COMPOSITIO MATHEMATICA

B. H. NEUMANN

Ascending derived series

Compositio Mathematica, tome 13 (1956-1958), p. 47-64

<http://www.numdam.org/item?id=CM_1956-1958__13__47_0>

Ascending derived series

To my parents, for their 80th birthdays

by

B. H. Neumann

Manchester

§ 1. Introduction.

The term "derived group" and the notation H' for the commutator subgroup of a group H suggests an analogy — albeit tenuous — to the derivative of a function; and this in turn suggests the problem of "integration": Given a group G, to find a group H whose derived group is isomorphic to G. There may not be such a group H at all: the symmetric group of degree 3, of order 6, is not the derived group of any group [1]), and it is not difficult to make other similar examples. Thus one shows easily that no symmetric group of finite degree > 2 and no properly metacyclic group [2]) can be the derived group of any group. But beyond this, little seems to be known.

The iterated commutator groups form the derived series

$$H \supseteq H' \supseteq H'' \supseteq \ldots, \tag{1.1}$$

and this leads naturally to the more general question: given a group G and a positive integer n, to find a group H such that G is isomorphic to the nth derived group $H^{(n)}$; or to show that no such group exists. Next one may ask whether solutions exist for all n; it is with this problem that the present paper is largely concerned.

We are thus led to consider series of groups

$$G_0 \subset G_1 \subset G_2 \subset \ldots \tag{1.2}$$

such that $G_i = G'_{i+1}$ for $i = 0, 1, 2, \ldots$, and where we shall always assume that the first (and therefore every) inclusion is proper. We call such a series an *ascending derived series*, and for clearer distinction we call a series (1.1) a *descending derived series*.

A descending derived series may become stationary after a finite

---

[1]) van Aardenne-Ehrenfest [1]; Lips [6].

[2]) Cf. e.g. Zassenhaus [11], p. 138, Satz 9.

number of steps, when we arrive at a group $H^{(n)}$ that is its own derived group. This happens if $H^{(n)}$ is the trivial group, when H is called soluble of (derived) length n. If, on the other hand, $H^{(n)} = H^{(n+1)} \neq \{1\}$, or if the descending derived series has infinite length, then H is insoluble.

An ascending derived series can not become stationary. It can break off if we arrive at a group G_n that is not the derived group of any group; we then call n the length of the series. An ascending derived series can, however, also go on indefinitely, that is have infinite length. If G_0 is not its own derived group, then the series (1.2) can be continued to the left, possibly for an infinite number of terms. By taking infinite unions or intersections one can easily introduce derived series of various transfinite order types; but for the present we shall be content with finite series and infinite series of type ω.

A finite derived series can be considered either as descending or as ascending, according as one thinks of one end or the other as given. We shall here look upon the series as ascending, that is we shall derive properties of the series (1.2) from properties of the groups with which it begins. Our principal result is that the series breaks off if G_2/G_0 is finitely generated. By contrast it is not sufficient to assume G_1/G_0 or G_2/G_1 finite to ensure that the series breaks off. It follows that *in an infinite ascending derived series no metabelian factor G_{i+2}/G_i can be finitely generated, whereas an abelian factor G_{i+1}/G_i can even be finite.*

We mention two finiteness conditions that occur in the formulation of some of the results and that would perhaps merit closer examination than they receive in the present paper: the *maximal condition for soluble subgroups*, and the *ascending derived chain condition for subgroups*. They are introduced in § 4.

An auxiliary result may perhaps claim some independent interest: It is that a finitely generated strictly metabelian group (that is one that is not abelian) has a finite strictly metabelian homomorphic image. The same is not generally true of metabelian groups with an infinity of generators; nor is the analogous proposition valid for finitely generated soluble groups of length 3.

§ 2. Preliminaries.

The following two lemmas are stated for later reference. The reader will easily supply proofs.

LEMMA 2.1. *If θ is a homomorphism of the group G, then*

$$(G\theta)' = G'\theta.$$

Equivalently, if N is a normal subgroup of G, then

$$(G/N)' = G'N/N.$$

LEMMA 2.2. *If M is a subgroup of G and N a normal subgroup of G such that $MN = G$, then*

$$G' = M'N'[M, N] \subseteq M'N,$$

where $[M, N]$ is the group generated by all commutators $[a, b]$ with $a \in M$, $b \in N$.

We denote by G^* the union of the groups in an ascending derived series:

$$G^* = \cup G_i.$$

If the series has finite length n, then $G^* = G_n$; if the series is infinite, then G^* is its direct (or "inductive") limit. In this case G^* is evidently its own derived group.

If θ is a homomorphism of G^*, then, by Lemma 2.1,

$$G_i\theta = (G_{i+1}\theta)', \qquad i = 0, 1, 2, \ldots.$$

Thus if $G_i\theta \neq G_{i+1}\theta$ for some i, then

$$G_i\theta \subset G_{i+1}\theta \subset G_{i+2}\theta \subset \ldots$$

is again an ascending derived series. In particular if N is a normal subgroup of G^* contained in G_i, then the groups

$$H_j = G_{i+j}/N \qquad (j = 0, 1, 2, \ldots)$$

form an ascending derived series

$$H_0 \subset H_1 \subset H_2 \subset \ldots$$

with the union

$$H^* = G^*/N,$$

and the factors

$$H_{j+1}/H_j \simeq G_{i+j+1}/G_{i+j} \qquad (j = 0, 1, 2, \ldots).$$

In order to ensure that N is a normal subgroup of G^* it suffices to take N as a characteristic subgroup of G_i; for as G_i is characteristic in G_{i+1}, then N is also characteristic in G_{i+1}, and inductively in all G_{i+j}. Then N is *a fortiori* normal in all G_{i+j}, and thus normal [3]) in G^*. We shall often find it useful to take $N = G_0$; the resulting series starts with the trivial group and consists of soluble groups.

---

[3]) It is not permissible to conclude that N is characteristic in G^*; cf. infra.

To make an example [4]) we take G_0 as a free group of countably infinite rank. This is isomorphic to, but distinct from, its derived group; we may therefore consider it as the derived group of another free group G_1 of countably infinite rank; and so we continue, to obtain an infinite ascending derived series. The inductive limit G^* is a locally free group which coincides with its own derived group. The factor groups G_{i+1}/G_i are free abelian groups of countably infinite rank. If we reduce modulo [5]) G_0, that is if we consider the groups $H_i = G_i/G_0$, then we obtain an ascending derived series

$$\{1\} = H_0 \subset H_1 \subset H_2 \subset \ldots$$

consisting of soluble groups. The limit group H^* is locally soluble but coincides with its own derived group.

§ 3. Automorphisms induced in a series

We take i fixed for the present and consider the effect on G_i of the inner automorphisms of G^*. As a normal subgroup G_i admits the inner automorphisms of G^*, and they therefore induce a certain group Γ_i^* of automorphisms of G_i. We obtain Γ_i^* from G^* by a homomorphism θ_i with the centralizer C_i^* of G_i in G^* as kernel. Transformation by the elements of G_k induces a subgroup of Γ_i^* on G_i, namely

$$\Gamma_i^k = G_k\theta_i \simeq G_k C_i^*/C_i^*. \tag{3.1}$$

It follows that

$$\Gamma_i^0 \subseteq \Gamma_i^1 \subseteq \Gamma_i^2 \subseteq \ldots \tag{3.2}$$

becomes an ascending derived series after the omission of possibly equal terms at the beginning — provided, of course, that not all the terms of (3.2) are equal.

We note in passing that if $i<j$, then $C_i^* \supseteq C_j^*$, and thus $\theta_i \geqq \theta_j$. This means that Γ_i^k is a homomorphic image of Γ_j^k. We also remark that Γ_i^k consists of inner automorphisms of G_i when $k \leqq i$. An incidental consequence of the following lemma is that when $k > i \geqq 2$, Γ_i^k can never consist of inner automorphisms only. The lemma describes what happens when not all the inclusions in (3.2) are proper.

---

[4]) The construction is well known.

[5]) There is an obvious automorphism of G^* mapping each G_i onto G_{i+1}. Hence G_0, though of course normal, is not characteristic in G^*; cf. footnote 3).

LEMMA 3.3. *Let* $\Gamma_i^{k+1} = \Gamma_i^k$ *for some values* i, k. *Then either* $i = 0$, *or* $k = 0$, *or* $i = 1$ *and* $G_0' = G_0$ *and* G_1/G_0 *is contained in the centre of* G_{k+1}/G_0.

PROOF. We suppose both $i > 0$ and $k > 0$, as otherwise there is nothing to prove. As in (3.2) every term is the derived group of its successor, it follows that

$$\Gamma_i^0 = \Gamma_i^1 = \ldots = \Gamma_i^{k+1}.$$

Using (3.1), we obtain

$$G_0 C_i^* = G_{k+1} C_i^*, \tag{3.4}$$

and modulo G_0

$$G_0 C_i^*/G_0 = G_{k+1} C_i^*/G_0 = (G_{k+1}/G_0)(G_0 C_i^*/G_0).$$

Here the left-hand side is contained in the centralizer of G_i/G_0 in G^*/G_0. We see that G_{k+1}/G_0 is contained in the centralizer of G_i/G_0 in G^*/G_0. If $i \geqq k+1$, this would imply that G_{k+1}/G_0 is abelian, hence that G_k/G_0 is trivial: but this is incompatible with our supposition that $k > 0$. Thus $i < k+1$, and G_i/G_0 is contained in the centre of G_{k+1}/G_0. This in turn implies that G_i/G_0 is abelian, and thus $i = 1$. Finally, intersecting both sides of (3.4) with G_1, we obtain

$$G_0 Z = G_1,$$

where $Z = C_1^* \cap G_1$ is the centre of G_1. We form the derived group, using Lemma 2.2:

$$G_0 = G_1' = G_0' Z'[Z, G_0] = G_0'.$$

This completes the proof of the lemma.

§ 4. Some corollaries

The lemma gives rise to sufficient conditions for an ascending derived series to break off. To formulate one of them we remind ourselves of the definition of the *automorphism class group* of a group G: this is the factor group of the group of inner automorphisms of G in the group of all automorphisms of G. We denote it by $\Lambda(G)$.

COROLLARY 4.1. *If* $i \geqq 2$ *and if the automorphism class group* $\Lambda(G_i)$ *satisfies the maximal condition for soluble subgroups, then the ascending derived series*

$$G_0 \subset G_1 \subset G_2 \subset \ldots$$

breaks off.

The groups

$$\Gamma_i^i/\Gamma_i^i \subseteq \Gamma_i^{i+1}/\Gamma_i^i \subseteq \ldots \tag{4.2}$$

form an ascending sequence of soluble subgroups of $A(G_i)$; no two successive terms can become equal, by Lemma 3.2, as we have assumed $i \geqq 2$. The maximal condition for soluble subgroups of $A(G_i)$ then implies that the series breaks off.

The maximal condition for soluble subgroups of a group has apparently not yet received much attention. For soluble groups it coincides with the maximal condition for subgroups, and thus singles out the *polycyclic groups* [6]). But for groups in general it is much less restrictive; it is e.g. satisfied in all free groups and locally free groups, and in a free product provided it is satisfied in the free factors.

COROLLARY 4.3. *If G_2/G_0 is finite, then the ascending derived series breaks off.*

This is an immediate consequence of the preceding corollary, coupled with the remark (cf. § 2) that G_0 can be factored out.

We shall say that a group G satisfies the *ascending derived chain condition for subgroups* if every ascending derived series

$$G_0 \subset G_1 \subset G_2 \subset \ldots \subseteq G$$

breaks off. This condition neither includes, nor is included by, the maximal condition for soluble subgroups: The example of the locally free group G^* of § 2 shows that a group can satisfy the latter without satisfying the former; a free abelian group of infinite rank, on the other hand, satisfies the former and not the latter.

COROLLARY 4.4. *If $i \geqq 2$ and if the automorphism class group $A(G_i)$ satisfies the ascending derived chain condition for subgroups, then the given ascending derived series breaks off.*

For the sequence (4.2), like (3.2), is an ascending derived series, and by the assumption on $A(G_i)$ it breaks off.

§ 5. Finitely generated metabelian groups.

We interrupt the discussion of ascending derived series to supply ourselves, for subsequent use, with some facts on finitely generated metabelian groups. Some known facts have been re-derived here. The techniques employed are, of course, well-known; cf. e.g. Jacobson [4], chapter VI.

---

[6]) These groups were first studied by K. A. Hirsch in a series of papers [3]. The name "polycyclic" is due to P. Hall [2].

If G is a metabelian [7]) group, we may consider G' as an Ω-module, where Ω is the ring generated by the operators induced by G/G' in G'. To facilitate the distinction between G' *qua* derived group of G (written multiplicatively) and the Ω-module (in which the same operation appears as addition), we write $\mathfrak{M}$ for the latter. Thus to every element $g \in G$ we have an operator $\alpha \in \Omega$ such that if $x, y \in G'$ and $g^{-1}xg = y$, and if u, v are the elements of $\mathfrak{M}$ corresponding to x, y respectively, under an isomorphism — fixed once and for all — between G' and $\mathfrak{M}$, then $u\alpha = v$; moreover, if g and h induce the same automorphism of G', the same $\alpha \in \Omega$ corresponds to both. Thus one and the same operator corresponds to the elements of a coset of G'. The operator ring Ω may be thought of as obtained from the group ring over the integers of G/G' by reducing it [8]) modulo the annihilator ideal of $\mathfrak{M}$.

To every submodule of $\mathfrak{M}$ there corresponds a subgroup of G' which admits G, in other words a normal subgroup of G contained in G'; this correspondence is one-to-one; if a subgroup of G' is normal in G, then the isomorphism maps it onto a subgroup of $\mathfrak{M}$ that admits Ω, that is a submodule of $\mathfrak{M}$.

If G is finitely generated, let us say by $g_1, g_2, \ldots, g_d$, then G' is generated by the finitely many commutators $[g_i, g_j]$ and their conjugates; thus $\mathfrak{M}$ is then finitely generated *qua* Ω-module. It follows [9]) that $\mathfrak{M}$ contains maximal proper submodules; in fact every proper submodule is contained in a maximal one.

A maximal proper submodule of $\mathfrak{M}$ corresponds to a maximal proper subgroup N of G' normal in G. The factor group

$$H = G/N,$$

whose derived group is

$$H' = G'/N,$$

has the property that every normal subgroup either contains H' or meets H' in the trivial subgroup only. Denote by Z a maximal normal subgroup of H meeting H' trivially:

$$H' \cap Z = \{1\}.$$

---

[7]) For the sake of brevity we use "metabelian" in the strict sense, that is excluding "abelian"; thus we here call G metabelian only if G' is a *non-trivial* abelian group.

[8]) This reduction is not essential, and we could use the whole group ring, at any rate for the present.

[9]) by a familiar argument which we do not present; cf. [7]. We could instead use the maximal condition for submodules of $\mathfrak{M}$ which follows from Jacobson [4], p. 171; or the maximal condition for normal subgroups of G proved by Hall [2].

The existence of such a subgroup again follows by familiar arguments [10]). A normal subgroup that has trivial intersection with the derived group must lie in the centre; thus Z lies in the centre of H. The factor group $K = H/Z$ has the property that every non-trivial normal subgroup contains the derived group K'; differently put, K is a metabelian group but all its proper homomorphic images are abelian. Such a group may be called *only just strictly metabelian*; it will here be called, somewhat ambiguously but more briefly, *just metabelian*. We have thus proved the following result.

LEMMA 5.1. *Every finitely generated metabelian group has a just metabelian factor group.*

It may be remarked that this is not true of metabelian groups in general. If an abelian group of type p^∞ is extended by an abelian automorphism group, say by the automorphism of order 2 which maps each element on its inverse, then the resulting group is metabelian but has no just metabelian homomorphic image: every metabelian factor group is isomorphic to it and has another, still metabelian, proper factor group. This example shows incidentally that it does not suffice to assume G/G' finitely generated, or even finite.

§ 6. Just metabelian groups.

We now consider a just metabelian group, which we again denote by G; the ring Ω and the Ω-module $\mathfrak{M}$ are defined as before. Then $\mathfrak{M}$ has no proper non-zero submodule, in other words $\mathfrak{M}$ is a *simple* Ω-module.

LEMMA 6.1.[11]) *If $\mathfrak{M}$ is a simple Ω-module where Ω is a commutative ring and the annihilator of $\mathfrak{M}$ is (0), then Ω is a field or consists of 0 only.*

PROOF. We may suppose that Ω contains a non-zero element; any such element does not annihilate $\mathfrak{M}$, and thus also $\mathfrak{M}$ contains non-zero elements. The elements of $\mathfrak{M}$ annihilated by Ω form a submodule, and this then consists of 0 only. Now if $u \in \mathfrak{M}$, $u \neq 0$, then $u\Omega$ is a non-zero submodule, and thus

$$u\Omega = \mathfrak{M}.$$

If $\omega \in \Omega$ and $u\omega = 0$, then

$$0 = u\omega\Omega = u\Omega\omega = \mathfrak{M}\omega,$$

[10]) Cf. the preceding footnote.

[11]) This is an almost immediate consequence of Schur's Lemma; cf. e.g. Jacobson [5], p. 271.

whence $\omega = 0$. Next let α, $\beta \epsilon \Omega$ and $\alpha \neq 0$. Then $u\alpha \neq 0$ and thus

$$u\alpha\Omega = \mathfrak{M}.$$

There exists then an element $\xi \epsilon \Omega$ such that in turn

$$u\alpha\xi = u\beta,$$
$$u(\alpha\xi - \beta) = 0,$$
$$\alpha\xi - \beta = 0,$$
$$\alpha\xi = \beta.$$

This shows that division by non-zero elements can be carried out in Ω, and the lemma follows.

LEMMA 6.2. *If a finitely generated ring is a field, then it is finite.*

PROOF. Let Ω be a ring generated by $\alpha_1, \alpha_2, \ldots, \alpha_d$, and assume Ω is a field. Then every element of Ω is a polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_d$ with coefficients in the prime field if Ω has prime characteristic, and with rational integer coefficients if Ω has zero characteristic. We first show that Ω is algebraic over its prime field P; suppose not, and let $\tau_1, \tau_2, \ldots, \tau_e$ be a basis of transcendental elements of Ω. Then there are polynomials $\pi_1(\tau_1, \ldots, \tau_e), \ldots, \pi_d(\tau_1, \ldots, \tau_e)$ such that

$$\alpha_1\pi_1, \ \alpha_2\pi_2, \ \ldots, \ \alpha_d\pi_d$$

are integral over $\mathsf{P}(\tau_1, \ldots, \tau_e)$. Then to every polynomial α in $\alpha_1, \ldots, \alpha_d$ there is an integer m (e.g. its degree) such that $\alpha \cdot (\pi_1\pi_2 \ldots \pi_d)^m$ is integral over $\mathsf{P}(\tau_1, \ldots, \tau_e)$. But if π_0 is an irreducible polynomial in $\tau_1, \ldots, \tau_e$ which does not divide any of $\pi_1, \ldots, \pi_d$, then $\frac{1}{\pi_0}(\pi_1\pi_2 \ldots \pi_d)^m$ is not integral over $\mathsf{P}(\tau_1, \ldots, \tau_e)$ for any m, and thus $\frac{1}{\pi_0}$ is not generated by $\alpha_1, \ldots, \alpha_d$. This contradicts the field property of Ω, and we conclude that $\alpha_1, \ldots, \alpha_d$ are absolutely algebraic, and that Ω is a finite algebraic extension of its prime field P. It only remains to prove that P is finite, that is that Ω can not have characteristic 0. Assume the contrary. Let $s_1, \ldots, s_d$ be positive rational integers such that $s_1\alpha_1, s_2\alpha_2, \ldots, s_d\alpha_d$ are algebraic integers. Then every polynomial in $\alpha_1, \ldots, \alpha_d$ with integral coefficients, that is every element of Ω, is of the form $\frac{\beta}{(s_1 s_2 \ldots s_d)^m}$, where β is an algebraic integer and m a suitable rational integer (e.g. the degree of the polynomial). But if p is a prime that divides none of $s_1, s_2, \ldots, s_d$, then $1/p$ is not of this form, hence not in Ω. This

again contradicts the assumption that Ω is a field, and the lemma follows.

More briefly one can argue as follows: If V is a non-trivial non-Archimedean valuation on Ω, then not all of $\alpha_1, \ldots, \alpha_d$ can be integers with respect to V. Unless Ω is modular and absolutely algebraic, it has an infinity of non-trivial non-Archimedean valuations, and no element can be a non-integer for more than a finite number of them. If follows that Ω is modular and algebraic, and being finitely generated, it must be finite.

THEOREM 6.3. *A finitely generated just metabelian group is finite.*

PROOF. By Lemma 6.1 the operator ring Ω is either trivial or a field. Assume first that $\Omega = (0)$. Then $\mathfrak{M}$, being a simple Ω-module, is cyclic of prime order. Hence in this case G' is a cyclic group of prime order p, say. Let Z be the centre of G. Then Z has no non-trivial subgroup with trivial intersection with G'. It follows that Z is cyclic of order a power of p. Now in a finitely generated group with finite derived group the centre has finite index [12]), and it follows that G is finite. Next assume that Ω is a field; then Ω is finitely generated *qua* ring, as it is a homomorphic image of the — evidently finitely generated — group ring of G/G'. Hence by Lemma 6.2 Ω is finite, and $\mathfrak{M}$, being of the form $u\Omega$ for some $u \neq 0$, has the same finite order as Ω. In this case the centre Z is in fact trivial; for if it were not, then there would be an element $y \in G' \cap Z$, $y \neq 1$, and thus an element $v \neq 0$ of $\mathfrak{M}$ annihilated by the whole of Ω. This is impossible (cf. the proof of Lemma 6.1.) Using again the fact that in a finitely generated group with finite derived group the centre has finite index, we see that also in this case G is finite, and the theorem follows.

Before we apply the theorem, we briefly indicate that it is in two ways a "best possible" result.

It would be unreasonable to expect the assumption that the group is finitely generated to be superfluous. In fact the group of linear inhomogeneous substitutions

$$z \to \beta + z\alpha,$$

with β ranging over an infinite field Ω and α over its non-zero elements, is an example of a just metabelian group that is infinite and thus can not be finitely generated.

It is, on the other hand, conceivable that "metabelian" can be replaced by "soluble of length 3". However this is not the case.

[12]) By [9], Corollary 5.41.

Let G be the group with generators a, b, c and defining relations

$$[b^{-i}cb^i, c] = 1 \qquad (i = \pm 1, \pm 2, \ldots),$$
$$[w(b, c)^{-1}aw(b, c), a] = 1 \qquad (w(b, c) \in \{b, c\}),$$
$$b^{-i}c^{-1}b^iab^{-i}cb^i = a^{p(i)},$$

where $p(i)$ is the ith prime when i is positive, and $p(i) = 1$ when i is zero or negative [13]). This group is "just soluble of length 3", that is it is soluble of length 3 but all its proper homomorphic images are metabelian or abelian. It is infinite, though it is finitely generated. We omit the proof, which is not difficult. Using deeper results due to Hall [2], one can also show the existence of infinite two-generator groups that are "just centre-extended-by-metabelian".

§ 7. Application to ascending derived series.

An immediate consequence of Lemma 5.1 and Theorem 6.3 is the following fact.

COROLLARY 7.1. *Every finitely generated metabelian group has a finite metabelian factor group.*

This can be slightly strengthened by remarking that in a finitely generated group G every normal subgroup N of finite index contains a subgroup N^* which is characteristic in G and also of finite index; for there are only finitely many normal subgroups of G whose factor groups are isomorphic to G/N: their intersection will serve as N^*. Thus we deduce from Corollary 7.1:

LEMMA 7.2. *Every finitely generated metabelian group contains a characteristic subgroup whose factor group is finite and metabelian.*

We apply this to extend Corollary 4.3.

THEOREM 7.3. *If*

$$G_0 \subset G_1 \subset G_2 \subset \ldots$$

is an ascending derived series and if G_2/G_0 is finitely generated, then the series breaks off.

PROOF. We may assume the series reduced modulo G_0, or equivalently $G_0 = \{1\}$. Then G_2 is a finitely generated metabelian group, hence by the preceding lemma it contains a characteristic subgroup N of finite index whose factor group is still metabelian. Then N is normal in G^*, and reducing modulo N we obtain a new series

$$\{1\} \subset H_1 \subset H_2 \subset \ldots$$

[13]) This group is adapted from an example constructed elsewhere [8].

with $H_2 = G_2/N$ finite. By Corollary 4.3 the series breaks off, and so then does our original series, and the theorem follows.

COROLLARY 7.4. *If G is a finitely generated group and $G'' \neq G'$, then every ascending derived series through G breaks off.*

This corollary deals e.g. with free groups of finite rank and more generally with free products of finitely many finitely generated groups.

COROLLARY 7.5. *In an infinite ascending derived series no metabelian factor G_{i+2}/G_i can be finitely generated.*

§ 8. Crown products.

One might expect that it is sufficient to assume G_1/G_0 finitely generated in order to ensure that the ascending derived series breaks off; this is, however, not the case, as an example will show.

The construction uses the iterated "wreath product" of two groups, with a slight modification. Given an abstract group G and a transitive permutation group Γ defined on a set Σ, we first form the restricted direct product K of isomorphic copies $G_\sigma = G\theta_\sigma$ of G, indexed by the elements σ of Σ; then K has an obvious automorphism group permuting the direct factors according to Γ; specifically if

$$k = \prod_{\sigma \in \Sigma} g_\sigma, \qquad (g_\sigma \in G_\sigma),$$

and if $\gamma \in \Gamma$, then we put

$$k^\gamma = \prod g_{\sigma\gamma}.$$

The splitting extension of K by this automorphism group (isomorphic to and again denoted by) Γ is then known as the wreath product of G and Γ. We denote it by G Wr Γ. Now assume we are also given a subgroup Z of the centre of G. The isomorphism θ_σ maps Z onto the subgroup Z_σ in the centre of G_σ, and the product of all the Z_σ is a subgroup of the centre of K. Denote by N the set of those elements

$$z^* = \prod_{\sigma \in \Sigma} z_\sigma \qquad (z_\sigma \in Z_\sigma)$$

for which [14])

$$\prod_\sigma z_\sigma \theta_\sigma^{-1} = 1.$$

Then N is easily seen to be a normal subgroup of the wreath

[14]) The product has only a finite number of factors $\neq 1$, because we are using *restricted* direct products.

product. It is generated by all quotients $(z\theta_\sigma)(z\theta_\tau)^{-1}$; hence modulo N all the different copies Z_σ of Z coalesce to a single group, which we again denote by Z. Let the factor group thus obtained be

$$P = (G \operatorname{Wr} \Gamma)/N;$$

we might call it a "wreath product with amalgamation", or a "centrally extended wreath product": It could have been obtained by forming the (restricted) direct product of the G_σ with amalgamated subgroups [15]) Z_σ, and letting Γ operate on that; alternatively it can be thought of as an extension of the centre Z by the wreath product of G/Z and Γ. Let us call it the *crown product*, and denote it by

$$P = G \operatorname{Cr}_Z \Gamma.$$

Like G, it contains Z in its centre; we can therefore form the crown product again, with another permutation group Δ. The wreath product can be considered as the special case of the crown product in which the central subgroup Z is trivial.

LEMMA 8.1. *If Γ is an abelian group of order greater than 2, then the second derived group P'' of the crown product*

$$P = G \operatorname{Cr}_Z \Gamma$$

contains a subgroup isomorphic to G' and is a homomorphic image of a direct power of G'.

PROOF. We first notice that P' is contained in the direct product of the G_σ with amalgamated Z; this is the group K/N of our construction, which we shall now denote by L. Clearly L is normal in P, and

$$P/L \simeq \Gamma,$$

and as Γ is assumed to be abelian, $P' \subseteqq L$. It follows that

$$P'' \subseteqq L'. \tag{8.2}$$

We also note that L' is the direct product of the derived groups G'_σ with their intersections with Z amalgamated, and thus a homomorphic image of a direct power of G'.

Our assumption on the order of Γ implies that the cardinal of the set Σ which Γ permutes is at least 3. Let ϱ, σ, τ be three distinct elements of Σ, and let γ, $\delta \in \Gamma$ be so chosen that

$$\sigma\gamma = \varrho,$$
$$\sigma\delta = \tau.$$

---

[15]) For this notion, cf. [10].

This is possible because Γ acts transitively on Σ. Next let g, h be two arbitrary elements of G, and denote again by $g_\sigma, h_\sigma, \ldots$ their copies in $G_\sigma, \ldots$. Consider the second commutator

$$\begin{aligned}[[[\gamma, g_\sigma], [\delta, h_\sigma]] &= [\gamma^{-1} g_\sigma^{-1} \gamma g_\sigma, \ \delta^{-1} h_\sigma^{-1} \delta h_\sigma] \\ &= [g_\varrho^{-1} g_\sigma, \ h_\tau^{-1} h_\sigma] = [g_\sigma, h_\sigma].\end{aligned}$$

(Note that g_ϱ commutes with g_σ, h_σ, h_τ, and h_τ with g_ϱ, g_σ, h_σ.) This belongs to P''; but as g, h were quite arbitrary elements of G, then the whole of G'_σ is contained in P'', that is P'' contains a subgroup isomorphic to G'. Again, as σ was arbitrary,

$$L' \subseteqq P'',$$

and combining this with (8.2) we have

$$P'' = L'. \tag{8.3}$$

The lemma then follows.

COROLLARY 8.4. *If Γ is abelian and of order greater than 2, and if G is soluble of length n, then the crown product $G \operatorname{Cr}_Z \Gamma$ is soluble of length $n+1$.*

§ 9. Construction of an example.

We are now in a position to construct the example announced at the beginning of § 8; the reader can easily satisfy himself that we have sufficient freedom in each step of the procedure to make continuously many different examples: but we shall in fact describe only one.

Let H_2 be the quaternion group, and

$$H_1 = H_2' = Z$$

the cyclic group of order 2, which is its centre. We take Γ_i ($i = 2, 3, \ldots$) all as the cyclic permutation group of order 4, which is transitive on the set $\Sigma = \{0, 1, 2, 3\}$. Then we form inductively the successive crown products

$$H_{i+1} = H_i \operatorname{Cr}_Z \Gamma_i \qquad (i \geqq 2).$$

Thus H_{i+1} is obtained by forming the direct product of groups H_{i0}, H_{i1}, H_{i2}, H_{i3} with amalgamated Z, and extending by the cyclic group of order 4 as described in § 8. The four groups $H_{i\sigma}$ ($\sigma = 0, 1, 2, 3$) are isomorphic to H_i, and we can identify H_i with one of them, say H_{i0}. Then

$$H_2 \subset H_3 \subset H_4 \subset \ldots \tag{9.1}$$

becomes an ascending sequence of groups, and we put

$$G^* = \cup H_i .$$

The centre Z has been preserved in each step and thus Z is also a subgroup of the centre of G^* (it is in fact the whole centre of G^*). We put $G_0 = \{1\}$ and $G_1 = Z$, and define G_2 as the group generated by all the isomorphic copies that arise from H_2 by forming the successive direct products with amalgamated Z. There are infinitely many such copies. More generally we define, for *even* $i = 2j$ *only*, G_i to be the group generated by the infinitely many copies of H_i that are created in the process of construction: thus G_i, for $i = 2, 4, 6, \ldots$, is generated by all $\gamma^{*-1} H_i \gamma^*$, where γ^* ranges over the permutations generated by $\Gamma_{i+1}, \Gamma_{i+2}, \ldots$ in G^*. Differently put, G_{2j} is the normal closure of H_{2j} in G^*. We also define the odd-numbered G_i simply as the derived groups of their even-numbered successors:

$$G_{2j-1} = G'_{2j} . \tag{9.2}$$

It is evident that

$$\{1\} = G_0 \subset G_1 \subset G_2 \subset \ldots, \tag{9.3}$$

and that the inductive limit of this series is

$$G^* = \cup G_i .$$

Now we observe that if i is even, then

$$G''_i = G_{i-2}; \tag{9.4}$$

for it follows from Lemma 8.1, or rather from its proof (cf. especially (8.3)), that H''_i is the direct product of several (namely 16) copies of H_{i-2}, with $Z = H_1$ amalgamated. Combining (9.4) with the definition (9.2), we see that

$$G'_i = G_{i-1} \qquad (i = 1, 2, \ldots),$$

that is (9.3) is an ascending derived series. Thus we have shown that *the cyclic group of order* 2 *can be made the first non-trivial group of an infinite ascending derived series.* It is not difficult to make similar examples with other abelian groups. The inductive limit group G^* of our example is, incidentally, locally finite-of-order-a-power-of-2, and thus also locally nilpotent; it is its own derived group, and hence its upper central series must terminate with the centre G_1, as is indeed easily verified directly.

§ 10. A further example.

It is now natural to ask whether it suffices to suppose G_2/G_1 finite, or even finitely generated only, to ensure that the ascending derived series breaks off. This is, however, not the case, as we now show by further elaborating the example of § 9. In order to have the notation

$$G_0 \subset G_1 \subset G_2 \subset \ldots$$

available for the ascending derived series that is to be constructed, we now denote by F_0, F_1, F_2, ... and F^* respectively the groups G_0, G_1, G_2, ... and G^* of § 9.

Let A denote the abelian group of type 2^∞, and form the wreath product

$$B = A \text{ Wr } F^*,$$

where F^* is taken in its regular permutation representation. We denote by P the (restricted) direct product of the isomorphic copies of A, indexed by the elements of F^*, that go into the construction of B. The direct factors can be bracketed together in pairs

$$A_f, \; A_{fc} = A_{cf},$$

where c is the element of order 2 in the centre of F^*, and f ranges over F^*. We now amalgamate the partners in every pair as follows.

Consider the elements

$$a_f a_{fc} \in A_f \times A_{fc}, \tag{10.1}$$

where a ranges over A and f over F^*. (Every element (10.1) arises twice, namely again as $a_{fc}a_{fc^2} = a_{fc}a_f$). The group N generated by all these elements is normal in B; for all conjugates of an element (10.1) are obtained by transforming by elements $f^* \in F^*$, and

$$f^{*-1}(a_f a_{fc})f^* = a_{ff^*}a_{fcf^*} = a_{ff^*}a_{ff^*c}$$

is again of the form (10.1). Reducing modulo N we amalgamate paired groups A_f, A_{fc} in such a way that elements corresponding to the same $a \in A$ become inverse to each other in the amalgamated copy.

We thus define the groups

$$G_0 = \{1\}, \; G_1 = P/N, \; G_2 = \{P, F_1\}/N,$$

and generally

$$G_i = \{P, \; F_{i-1}\}/N.$$

Then clearly

$$G_0 \subset G_1 \subset G_2 \subset \ldots, \tag{10.2}$$

and the inductive limit of this sequence is

$$G^* = B/N.$$

It remains to show that (10.2) is an ascending derived series. It is indeed clear that $G_0 = G_1'$, for G_1 is abelian; also $G_i = G_{i+1}'$ for $i = 2, 3, \ldots$, because the factor groups G_i/G_1 are isomorphic to F_{i-1}, for $i = 1, 2, \ldots$, and because, as we shall show presently, $G_1 \subseteqq G_i'$ for $i \geqq 2$. Thus we need only examine the relation between G_1 and G_2.

Clearly

$$G_2' \subseteqq G_1, \tag{10.3}$$

as $G_2/G_1 \simeq F_1$ is abelian. On the other hand every element $a_f \in A_f$ is a square of another element in A_f, by our choice of A as a group of type 2^∞, say

$$a_f = b_f^2.$$

Now

$$b_f^2 = b_{fc}^{-1} b_f^2 b_{fc} = c^{-1} b_f^{-1} c b_f \,.\, b_f b_{fc} \equiv [c, b_f] \pmod{N}.$$

Thus the elements of P are generated modulo N by commutators of c and elements of P, that is

$$G_1 = P/N \subseteqq (\{P, c\}/N)' = G_2'.$$

Combining this with (10.3), we see that also $G_1 = G_2'$, and thus (10.2) is indeed an ascending derived series. It is an infinite series, though G_2/G_1 is finite, namely cyclic of order 2. It may be added that G^* is again locally finite-of-order-a-power-of-2, and in particular locally nilpotent; its centre is trivial and its derived group coincides with the whole of it.

It is clear that the example can be modified in a great number of ways. We only mention that one can similarly construct infinite ascending derived series in which some other factor G_{i+1}/G_i is an arbitrarily prescribed finite abelian group, but G_{i+2}/G_{i+1} and G_i/G_{i-1} then can not, by what we have shown, be finitely generated.

REFERENCES

Mevr. Dr. T. van Aardenne-Ehrenfest

[1] Vraagstuk 3. Wiskundige Opgaven 19 (1949). Cf. also [6].

P. Hall

[2] Finiteness conditions for soluble groups. Proc. London Math. Soc. (3) 4, 419—436 (1954).

K. A. Hirsch

[3] On infinite soluble groups, I—V. Proc. London Math. Soc. (2) 44, 53—60 (1938); 44, 336—344 (1938); 49, 184—194 (1946); J. London Math. Soc. 27, 81—85 (1952); 29, 250—251 (1954).

Nathan Jacobson

[4] Lectures in Abstract Algebra, vol. I: Basic Concepts. Van Nostrand, New York &c. 1951.

Nathan Jacobson

[5] Lectures in Abstract Algebra, vol. II: Linear Algebra. Van Nostrand, New York &c. 1953.

L. Lips

[6] Oplossing van Vraagstuk 3. Wiskundige Opgaven met de Oplossingen: Wiskundig Genootschap te Amsterdam, 19, 4—6 (1950).

B. H. Neumann

[7] Some remarks on infinite groups. J. London Math. Soc. 12, 120—127 (1937).

B. H. Neumann

[8] On a special class of infinite groups. Nieuw Archief voor Wiskunde 23, 117—127 (1950).

B. H. Neumann

[9] Groups with finite classes of conjugate elements. Proc. London Math. Soc. (3) 1, 178—187 (1951).

B. H. Neumann and Hanna Neumann

[10] A remark on generalized free products. J. London Math. Soc. 25, 202—204 (1950).

Hans Zassenhaus

[11] Lehrbuch der Gruppentheorie, 1. Band. Teubner, Leipzig & Berlin 1937.

The University, Manchester, 13.

(Oblatum 1-8-55).