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# Proximate Orders and Distribution of $a$ -points of Meromorphic Functions

by

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§ 1. Let  $f(z)$  be a meromorphic function of order  $\rho$  ( $0 < \rho < \infty$ ) and lower order  $\lambda$  ( $0 \leq \lambda < \infty$ ). Let  $M(r, f)$ ,  $T(r, f)$ ,  $n(r, a)$ ,  $N(r, a)$  have their usual meanings.

We define  $\rho(r)$  to be proximate order  $D$  of  $f(z)$  for  $T(r, f)$ , having the following properties;

- 1.1  $\rho(r)$  is real, continuous and piecewise differentiable;
- 1.2  $\rho(r) \rightarrow \rho$  as  $r \rightarrow \infty$ ,
- 1.3  $r\rho'(r) \log r \rightarrow 0$  as  $r \rightarrow \infty$ ,
- 1.4  $T(r, f) \leq r^{\rho(r)}$  for  $r \geq r_0$   
 $= r^{\rho(r)}$  for a sequence of values of  $r \rightarrow \infty$ .

For the existence of this proximate order see [7] where  $\rho(r)$  is constructed with  $\log M(r, f)$  and  $f(z)$  is an entire function. The same reasoning may be applied to construct  $\rho(r)$  with the above properties. From the properties 1.1 to 1.4 we can deduce the following,

- 1.5  $r^{\rho(r)}$  is an increasing function of  $r \geq r_0$ .
- 1.6  $(ur)^{\rho(ur)} \sim u^{\rho} r^{\rho(r)}$  for  $r \geq r_0$ .
- 1.7  $n(r, a) < K r^{\rho(r)}$  for all  $r \geq r_0$ . [13]

§ 2. We define  $\lambda(r)$  to be proximate order  $L$  for  $f(z)$  for  $T(r, f)$  having the following properties.

- 2.1  $\lambda(r)$  is non-negative, continuous function of  $r$  for  $r \geq r_0$ .
- 2.2  $\lambda(r)$  is differentiable except at isolated points at which  $\lambda'(r-0)$  and  $\lambda'(r+0)$  exist.
- 2.3  $\lambda(r) \rightarrow \lambda$  as  $r \rightarrow \infty$ .
- 2.4  $r\lambda'(r) \log r \rightarrow 0$  as  $r \rightarrow \infty$ .
- 2.5  $T(r, f) \geq r^{\lambda(r)}$  for  $r \geq r_0$ .  
 $= r^{\lambda(r)}$  for a sequence of values of  $r \rightarrow \infty$ .

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For the existence of this proximate order see [8] where  $\lambda(r)$  is constructed with  $\log M(r, f)$  and  $f(z)$  is an entire function. The same argument may be applied to construct  $\lambda(r)$  with the above properties.

From properties 2.1—2.5 we can easily deduce the following

2.6  $r^{\lambda(r)}$  is an increasing function of  $r \geq r_0$ .

2.7  $(ur)^{\lambda(ur)} \sim u^{\lambda} r^{\lambda(r)}$  for  $r \geq r_0$ . [4]

§ 3. Applying the properties of  $\rho(r)$  and  $\lambda(r)$  we prove a number of results. For convenience we set

3.1  $n(r) = n(r, a) + n(r, b)$

3.2  $N(r) = N(r, a) + N(r, b)$

where  $a \neq b, 0 \leq a \leq \infty, 0 \leq b \leq \infty$  and prove the following theorems

THEOREM 1. If

3.3  $\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} = \alpha < \infty$

and

3.4  $\frac{N(r)}{r^{\lambda(r)}} \rightarrow 0$  as  $r \rightarrow \infty$ .

Then for  $x \neq a, b$

$$1 = \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq \limsup_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq \alpha < \infty.$$

By putting  $b = \infty$ , we can easily deduce from this theorem the analogous result for entire functions. Also consider the following function

$$f(z) = \prod_1^{\infty} \left( 1 + \frac{z}{A_n} \right)^{k u_n}$$

where

$$\begin{aligned} k &= [\rho] + 1 \\ U_n &= A_n^{\rho+n} \\ A_n &= n^{n^n} \end{aligned}$$

then

$$\limsup_{r \rightarrow \infty} \frac{n(r, 0)}{\log m(r, f)} = \infty. \quad [6]$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{n(r, 0)}{r^{\lambda(r)}} = \infty$$

so that

$$\limsup_{r \rightarrow \infty} \frac{N(r, 0)}{r^{\lambda(r)}} = \infty. \quad [3]$$

Hence the condition 3.3 is essential.

**THEOREM 2.** If

$$3.6 \quad \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}} = \beta > 0$$

and

$$3.7 \quad \frac{N(r)}{r^{\rho(r)}} \rightarrow 0 \text{ as } r \rightarrow \infty$$

Then for  $x \neq a, b$ ,

$$3.8 \quad 0 < \beta \leq \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\rho(r)}} \leq \limsup_{r \rightarrow \infty} \frac{N(r, x)}{r^{\rho(r)}} \leq 1.$$

And since [3]

$$3.9 \quad 0 < \limsup_{r \rightarrow \infty} \frac{n(r, a)}{r^{\rho(r)}} < \infty$$

if and only if  $0 < \limsup_{r \rightarrow \infty} \frac{N(r, a)}{r^{\rho(r)}} < \infty$

we can easily deduce analogous results for entire functions by putting  $b = \infty$  and replacing  $N(r, a)$  by  $n(r, a)$ . See [13].

**§ 4.** To see whether the converse of theorem 1 and 2 is true or not we note that if  $N(r, x)/r^{\lambda(r)} \rightarrow \infty$ , then  $T(r, f)/r^{\lambda(r)} \rightarrow \infty$  as  $r \rightarrow \infty$  also. Hence without any restrictions on  $N(r, x)/r^{\lambda(r)}$  we cannot prove anything, in general. We prove the following

**THEOREM 3.**

If

$$4.1 \quad \limsup_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} < \infty \quad \text{for } x = a, b, c.$$

Then

$$4.2 \quad \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} < \infty.$$

Imposing more restrictions on  $f(z)$  we prove the following

**THEOREM 4.**

If  $f(z)$  is a meromorphic function of non-integral order where  $p(p \geq 1)$  is the genus and

$$4.3 \quad \limsup_{r \rightarrow \infty} \frac{N(r)}{r^{\lambda(r)}} = \alpha < \infty.$$

Then

$$4.4 \quad \frac{\alpha}{2} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \leq 3e(p+1)^2 \alpha (2 + \log p) \pi \operatorname{cosec} \pi(\lambda - p).$$

**THEOREM 5.**

If  $f(x)$  is a meromorphic function of non-integral order and genus  $p \geq 1$ , then

$$4.5 \quad \limsup_{r \rightarrow \infty} \frac{N(r)}{T(r, f)} \geq \frac{\sin \pi(\rho - p)}{3e\rho(2 + \log p)(1 + p)\pi}$$

$$4.6 \quad \geq \frac{\sin \pi(\rho - p)}{3e(2 + \log p)(1 + p)^2\pi}.$$

§ 5. S. K. Singh [10] has proved

If  $f(z)$  be an entire function of non-integral order, then

$$5.1 \quad \limsup_{r \rightarrow \infty} \frac{N(r, a)}{\log M(r, f)} > 0 \text{ for all } a, (0 \leq |a| < \infty).$$

S. M. Shah [8] has proved that for functions of order less than one

$$5.2 \quad \limsup_{r \rightarrow \infty} \frac{N(r, a)}{\log M(r, f)} \geq 1 - \rho$$

We here prove

**THEOREM 6.**

If  $f(z)$  be an entire function of non-integral finite order and genus  $p$ , and

$$5.3 \quad \limsup_{r \rightarrow \infty} \frac{N(r, a)}{r^{\lambda(r)}} = \alpha < \infty.$$

Then

$$5.4 \quad \frac{\alpha}{\lambda} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} \leq \pi \alpha 3e(p+1)^2 (2 + \log p) \operatorname{cosec} \pi(\lambda - p).$$

**THEOREM 7.**

If  $f(z)$  is an entire function of genus zero and  $0 < \lambda < 1$  and

$$5.5 \quad \limsup_{r \rightarrow \infty} \frac{n(r, a)}{r^{\lambda(r)}} = \alpha < \infty.$$

Then

$$5.6 \quad \frac{\alpha}{\lambda} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} \leq \pi \alpha \operatorname{cosec}(\pi \lambda).$$

**THEOREM 8.**

If  $f(z)$  is an entire function of non-integral order  $\rho$  and genus  $p$ , then

$$5.7 \quad \limsup_{r \rightarrow \infty} \frac{N(r, a)}{\log M(r, f)} \geq \frac{\sin \pi(\rho - p)}{3e(p + 1)^2(2 + \log p)\pi}.$$

**THEOREM 9.**

If  $f(z)$  is an entire function of order  $\rho$ ,  $0 < \rho < 1$  and genus zero, then

$$5.8 \quad \limsup_{r \rightarrow \infty} \frac{N(r, a)}{\log M(r, f)} \geq \frac{\sin \pi \rho}{\pi \rho}.$$

This theorem has been proved by Valirom [12], but we give a different proof by using proximate orders.

**§ 6. PROOF OF THEOREM 1.**

By 2.5 we have

$$6.1 \quad \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} = 1.$$

Also for  $x \neq a, b$

$$T(r, f) < N(r) + N(r, x) + o(\log r).$$

Hence

$$\begin{aligned} 1 &= \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \leq \liminf_{r \rightarrow \infty} \frac{N(r)}{r^{\lambda(r)}} + \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \\ &\leq \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \\ &\leq \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \\ &= 1 \end{aligned}$$

and the left hand equality follows.

The right hand inequality follows from the fact that  $N(r, x) \leq T(r, f)$  for all  $x$  and the theorem is proved.

**PROOF of THEOREM 2.**

By 1.4 we have

$$6.2 \quad \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}} = 1$$

and so the right hand inequality is obvious.

To prove the left hand inequality, suppose if possible

$$\liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\rho(r)}} = 0 \quad \text{for } x \neq a, b.$$

Hence

$$\left[ \frac{N(r)}{r^{\rho(r)}} + \frac{N(r, x)}{r^{\rho(r)}} \right] \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

and so

$$\frac{T(r, f)}{r^{\rho(r)}} \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

and this contradicts 3.6 and the theorem follows.

**PROOF of THEOREM 3.**

Let

$$\limsup_{r \rightarrow \infty} \frac{N(r, x_i)}{r^{\lambda(r)}} = \alpha_i \quad (i = 1, 2, 3).$$

Then

$$N(r, x_i) < (\alpha_i + \varepsilon_i) r^{\lambda(r)} \quad (i = 1, 2, 3).$$

We have

$$\begin{aligned} T(r, f) &\leq \sum_{i=1}^3 N(r, x_i) + o(\log r) \\ &\leq \sum_{i=1}^3 (\alpha_i + \varepsilon_i) r^{\lambda(r)} + o(\log r) \\ &= \beta r^{\lambda(r)} + o(\log r) \quad (\beta < \infty). \end{aligned}$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \leq \beta < \infty$$

and the Theorem follows.

**PROOF OF THEOREM 4.**

Since

$$T\left(r, \frac{\alpha f + \beta}{r f + \delta}\right) = T(r, f) \mathcal{O}(1)$$

we may suppose  $a = 0$ , and  $b = \infty$ , without any loss of generality and so we have

$$6.3 \quad n(r) = n(r, 0) + n(r, \infty)$$

$$6.4 \quad N(r) = N(r, 0) + N(r, \infty).$$

Also we know [5] that

$$6.5 \quad T(r, f) \leq \mathcal{O}(r^p) + 3e(2 + \log p)(1 + p) \int_0^\infty \frac{n(t)r^{p+1}dt}{t^{p+1}(t+r)}$$

By lemma 1 [2] we have

$$6.6 \quad \int_0^\infty \frac{n(t)r^{p+1}dt}{t^{p+1}(t+r)} \leq (p+1) \int_0^\infty \frac{N(t)r^{p+1}dt}{t^{p+1}(t+r)}.$$

Setting  $S = 3e(2 + \log p)(1 + p)^2$  and since from 4.3

$$N(r) \leq (\alpha + \varepsilon)r^{\lambda(r)} = \beta r^{\lambda(r)} \quad (\beta < \infty)$$

we get

$$T(r, f) \leq S\beta \int_0^\infty \frac{t^{\lambda(t)}r^{p+1}dt}{t^{p+1}(t+r)} + \mathcal{O}(r^p).$$

Put  $t = ur$

$$T(r, f) \leq S\beta \int_0^\infty \frac{(ur)^{\lambda(ur)}r^{p+1}r du}{(ur)^{p+1}(ur+r)} + \mathcal{O}(r^p)$$

$$\sim S\beta \int_0^\infty r^{\lambda(r)} \frac{u^{\lambda-p-1}}{u+1} du + \mathcal{O}(r^p), \quad \text{by 2.7}$$

$$\sim S\beta r^{\lambda(r)} \pi \operatorname{cosec} \pi(\lambda - p) + \mathcal{O}(r^p), \quad \text{since } 0 < \lambda - p < 1.$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \leq S\alpha\pi \operatorname{cosec} \pi(\lambda - p)$$

and the right hand inequality is proved.

The left hand inequality is obvious since  $N(r) \leq 2T(r, f)$  and the theorem follows.



**PROOF OF THEOREM 5.**

From 1.7 we have

$$\limsup_{r \rightarrow \infty} \frac{n(r)}{r^{\rho(r)}} = H_1 < \infty.$$

Also since

$$6.7 \quad \int_{r_0}^r t^{\rho(t)-1} dt \sim \frac{r^{\rho(r)}}{\rho} \tag{1}$$

$$6.8 \quad N(r) \leq \frac{H}{\rho} r^{\rho(r)}.$$

From [5] we have

$$6.9 \quad T(r, f) \leq O(r^{\rho}) + 3e(2 + \log p)(1 + p) \int_0^{\infty} \frac{n(t)r^{\rho+1}dt}{t^{\rho+1}(t + r)}.$$

Applying lemma 1 [2] we get

$$6.10 \quad T(r, f) \leq O(r^{\rho}) + 3e(2 + \log p)(1 + p)^2 \int_0^{\infty} \frac{N(t)r^{\rho+1}}{t^{\rho+1}(t + r)} dt.$$

In 6.10, set  $S = 3e(2 + \log p)(1 + p)^2$ .

Using 6.8 we have

$$\begin{aligned} T(r, f) &\leq O(r^{\rho}) + S \int_0^{\infty} \frac{H}{\rho} \frac{t^{\rho(t)} r^{\rho+1}}{t^{\rho+1}(t + r)} dt \\ &\leq O(r^{\rho}) + \frac{S.H.}{\rho} \int_0^{\infty} \frac{(ur)^{\rho(ur)} r^{\rho+1}}{(ur)^{\rho+1}(ur + r)} du \\ &\sim O(r^{\rho}) + \frac{S.H.}{\rho} r^{\rho(r)} \int_0^{\infty} \frac{u^{\rho-p-1}}{u + 1} du. \end{aligned}$$

Hence

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}} &\leq S.\pi. \operatorname{cosec} \pi (\rho - p) \frac{H}{\rho} \\ &\leq S.\pi. \operatorname{cosec} \pi(\rho - p) \cdot \limsup_{r \rightarrow \infty} \frac{N(r)}{r^{\rho(r)}}. \end{aligned}$$

So

$$\liminf_{r \rightarrow \infty} \frac{T(r, f)}{N(r)} \leq \frac{\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}}}{\limsup_{r \rightarrow \infty} \frac{N(r)}{r^{\rho(r)}}} \leq S.\pi. \operatorname{cosec} \pi(\rho - p)$$

and 4.6 follows.

Starting with 6.9 and proceeding similarly we have 4.5 and we note that 4.6 is a better inequality than 4.5, since  $\rho < p + 1$ . Proofs of Theorems 6 and 8 are omitted since they are similar to the proofs of Theorems 4 and 5.

**PROOF OF THEOREM 7**

$$\log f(z) \leq r \int_0^{\infty} \frac{n(t, a)}{t(t+r)} dt. \quad [11]$$

From 5.6,

$$n(r, a) \leq (\alpha + \varepsilon)r^{\lambda(r)} = \beta r^{\lambda(r)}, \quad \beta < \infty.$$

Hence

$$\begin{aligned} \log M(r, f) &\leq r\beta \int_0^{\infty} \frac{t^{\lambda(r)}}{t(t+r)} dt \\ &\sim \beta r^{\lambda(r)} \int_0^{\infty} \frac{u^{\lambda}}{u(u+1)} dt && \text{by 2.7.} \\ &= \beta r^{\lambda(r)} \frac{\pi}{\sin \pi\lambda} \end{aligned}$$

$$\limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} \leq \frac{\alpha\pi}{\sin \pi\lambda}.$$

Left hand inequality is obvious.

**PROOF OF THEOREM 9.**

From 1.4 we have

$$N(r, a) \leq T(r, f) \leq r^{\rho(r)}.$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{N(r, a)}{r^{\rho(r)}} = \alpha \leq 1$$

we have [11]

$$\begin{aligned} \log M(r, f) &\leq \int_0^{\infty} \frac{n(t)r}{t(t+r)} dt \\ &\leq \int_0^{\infty} \frac{N(t)r}{(t+r)^2} dt \\ &\leq \int_0^{\infty} \alpha \frac{t^{\rho(t)}r}{(t+r)^2} dt \\ &\sim \alpha \int_0^{\infty} \frac{r^{\rho(r)}u^{\rho}}{(u+1)^2} du. \end{aligned}$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\rho(r)}} \leq \frac{\alpha \pi \rho}{\sin \pi \rho}$$

$$\liminf_{r \rightarrow \infty} \frac{\log M(r, f)}{N(r, a)} \leq \frac{\limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\rho(r)}}}{\limsup_{r \rightarrow \infty} \frac{N(r, a)}{r^{\rho(r)}}} \leq \frac{\pi \rho}{\sin \pi \rho}.$$

Lastly we note that if we use the properties of lower proximate order and assume

$$\limsup_{r \rightarrow \infty} \frac{N(r, a)}{r^{\lambda(r)}} < \infty.$$

Then we have

$$\limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{N(r, a)} \leq \frac{\pi \lambda}{\sin \pi \lambda}$$

and since

$$\frac{\pi \lambda}{\sin \pi \lambda} \leq \frac{\pi \rho}{\sin \pi \rho}$$

and so in one way we have a better inequality.

#### REFERENCES

M. L. CARTWRIGHT

[1] Integral functions. Cambridge 1958. pp. 58.

S. H. DWIVEDI

[2] On entire functions of finite order. The Math. Student, Vol. 26, No. 4, 1958. pp. 169—172.

S. H. DWIVEDI

[3] Proximate orders and distribution of  $a$ -points of entire function. M.R.C. Technical report No. 259. 1961.

S. H. DWIVEDI and S. K. SINGH

[4] The distribution of  $a$ -points of an entire function. Proc. Amer. Math. Soc. Vol. 9, No. 4, 1958. pp. 562—568.

R. NEVANNLINNA

[5] Eindeutige Analytische Funktionen 2 Aufl. 1953, pp. 227.

S. M. SHAH

[6] A note on maximum modulus and zeros of an integral function. Bull. Amer. Math. Soc. Vol. 46, 1940, pp. 909—912.

S. M. SHAH

- [7] On proximate orders of integral functions. *Bull. Amer. Math. Soc.* Vol. 52, 1942. pp. 326—328.

S. M. SHAH

- [8] A note on meromorphic functions. *The Math. Student.* Vol. 12, 1944.

S. M. SHAH

- [9] A note on lower proximate orders. *J. Indian Math. Soc.* Vol. 12, 1948, pp. 31—32.

S. K. SINGH

- [10] A note on entire and meromorphic functions. *Proc. Amer. Math. Soc.* Vol. 9, No. 1, 1958.

E. C. TITCHMARSH

- [11] *The theory of functions*, 1950, pp. 271.

G. VALIRON

- [12] Sur le minimum, du module des fonctions entières d'ordres inférieurs a un, *Mathematica*, Vol. 11, 1935, pp. 264—269.

G. VALIRON

- [13] *The general theory of integral functions*, Chelsia 1949, pp. 68.

University of Wisconsin Milwaukee.

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