of $2^{2^{\mathbf{m}}}$ sets $\{F_\alpha\}$, each of weight $\mathbf{m}$ and dimension 0, such that F_γ is rigid for each γ, and the F_γ are topologically distinct.

2.8. Problem. The rigid spaces constructed in the preceding example are proper dense subsets of a compact space, hence they are not themselves compact. We have not been able to construct examples of compact, rigid 0-dimensional spaces of arbitrarily high weight; such spaces would be of interest in the study of Boolean rings (see, for example, [2; § 8.1]).

III. Spaces whose Autohomeomorphism Groups are S_α or $S_\alpha + \Sigma C_2$.

If α is finite, the discrete space of cardinal α has S_α as its autohomeomorphism group. This is not the case for α infinite, of course. In Example 3.1, however, we produce for each infinite α a space having α isolated points whose autohomeomorphism group is precisely S_α. In Example 3.2, we find spaces whose autohomeomorphism group is the direct sum of S_α and the sum of *continuously* many groups of order two; this group is then isomorphic to $S + \Sigma C_2$ if we assume the continuum hypothesis. In this connection one should recall the remark following the proof of Theorem 1.3; it is conceivable that $\aleph_1$ can be replaced by $\mathbf{c}$ throughout this paper.

In both 3.1 and 3.2, the spaces S_p which play a part in the construction can evidently be chosen to have arbitrarily high weight, hence the same is true for our examples.

3.1. Example. Let P be a discrete space of cardinal α, and let βP be its (0-dimensional) Čech-Stone compactification. With each $p \in P$, we associate a 0-dimensional space S_p such that

(1) for each $p \in P$, S_p is rigid and dense-in-itself,

and (2) if p and q are distinct elements of P, then no non-empty open subset of S_p is homeomorphic to an open subset of S_q.

Such a collection $\{S_p\}$ can be constructed by using Example 2.7, as follows: with each $p \in P$, we associate a cardinal α_p such that if $p \neq q$, $2^{\alpha_p} \neq 2^{\alpha_q}$. Taking $\alpha_p = \mathbf{m}$ in 2.1, we obtain a rigid space which we can denote by S_p such that each open subset of S_p contains 2^{α_p} points. The collection $\{S_p\}$, $p \in P$ evidently satisfies (1) and (2).

Now let

$$X = \bigcup_{p \in P} S_p \cup \beta P.$$

We topologize X by prescribing a base for the open sets, consisting of

(i) the sets $\{p\}$, $p \in P$,

(ii) the open-and-closed sets in S_p for each $p \in P$,

(iii) the sets

$$U \cup \bigcup_{p \in U} S_p$$

where U is open-and-closed in P.

The space X so defined is evidently a 0-dimensional completely regular Hausdorff space. The topology on each S_p as a subspace of X is the same as its original topology.

Every mapping of X onto X which permutes a finite number of the (isolated) points of P and leaves all other points of X fixed, is clearly a homeomorphism. These are the only autohomeomorphisms of X. For if an autohomeomorphism φ leaves each $p \in P$ pointwise fixed, then the points of βP are fixed, so

$$\bigcup_{p \in P} S_p$$

must be mapped topologically on itself. But from (1) and (2), this space is rigid, so φ is the identity map. On the other hand, if φ displaces an infinite subset D of P, then φ must move some point of $\beta P \backslash P$ (since the closures of D and $\varphi(D)$ in βP are non-empty and disjoint), hence there is a $p \in P$ such that $S_p \cap \varphi S_p = \phi$. But $\varphi S_p \cap \beta P = \phi$, since no open set in S_p contains an isolated point. It follows that $\varphi S_p \cap S_{p'} \neq \phi$ for some $p \neq p'$, contradicting (2).

3.2. Example. For each α, we construct a space T_α such that $A(T_\alpha)$ is precisely $S_\alpha + \Sigma C_2$ (assuming the continuum hypothesis). Let M be a 0-dimensional subset of the real numbers such that $A(M)$ is the direct sum of continuously many groups of order two [2; § 5, Example I], and let X be the space constructed in Example 3.1, so that $A(X) = S_\alpha$. Let $T_\alpha = X \cup M$. If φ is any autohomeomorphism of T, then $x \in M$ if and only if $\varphi(x) \in M$, since $x \in M$ if and only if the least cardinal of a base at x is $\aleph_0$. It follows that $A(T_\alpha) = A(X) + A(M) = S_\alpha + \Sigma C_2$.

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