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PAWAN KUMAR KAMTHAN

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A Theorem on the Zeros of an Entire Function

by

Pawan Kumar Kamthan, Birla College, Pilani

1. Our aim in this note is to prove the following theorem.

THEOREM: If $P(z)$ is a canonical product of genus p and order ρ ($\rho > p$) defined by:

$$P(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n} \right) \exp \left\{ z/z_n + \frac{1}{2}(z/z_n)^2 + \dots + \frac{1}{p} (z/z_n)^p \right\},$$

where z_1, z_2, \dots etc. are the zeros of $P(z)$ whose moduli r_1, r_2, \dots etc. form a non-decreasing sequence such that $r_n > 1$ for all n and where $r_n \rightarrow \infty$ as $n \rightarrow \infty$, then for z in a domain exterior to the circles of radius r_n^{-h} ($h > \rho$) described about the zeros z_n as centres, we have

$$\left| \frac{P'(z)}{P(z)} \right| < K \int_0^{\infty} \frac{n(x)r^{\rho}}{x^{\rho}(x+r)^2} dx,$$

where K is a constant independent of p and $P'(z)$ is the first derivative of $P(z)$ and $n(x)$ denotes the number of zeros within and on the circle $|z| = x$.

PROOF: It is sufficient to differentiate $\log P(z)$ in a region in which it is regular. Such a region can always be found out: and before we tackle this problem, we would, however, like to arrange the zeros in the following way.

Let $\kappa (> 1)$ and $\kappa' (> 1)$ be two numbers so suitably chosen that the zeros of moduli r_{N+1}, r_{N+2}, \dots etc. lie outside the circle with centre origin and radius κr and the zeros of moduli r_1, r_2, \dots, r_N lie inside the annular region of outer radius κr and inner radius $r(\kappa')^{-1}$ respectively (these later zeros may also lie on the outer circumference of the annulus).

Now we indent all the zeros by small circles of radii r_n^{-h} ($h > \rho$; $n = 1, 2, \dots$). But $\sum_{n=1}^{\infty} r_n^{-h}$ is convergent since $h > \rho$ and hence after exclusion of these circles we are still left with a domain which does not include these so drawn circles. This means that if we take a point z in this excluded region, then $|z - r_n| > r_n^{-h}$.

Now we return to the mathematical formulation of the problem.

We write

$$P(z) = P_N(z)Q(z), \quad (\text{A})$$

where

$$P_N(z) = \prod_{n=1}^N E(z/z_n, p),$$

and

$$Q(z) = \prod_{n=N+1}^{\infty} E(z/z_n, p),$$

$E(z/z_n, p)$ being Weierstrass's primary factor. Now from the expression for $P(z)$ we have

$$\log P(z) = \sum_{n=1}^{\infty} \left\{ \log \left(1 - \frac{z}{z_n} \right) + \left(z/z_n + \frac{1}{2}(z/z_n)^2 + \dots + \frac{1}{p}(z/z_n)^p \right) \right\}.$$

We can differentiate the above expression in the excluded region, for the right-hand side is regular, and uniformly and absolutely convergent. We have then

$$\begin{aligned} \frac{P'(z)}{P(z)} &= \sum_{n=1}^{\infty} \left\{ \frac{-1}{z_n(1-z/z_n)} + \frac{1}{z_n} \left(1 + \frac{z}{z_n} + \dots + \left(\frac{z}{z_n} \right)^{p-1} \right) \right\} \\ &= \sum_{r/\kappa' < r_n \leq \kappa r} + \sum_{r_n \geq \kappa r} = \sum_1 + \sum_2 \end{aligned} \quad (\text{1})$$

ESTIMATION OF \sum_1 : Let us write $r/r_n = u_n$. Then, since $|1-z/z_n| \geq |1-r/r_n|$ we have

$$|\sum_1| \leq \sum_{n=1}^N \left\{ \frac{1}{r_n|1-u_n|} + \frac{1}{r_n} (1+u_n + \dots + u_n^{p-1}) \right\},$$

where $N = n(\kappa r)$. Again in \sum_1 , $\kappa' > u_n \geq 1/\kappa$ and so

$$\begin{aligned} |\sum_1| &\leq \sum_{n=1}^N \frac{1}{r_n|1-u_n|} + \sum_{n=1}^N \frac{u_n^{p-1}}{r_n} \left(1 + \frac{1}{u_n} + \dots + \frac{1}{u_n^{p-1}} \right) \\ &\leq \sum_{n=1}^N \frac{1}{r_n|1-u_n|} + \sum_{n=1}^N \frac{u_n^{p-1}}{r_n} (1+\kappa + \dots + \kappa^{p-1}) \\ &\leq \sum_{n=1}^N \frac{1}{r_n|1-u_n|} + K_1 \sum_{n=1}^N \frac{u_n^{p-1}}{r_n} \end{aligned}$$

But $|1-u_n| > r_n^{-h-1}$.

Hence

$$\begin{aligned} |\Sigma_1| &< \sum_{n=1}^N r_n^h + K_1 \sum_{n=1}^N \frac{u_n^{p-1}}{r_n} < K_2 + K_1 \sum_{n=1}^N \frac{u_n^{p-1}}{r_n} \\ &< K_3 \sum_{n=1}^N \frac{u_n^{p-1}}{r_n} < K_4 \sum_{n=1}^N \frac{u_n^p}{r_n(1+u_n)^2} \end{aligned} \quad (2)$$

where K_4 depends on κ and κ' .

ESTIMATION OF Σ_2 : We have

$$\Sigma_2 = \sum_{n=N+1}^{\infty} \left\{ \frac{-1}{z_n(1-z/z_n)} + \frac{1}{z_n} \left(1 + \frac{z}{z_n} + \dots + \frac{z^{p-1}}{z_n^{p-1}} \right) \right\}.$$

But in Σ_2 , $|z/z_n| < 1/\kappa < 1$. Hence

$$\Sigma_2 = - \sum_{n=N+1}^{\infty} \left\{ (z/z_n)^p + (z/z_n)^{p+1} + \dots \right\} \frac{1}{z_n}.$$

Therefore,

$$\begin{aligned} |\Sigma_2| &\leq \sum_{n=N+1}^{\infty} \frac{1}{r_n} (u_n^p + u_n^{p+1} + \dots) \\ &< \sum_{n=N+1}^{\infty} \frac{u_n^p}{r_n} \left(1 + \frac{1}{\kappa} + \frac{1}{\kappa^2} + \dots \right) \\ &= \frac{\kappa}{\kappa-1} \sum_{n=N+1}^{\infty} \frac{u_n^p}{r_n}. \end{aligned}$$

But $(1+u_n)^2 < (1+1/\kappa)^2$. So we get:

$$\begin{aligned} |\Sigma_2| &< \frac{\kappa}{\kappa-1} (1+1/\kappa)^2 \sum_{n=N+1}^{\infty} \frac{u_n^p}{r_n(1+u_n)^2} \\ &= K_5 \sum_{n=N+1}^{\infty} \frac{u_n^p}{r_n(1+u_n)^2}. \end{aligned} \quad (3)$$

Hence from (1), (2) and (3) we get:

$$\begin{aligned} \left| \frac{P'(z)}{P(z)} \right| &< K_6 \sum_{n=1}^{\infty} \frac{u_n^p}{r_n(1+u_n)^2}, \quad K_6 = K_6(\kappa, \kappa') \\ &= K_6 \sum_{n=1}^{\infty} n \left\{ \frac{u_n^p}{r_n(1+u_n)^2} - \frac{u_{n+1}^p}{r_{n+1}(1+u_{n+1})^2} \right\} \\ &= K_6 \sum_{n=1}^{\infty} n \int_{r_n}^{r_{n+1}} d \left(\frac{-(r/x)^p}{x(1+r/x)^2} \right) \end{aligned}$$

$$= K_6 \int_0^{\infty} \frac{n(x)r^p}{x^p(x+r)^2} \left\{ \frac{x(p+1)+r(p-1)}{x+r} \right\} dx.$$

Now the expression written within the curly bracket inside the integral sign is bounded in $(0, \infty)$ and monotonic increasing. Hence, we have finally

$$\left| \frac{P'(z)}{P(z)} \right| < K \int_0^{\infty} \frac{n(x)r^p}{x^p(x+r)^2} dx.$$

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Mathematics Department,
Birla College, Pilani, India

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