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On Entire Functions of Infinite Order

by

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1. Introduction. The purpose of this paper is to extend to a class of entire functions of infinite order some theorems on entire functions of finite order.

Theorems 1 and 2 are formal analogues of two theorems [1] and [2] of Shah. Theorems 3, 4 and 5 are new; but they are closely connected with some theorems [3] of Shah. Theorem 6 is an analogue of a theorem of Lindelöf [4].

2. Definitions. We define the k-th order and the k-th lower order of an entire or meromorphic function as

\[ \varrho_k = \lim_{r \to \infty} \frac{k T(r)}{\log r} \]

and

\[ \lambda_k = \lim_{r \to \infty} \frac{k T(r)}{\log r} \]

Similarly, we define the k-th order and the k-th lower order of the zeros of \( f(z) \) as

\[ \sigma_k = \lim_{r \to \infty} \frac{k n(r)}{\log r} \]

and

\[ \delta_k = \lim_{r \to \infty} \frac{k n(r)}{\log r} \]

where \( T(r), n(r) \) have their usual meanings and \( l_1 x = \log x, l_2 x = \log \log x \), and so on.

3. Lemma (i) If \( \chi(x) \) is a positive function continuous almost everywhere in every interval \((r_0, r)\); and if

\[ \lim_{r \to \infty} \frac{k \xi(r)}{\log r} = \sigma_k \]

then

\[ \lim_{r \to \infty} \frac{\xi(r)l_1 \xi(r)l_2 \xi(r) \ldots l_{k-1} \xi(r)}{\chi(r)} \leq \frac{1}{\sigma_k} \]
where
\[ \xi(r) = \int_{r_0}^{r} \frac{\chi(x)}{x} \, dx. \]

**Lemma (ii)** If \( \chi(x) \) and \( \xi(r) \) are the same functions as before; and if
\[ \lim_{r \to \infty} \frac{l_k \xi(r)}{\log r} = \delta_k, \]
then
\[ \lim_{r \to \infty} \frac{\chi(r)}{\xi(r) l_1 \xi(r) \cdots l_{k-1} \xi(r)} \leq \delta_k. \]

**Proof.** If \( f(x) \) and \( g(x) \) are two positive functions which tend to infinity with \( x \); and if each of the functions is differentiable almost everywhere in every interval \((r_0, r)\), such that their derivatives \( f'(x) \) and \( g'(x) \) have a definite finite value at every point of this interval, then
\[ \lim_{r \to \infty} \frac{f(r)}{g(r)} \leq \lim_{r \to \infty} \frac{f'(r)}{g'(r)} \]
and
\[ \lim_{r \to \infty} \frac{f(r)}{g(r)} \geq \lim_{r \to \infty} \frac{f'(r)}{g'(r)}. \]

Now, putting \( f(r) = l_k \xi(r) \) and \( g(r) = \log r \), we get the required results.

4. **Theorem 1.** If \( f(z) \) is an entire function of infinite order; and if the \( k \)-th lower order of its zeros is \( \delta_k \), then

(i) \[ \lim_{r \to \infty} \frac{n(r)}{l_1 M(r) l_2 M(r) \cdots l_k M(r)} \leq \delta_k \]
and

(ii) \[ \lim_{r \to \infty} \frac{n(r)}{l_1 M(r) l_1 n(r) l_2 n(r) \cdots l_{k-1} n(r)} \leq \delta_k, \]
provided that
\[ \lim_{r \to \infty} \frac{\log n(r)}{l_2 r} = \infty. \]

These can be proved easily by putting \( \xi(r) = \int_{r_0}^{r} \frac{n(x)}{x} \, dx \) in Lemma (ii).

**Theorem 2.** If \( f(z) \) is an entire function of finite \( k_1 \)-th order but of infinite \((k_1-1)\)-th lower order, then
where \( \varrho_k \) is the \( k \)-th order of \( f(z) \).

**Proof.** Since, by hypothesis, \( f(z) \) is of finite \( \dot{k}_1 \)-th order but of infinite \( (k_1 - 1) \)-th lower order, we can very easily prove, by using the inequalities

\[
\frac{u(r)}{r} \leq M(r) \leq 3u(r)\nu(2r)
\]

that

\[
\lim_{r \to \infty} \frac{l_{k_1}v(r)}{\log r} < \infty
\]

and

\[
\lim_{r \to \infty} \frac{l_{k_1 - 1}v(r)}{\log r} = \infty.
\]

Now, we can very easily show that

\[
\lim_{r \to \infty} \frac{l_{k+1}v(2r)}{l_kv(ar)} = 0,
\]

where \( k \) is any positive integer or zero; and \( \alpha \) is any fixed positive number.

Also, putting \( \xi(r) = \log u(r) \) in Lemma (i); and using (1), we have

\[
\lim_{r \to \infty} \frac{l_1u(r)l_2u(r) \cdots l_ku(r)}{v(r)} \leq \frac{1}{\varrho_k}
\]

\( \varrho_k \) being the \( k \)-th order of \( f(z) \).

Lastly, by using (1), (2) and (3), we can easily prove the required result.

**Theorem 3.** If \( f(z) \) is an entire function of finite \( k_1 \)-th order but of infinite \( (k_1 - 1) \)-th lower order, then

\[
\lim_{r \to \infty} \frac{T(r)l_1T(r) \cdots l_{k-1}T(r)}{n(r, f-f_1)} \leq \frac{2}{\varrho_k}
\]

for every entire function \( f_1(z) \) of finite \( (k_1 - 1) \)-th order, with one possible exception, where \( T(r) \) refers to \( f(z) \), \( \varrho_k \) is the \( k \)-th order of \( f(z) \); and \( n(r, f-f_1) \) denotes the number of zeros of \( f(z) - f_1(z) \) in the region \( |z| \leq r \), every zero being counted according to its order.

**Proof.** By the second fundamental theorem of Nevanlinna

[5, § 34], we have

\[
T(r, \varphi) = T(r) < N(r, 0) + N(r, 1) + N(r, \infty) + 8 \log T(\alpha r) + O(\log r)
\]
for all sufficiently large $r$, where $c$ is a fixed number greater than 1. Putting $\varphi(z) = \frac{f(z) - f_1(z)}{f(z) - f_2(z)}$ in (4), we have

$$T(r, f) = T(r) < N(r, f - f_1) + N(r, f - f_2) + 8 \log T(cr) + aT(r, f_1) + bT(r, f_2) + O(\log r) \quad (5)$$

for all $r > r_0$, where $a$ and $b$ are certain positive constants.

Since, by hypothesis, $f(z)$ is of finite $k_1$-th order but of infinite $(k_1 - 1)$-th lower order; and each of the functions $f_1(z)$ and $f_2(z)$ is of finite $(k_1 - 1)$-th order, we have

$$\lim_{r \to \infty} \frac{\log T(cr)}{T(r)} = 0$$

and

$$\lim_{r \to \infty} \frac{T(r, F)}{T(r)} = 0,$$

where $F$ denotes each of the functions $f_1(z)$ and $f_2(z)$. Consequently, we have

$$l_k\{T(r) - 8 \log T(cr) - aT(r, f_1) - bT(r, f_2)\} < l_k\{N(r, f - f_1) + N(r, f - f_2)\} \quad (6)$$

Now, putting $\xi(r) = N(r, f - f_1) + N(r, f - f_2)$ in Lemma (i), we get

$$\xi_k \leq \lim_{r \to \infty} \frac{n(r, f - f_1) + n(r, f - f_2)}{\xi(r)l_1\xi(r) \ldots l_{k-1}\xi(r)} \quad (7)$$

Combining (6) and (7), we have

$$\xi_k \leq \lim_{r \to \infty} \frac{n(r, f - f_1) + n(r, f - f_2)}{T(r)l_1T(r) \ldots l_{k-1}T(r)}.$$ 

Therefore

$$\lim_{r \to \infty} \frac{T(r)l_1T(r) \ldots l_{k-1}T(r)}{n(r, f - f_1) + n(r, f - f_2)} \leq \frac{1}{\xi_k} \quad (8)$$

The required result follows easily from (8).

**Theorem 4.** If $f(z)$ is an entire function of finite $k_1$-th order but of infinite $(k_1 - 1)$-th lower order, for which the deficiency sum (excluding $\alpha = \infty$) $\sum \delta(\alpha) = \sigma > 0$; and if $n'(r, \alpha)$ denotes the number of simple zeros of the function $f(z) - \alpha$ in the region $|z| \leq r$, then

$$\lim_{r \to \infty} \frac{T(r)l_1T(r) \ldots l_{k-1}T(r)}{n'(r, \alpha)} \leq \frac{2}{\xi \cdot \sigma_k}.$$
for every finite value of $\alpha$, with one possible exception, where $\psi_k$ is the $k$-th order of $f(z)$.

**Proof.** If $N'(r, \alpha)$ and $N'(r, \beta)$ refer to $n'(r, \alpha)$ and $n'(r, \beta)$ respectively, we have

$$N(r, \alpha) + N(r, \beta) < N'(r, \alpha) + N'(r, \beta) + 2N_1(r) + 0 \log r.$$ 

Also, by the theorem of Nevanlinna (loc. cit.), we have

$$T(r, f) < N(r, \alpha) + N(r, \beta) - N_1(r) + 8 \log T(cr) + 0 \log r$$

$$< N'(r, \alpha) + N'(r, \beta) + N_1(r) + 8 \log T(cr) + 0 \log r \quad (9)$$

for all sufficiently large $r$, where $N_1(r)$ has the same meaning as in [6, § 33, (16)].

Further, by the same theorem, we have

$$\sum \delta(\alpha) + \lim_{r \to \infty} \frac{N_1(r)}{T(r)} \leq 1 + \lim_{r \to \infty} \frac{\log T(cr)}{T(r)}.$$

But, under the conditions of the theorem, we have

$$\lim_{r \to \infty} \frac{\log T(cr)}{T(r)} = 0.$$

Therefore

$$\lim_{r \to \infty} \frac{N_1(r)}{T(r)} \leq 1 - \sigma. \quad (10)$$

By (9), we have

$$l_k\{T(r) - N_1(r) - \log T(cr) - 0 \log r\} < l_k\{N'(r, \alpha) + N'(r, \beta)\}.$$ 

The rest of the proof, now, depends on (10) and follows the same lines as that of the preceding theorem.

**Theorem 5.** If $f(z)$ is a meromorphic function of finite $k_1$-th order but of infinite $(k_1-1)$-th lower order, then

$$\lim_{r \to \infty} \frac{T(r)l_1(Tr) \ldots l_{k-1}T(r)}{n(r, f-f_1)} \leq \frac{3}{\psi_k}$$

for every meromorphic function $f(z)$ of finite $(k_1-1)$-th order, with two possible exceptions, where $n(r, f-f_1)$ and $\psi_k$ have the same meanings as before.

The proof of this is similar.

4. We define the type of an entire function $f(z)$ of finite $k$-th order as

$$T_k = \lim_{r \to \infty} \frac{l_k M(r)}{r^{\psi_k}}.$$

LEMMA. If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is an entire function of finite \( k \)-th order \( q_k, k > 1 \), then

\[
T_k = \lim_{n \to \infty} l_{k-1} n \cdot |a_n|^n.
\]

PROOF. Let

\[
v_k = \lim_{n \to \infty} l_{k-1} n \cdot |a_n|^n.
\]

We have

\[
|a_n| \geq \left( \frac{v_k - \epsilon}{l_{k-1} n} \right)^{\frac{n}{q_k}}
\]

for an infinity of \( n \).

Therefore, by Cauchy's inequality, we have

\[
M(r) \geq \left( \frac{v_k - \epsilon}{l_{k-1} n} \right)^{\frac{n}{q_k}} \cdot r^n
\]

for an infinity of \( n \). Choose \( r \) such that

\[
r^{q_k} = \frac{a \cdot l_{k-1} n}{v_k - \epsilon},
\]

where \( a \) is any fixed number greater than 1.

Consequently, we have

\[
M(r) \geq \left( \frac{v_k - \epsilon}{l_{k-1} n} \right)^{\frac{n}{q_k}} \left( \frac{a \cdot l_{k-1} n}{v_k - \epsilon} \right)^{\frac{n}{q_k}}
\]

\[
= \frac{1}{a^{q_k}} \cdot \left( \frac{v_k - \epsilon}{l_{k-1} n} \right)^{\frac{n}{q_k}}
\]

Proving thereby that

\[
aT_k \geq v_k - \epsilon.
\]

Making \( a \) and \( \epsilon \) tend to unity and zero respectively, we have

\[
T_k \geq v_k. \quad (11)
\]

Also, we have

\[
|a_n| \leq \left( \frac{v_k + \epsilon}{l_{k-1} n} \right)^{\frac{n}{q_k}}
\]

for all sufficiently large \( n \).

Therefore

\[
|f(z)| \leq \sum_{n=0}^{\infty} |a_n| r^n
\]

\[
\leq \sum_{n=n_0}^{\infty} r^n \left( \frac{v_k + \epsilon}{l_{k-1} n} \right)^{\frac{n}{q_k}} + o(r^{n_0}).
\]
Now, \( r^x \left( \frac{v_k + \varepsilon}{l_k-1 x} \right)^{\varepsilon_k} \) is maximum for a value of \( x \), say \( x_1 \), which satisfies the equation
\[
(v_k + \varepsilon) r^{\varepsilon_k} = l_{k-1} x_1 \cdot e^{\varepsilon_k \frac{1}{1+\varepsilon}} \cdot l_{k-2} \cdot \ldots \cdot l_1 .
\]
We can take \( x_1 \) sufficiently large, by choosing \( r \) to be large. Therefore, we have
\[
e_{k-1} \left( \frac{(v_k + \varepsilon) r^{\varepsilon_k}}{1+\varepsilon} \right) \leq x_1 \leq e_{k-1} \left( \frac{(v_k + \varepsilon) r^{\varepsilon_k}}{1-\varepsilon} \right),
\]
where \( \varepsilon_1 \) is arbitrarily small.

Let \( m = e_{k-1} \{(v_k + 2\varepsilon) r^{\varepsilon_k}\} \). We have
\[
|f(z)| \leq \sum_{n \leq m} |a_n| r^n + \sum_{n > m} |a_n| r^n
\]
\[
\leq e_{k-1} \{(v_k + 2\varepsilon) r^{\varepsilon_k}\} (1 + \varepsilon_1)^{\frac{1}{v_k}} \frac{(v_k + \varepsilon) r^{\varepsilon_k}}{1-\varepsilon_1} + \sum_{n=0}^{\infty} \frac{(v_k + \varepsilon) r^{\varepsilon_k}}{v_k + 2\varepsilon}.
\]
\[
= e_{k-1} \{(v_k + 2\varepsilon) r^{\varepsilon_k}\} (1 + \varepsilon_1)^{\frac{1}{v_k}} \frac{(v_k + \varepsilon) r^{\varepsilon_k}}{1+\varepsilon_1} + o(1).
\]

Therefore, we have
\[
T_k \leq v_k . \quad (12)
\]

Hence, combining \((11) \) and \((12)\), we have
\[
T_k = v_k .
\]

**Theorem 6.** If \( P(z) = \prod_{n} E \left( \frac{z}{z_n}, p_n \right) \) is a product of primary factors of finite \( k \)-th order, having zeros \( (z_n) \ n = 1, 2, 3, \ldots \), where \( p_n \leq \log n < p_{n+1} \); and if
\[
L_k = \lim_{r \to \infty} \frac{l_{k-1} n(r)}{r^{\varepsilon_k}},
\]
then
\[
L_k \leq T_k \leq AL_k,
\]
where \( n(r) \) has its usual meaning and \( A \) is a constant.

**Proof.** When \( p_n > 0 \) and \( |z| \geq \frac{1}{2} \), we have
\[
\log |E(z, p_n)| \leq \log (1 + |z|) + |z| + \frac{|z|^2}{2} + \ldots + \frac{|z|^{p_n}}{p_n}
\]
\[
\leq 2|z| + \frac{|z|^2}{2} + \ldots \frac{|z|^{p_n}}{p_n}
\]
\[
\leq 2(2|z|)^{p_n}.
\]
Similarly, we have
\[
\log |E(z, p_n)| \geq \log |1 - z| - |z| - \frac{|z|^2}{2} - \ldots - \frac{|z|^{p_n}}{p_n}
\geq \log |1 - z| - 2|z|^{p_n}.
\]

Let \( N \) be a positive integer such that \(|z_N| \leq 2|z| < |z_{N+1}|\). The product of primary factors is
\[
P(z) = \prod_{1}^{N} E \left( \frac{z}{z_n}, p_n \right) \cdot \prod_{N+1}^{\infty} E \left( \frac{z}{z_n}, p_n \right) = \Pi_1 \cdot \Pi_2 ,
\]
say. We denote \(|z|, |z_n|, \left| \frac{z}{z_n} \right|\) by \( r, r_n, u_n \) respectively.

If \( p_n > 0 \), when \( n > n_0 \), we have
\[
\sum_{n_0+1}^{N} \log \left| 1 - \frac{z}{z_n} \right| - 2 \sum_{n_0+1}^{\infty} (2u_n)^{p_n} \leq \log \left| \prod_{n_0+1}^{N} E \left( \frac{z}{z_n}, p_n \right) \right|
\leq 2 \sum_{n_0+1}^{\infty} (2u_n)^{p_n}
\]
since \( u_n \geq \frac{1}{2} \) in \( \Pi_1 \).

In \( \Pi_2 \), we have \( u_n < \frac{1}{2} \) and so
\[
|\log |\Pi_2|| \leq |\log \Pi_2| \leq \sum_{N+1}^{\infty} \left| \log E \left( \frac{z}{z_n}, p_n \right) \right| \leq 2 \sum_{N+1}^{\infty} u_n^{p_n+1}.
\]

Combining the two inequalities, we have
\[
\sum_{1}^{N} \log \left| 1 - \frac{z}{z_n} \right| - 2 \sum_{n_0+1}^{N} (2u_n)^{p_n} - 2 \sum_{N+1}^{\infty} u_n^{p_n+1} \leq \log |P(z)|
\leq 2 \sum_{n_0+1}^{N} (2u_n)^{p_n} + 2 \sum_{N+1}^{\infty} u_n^{p_n+1} + 0(\log r). \tag{14}
\]

Let us suppose that the second order of \( P(z) \) is \( \varrho_2 \), where \( \varrho_2 \) is finite; and let \( L_2 = \lim_{r \to \infty} \frac{\log n(r)}{r^{\varrho_2}} < \infty \). We have
\[
r_n > \left( \frac{\log n}{H} \right)^a,
\]
when \( n > n_1 \), where \( a = 1/\varrho_2 \); and \( H \) is any fixed positive number greater than \( L_2 \).

If \( m \) denotes the greater of the two numbers \( n_0 \) and \( n_1 \), we have
We can easily see that the function \( \frac{p^x}{x^{\alpha x}} \) is steadily increasing or steadily decreasing, according as \( x < \frac{1}{Hr_a} \) or \( x > \frac{1}{Hr_a} \). Putting \( R = e^{1/e} \), \( R_1 = e^{s_1/e} \), we have

\[
I < 2 \sum_{n=1}^{n<R} \frac{(2rH_a)^n}{n^{a_1}} + 2 \frac{(2rH_a)^{\log R}}{(\log R)^{a_1}} + 2 \sum_{n>R} \frac{(2rH_a)^p_n}{p_n^{a_1}}
\]

\[
+ 2 \sum_{n=1}^{n<R_1} \frac{(rH_a)^n}{n^{a_1}} + 2 \frac{(rH_a)^{\log R_1}}{(\log R_1)^{a_1}} + 2 \sum_{n>R_1} \frac{(rH_a)^p_n}{p_n^{a_1}}.
\]

Now, if \([x]\) denotes the integral part of the positive number \(x\); and if \(s_1 = \left[\frac{s}{e}\right]\), where \(s\) is a positive integer, not less than \(e\), we have

\[
p_{3s} = [\log 3s] \geq [\log s] + 1
\]

\[
p_{s_1} = [\log s_1] = [\log s] - 1.
\]

Therefore, the number of times an integer \(p_s\) can be repeated is less than \(\frac{s(3e-1)}{e}\); and this is less than \((3e-1)e^{p_s}\). Consequently, we have

\[
I < \sum_{1}^{\infty} \frac{(2rH_a)^n}{n^{a_1}} + 2 \frac{(2rH_a)^{\log R}}{(\log R)^{a_1}} + 2(3e-1) \sum_{1}^{\infty} \frac{(2eH_a r)^n}{n^{a_1}}
\]

\[
+ 2 \sum_{1}^{\infty} \frac{(rH_a)^n}{n^{a_1}} + 2 \frac{(rH_a)^{\log R_1}}{(\log R_1)^{a_1}} + 2(3e-1) \sum_{1}^{\infty} \frac{(rH_a)^p_n}{p_n^{a_1}}.
\]

where \(A\) is a constant.
Since the type [7, § 2.2.9] of the entire function \( \sum_{n=1}^{\infty} \frac{(2eH^a r)^n}{n^m} \) is \((2e)^a \cdot H\), we have proved that
\[
I \leq e^{A_2 H r^2}
\]
(15)
for all sufficiently large \( r \), where \( A \) is an absolute constant.

By (14) and (15), we can easily show that
\[
T_2 \leq A_2 L_2.
\]
But, by Jensen’s theorem, we have
\[
L_2 \leq T_2.
\]
Combining the two, we have
\[
L_2 \leq T_2 \leq A_2 L_2.
\]

Next, let us suppose that the 3rd. order of \( f(z) \) is \( \varrho_3 \), where \( \varrho_3 \) is finite; and let
\[
L_3 = \lim_{r \to \infty} \frac{l_n(r)}{r^{\varrho_3}} < \infty.
\]
We have
\[
r_n > \left( \frac{l_n}{H} \right)^a,
\]
when \( n > n_2 \), where \( H \) is any fixed positive number greater than \( L_3 \) and \( a = 1/\varrho_3 \).

If \( m_1 \) be a positive integer greater than \( n_0 \) and \( n_2 \), such that \( \log \log m_1 > 1 \), we have
\[
I = 2 \sum_{m_1+1}^{N} (2u_n)^{n+1} + 2 \sum_{N+1}^{\infty} u_n^{n+1}
\]
\[
< 2 \sum_{m_1+1}^{N} (2u_n)^{n+1} + 2 \sum_{N+1}^{\infty} u_n^{n+1}
\]
\[
< 2 \sum_{m_1+1}^{N} \frac{(2H^a r)^{\log n}}{(\log \log n)^{a\log n}} + 2 \sum_{N+1}^{\infty} \frac{(H^a r)^{\log n}}{\log \log n)^{a\log n}}.
\]

Now, the function \( \frac{r^a}{(\log x)^{ax}} \) is steadily increasing or steadily decreasing, according as
\[
\log r \begin{cases} > & a \log \log x + \frac{a}{\log x} \\ < & \end{cases}
\]
Let \( r > 1 \). If \( n = R \) be a root of the equation
\[
\log (r^a) = al_3 n + \frac{a}{l_2 n},
\]
when \( n > m_1 \); and \( n = R_3 \) be a root of the same equation with \( r \) replaced by \( 2r \), then \( \log n < e^{Hr^a} \), when \( n = R_2 \) and \( \log n < e^{H(2r)^a} \), when \( n = R_3 \).

Consequently, if \( E_r \) be the set of values of \( r \), at which the inequality

\[
\log (rH^a) > al_n + \frac{a}{l_n}
\]

holds; and \( S_r \) the set at which the reverse inequality holds, then we have

\[
I < 2 \sum_{E_r} \frac{(2rH^a)^n}{(\log n)^a} + 2e_2 \left\{ H(2r)^a \right\} \cdot (2rH^a)^e + \n \]

\[
+ 2 \sum_{S_r} \frac{(2rH^a)^n}{(\log p_n)^a} + 2 \sum_{E_r} \frac{(rH^a)^n}{(\log n)^a} + 2e_2(Hr^a) \cdot (rH^a)^e + 2 \sum_{S_r} \frac{(rH^a)^n}{(\log p_n)^a} \n \]

\[
< 2 \sum_{m_{k+1}} \frac{(2rH^a)^n}{(\log p_n)^a} + 2e_2 \left\{ H(2r)^a \right\} \cdot (2rH^a)^e + \n \]

\[
+ 2 \sum_{m_{k+1}} \frac{(2rH^a)^n}{(\log p_n)^a} + 2(3e-1) \sum \frac{(2erH^a)^n}{(\log n)^a} + \n \]

\[
+ 2e_2 \left\{ H(2r)^a \right\} \cdot (2rH^a)^e + 2e_2(Hr^a) \cdot (rH^a)^e \n \]

\[
< A \sum \frac{(2erH^a)^n}{(\log n)^a} + 4e_2 \left\{ H(2r)^a \right\} \cdot (2rH^a)^e, \quad (16)
\]

where \( A \) is a constant.

It is easily seen, by putting \( k = 2 \) in the lemma, that the type of the series on the right-hand side is \( H(2e)^a \).

Therefore, by (14) and (16), we have

\[ T_3 \leq A_3 L_3. \]

Now, let us suppose that the \( k \)-th order of \( P(z) \) is \( \ell_k \), where \( \ell_k \) is finite; and let

\[ L_k = \lim_{r \to \infty} \frac{l_{k-1}n(r)}{r^{\ell_k}} < \infty. \]
We have
\[ r_n > \left( \frac{l_{k-1} n}{H} \right)^a, \]
when \( n > n_3 \), where \( H \) is any fixed positive number greater than \( L_k \) and \( a = 1/\ell_k \).

Let \( m_2 \) be a positive integer greater than \( n_0 \) and \( n_3 \), such that \( l_{k-2} m_2 > 1 \).

Proceeding in the same way as before, we can prove that
\[ I < A \sum_{m_2}^{\infty} \frac{(2erH^a)^n}{(l_{k-2} n)^a} + e_{k-1}(B r^a H), \]
where \( A \) and \( B \) are absolute constants.

The rest of the proof follows easily, if we put \((k-1)\) for \( k \) in the lemma.

**Corollary 1.** If \( f(z) = P(z)e^{Q(z)} \) is an entire function of finite \( k \)-th order, where \( P(z) \) is the product of primary factors of Theorem 6 formed with the zeros of \( f(z) \); and \( Q(z) \) is an entire function, then \( Q(z) \) is of finite or zero type, finite \((k-1)\)-th order, if \( f(z) \) is of finite or zero type.

**Proof.** By a slight modification of the proof of Theorem 6, it can be easily shown that the \( k \)-th order of the product of primary factors \( P(z) \) is equal to the \( k \)-th order of its zeros.

By (14), we have
\[ \log |P(z)| \geq \sum_{n=1}^{N} \log \left| 1 - \frac{z}{z_n} \right| - I, \]
where
\[ I = 2 \sum_{n_0+1}^{N} (2u_n)^{p_n} + 2 \sum_{N+1}^{\infty} u_n^{p_n+1}. \]

If \( f(z) \) is of finite type, \( L_k \) is finite.

Consequently, by Theorem (6), we have
\[ I < e_{k-1}(A r^{q_k}) \]
for all sufficiently large values of \( r \), where \( A \) is a constant.

Now, when \( r_n \leq 1 \), we have \( \left| 1 - \frac{z}{z_n} \right| > 1 \), provided that \( r > 2 \), and so
\[ \log \prod_{r_n \leq 1} \left| 1 - \frac{z}{z_n} \right| > 0. \]

But, when \( 1 < r_n \leq 2r \) and \( z \) lies outside all the small circles \( |z-z_n| = e^{-h e_z z_n (r_n^{q_k} + e)} \) for which \( r_n = |z_n| > 1 \), \( h \) being any
fixed number greater than 1, we have
\[ \left| 1 - \frac{z}{z_n} \right| = \frac{|z - z_n|}{r_n} \geq \frac{1}{r_n} \cdot e^{-he_{k-2}(r_n)_{\theta_k^+\epsilon}} \]
\[ \geq \frac{1}{2r} \cdot e^{-he_{k-2}(2r)_{\theta_k^+\epsilon}} \]

Therefore
\[ \log \prod_{1 > r_n \leq 2r} \left| 1 - \frac{z}{z_n} \right| \geq -N [he_{k-2}(2r)_{\theta_k^+\epsilon} + \log 2r] \]

Since \( L_k \) is finite, we have
\[ N < e_{k-1}(B_{\theta_k}) \]
for all sufficiently large \( r \), where \( B \) is a constant.

Combining these results, we have
\[ \log \prod_{1}^{N} \left| 1 - \frac{z}{z_n} \right| > -e_{k-1}(B_{\theta_k}) \cdot [he_{k-2}(2r)_{\theta_k^+\epsilon} + \log 2r]. \]

Consequently, we have
\[ \log |P(z)| > -e_{k-1}(B_{\theta_k}) [he_{k-2}(2r)_{\theta_k^+\epsilon} + \log 2r] - e_{k-1}(A_{r_{\theta_k}}) \]
\[ > -2e_{k-1}(c_{\theta_k}) \cdot e_{k-2}(2r)_{\theta_k^+\epsilon} \]
for all sufficiently large \( r \) such that the circle \(|z| = r\) intersects none of the small circles containing the zeros of \( f(z) \), \( c \) being any fixed number greater than each of \( A \) and \( B \).

Also, since \( f(z) \) is of finite type, we have
\[ |f(z)| < e_k(M_{r_{\theta_k}}) \]
for all sufficiently large \( r \), \( M \) being a constant.

Combining the two inequalities, we have
\[ |e^{Q(z)}| = \left| \frac{f(z)}{P(z)} \right| < e_k(M_{r_{\theta_k}}) \cdot e^{2e_{k-2}(c_{\theta_k}) \cdot e_{k-2}(2r)_{\theta_k^+\epsilon}} \]
\[ < e^{e_{k-1}(c_1 r_{\theta_k}) \cdot e_{k-2}(2r)_{\theta_k^+\epsilon}} \]
for a certain set of arbitrarily large values of \( r \), \( c_1 \) being an absolute constant.

Consequently, by the principle of the maximum modulus, it can be easily proved that
\[ |e^{Q(z)}| < e^{e_{k-1}(c_1 r_{\theta_k}) \cdot e_{k-2}(2r)_{\theta_k^+\epsilon}} \]
for all sufficiently large values of \( r \). Hence it follows that \( Q(z) \) is of finite type.

The proof for zero type follows the same lines.
COROLLARY 2(i). If \( f(z) = P(z)e^{Q(z)} \) is an entire function of finite 2nd. order, then a necessary and sufficient condition that \( f(z) \) be of finite or zero type is that \( L_2 \) be finite or zero and \( Q(z) \) satisfy the conditions of a theorem of Lindelöf (loc-cit.).

(ii) If \( f(z) = P(z)e^{Q(z)} \) is an entire function of finite 3rd. order, then a necessary and sufficient condition that \( f(z) \) be of finite or zero type is that \( L_3 \) be finite or zero and \( Q(z) \) satisfy the conditions of (i).

(iii) If \( f(z) = P(z)e^{Q(z)} \) is an entire function of finite \( k \)-th order, then a necessary and sufficient condition that \( f(z) \) be of finite or zero type is that \( L_k \) be finite or zero; and \( Q(z) \) satisfy the conditions for an entire function of finite \((k - 1)\)-th order to be of finite or zero type, where \( P(z) \) is a product of primary factors of Theorem 6, formed with the zeros of \( f(z) \).

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