COMPOSITIO MATHEMATICA

WERNER GAUTSCHI

Bounds of matrices with regard to an hermitian metric

Compositio Mathematica, tome 12 (1954-1956), p. 1-16
<http://www.numdam.org/item?id=CM_1954-1956__12__1_0>

© Foundation Compositio Mathematica, 1954-1956, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (http://http://www.compositio.nl/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Bounds of matrices with regard to an Hermitian metric [1])

by

Werner Gautschi.

§ 1. The bounds $\Omega_{H,K}$, $\omega_{H,K}$.

1. *Introduction.* In various questions concerning the solutions of systems of equations and the errors made by rounding off, the following definition of upper and lower bounds $\Omega(A)$, $\omega(A)$ of a matrix A has frequently been used:

$$\Omega(A) = \operatorname*{Max}_{\nu} \sqrt{\varrho_\nu}, \quad \omega(A) = \operatorname*{Min}_{\nu} \sqrt{\varrho_\nu}, \tag{1}$$

where ϱ_ν denote the eigenvalues of $\bar{A}'A$ (cf. e.g. [8] p. 1042 ff., or [9], p. 787, for the special case in which φ, χ are Euclidean lengths). In this paper we will discuss a generalization of this definition introducing as "parameters" two positive definite Hermitian matrices H, K. If H, K vary independently, the generalized bounds $\Omega_{H,K}(A)$, $\omega_{H,K}(A)$ can in general take values in the whole range $(0, \infty)$ (cf. § 1, section 3(vi)); to obtain appropriate values one has to couple H, K in some way. This can be done very naturally when A is an $n \times n$ matrix, by taking $K = H$. The bounds $\Omega_{H,H} \equiv \Omega_H$, $\omega_{H,H} \equiv \omega_H$ are in fact often more favourable for A than (1), but at the same time their actual calculation is considerably more difficult, as is shown by the examples given in § 2. If, however, A contains only a few non-vanishing elements, $\Omega_{H,K}(A)$ can fairly well be estimated from above by means of our theorem 2 in § 3, section 1, which generalizes a theorem due to W. Ledermann [7]. We will also make use of the theorem 2 in § 3, section 2, to determine both $\operatorname*{Inf}_{H} \Omega_H(A)$ and $\operatorname*{Sup}_{H} \omega_H(A)$, where H runs through all positive definite matrices. In sections 2 and 3 of § 1 we give the exact definitions and a few elementary properties of $\Omega_{H,K}(A)$ and $\omega_{H,K}(A)$, while in section 4 of § 1 a property not quite so trivial is proved.

---

[1]) This paper is part of the thesis for the Dr. phil.-degree at the University of Basle, Switzerland.

The idea of relating lengths of vectors to a positive definite Hermitian matrix H has recently been applied to the solution of linear equations $Ax = b$ by M. R. Hestenes and M. L. Stein [6]. Their main problem is to minimize the "H-length" of the residual vectors $r(x) = b - Ax$. Our definition of $\Omega_{H,K}(A)$, $\omega_{H,K}(A)$ involves a similar extremum problem, but (in contrast to [6]) with a *side condition.*

In defining the H-length of a vector we make use of the "scalar product" (x, y) with regard to H of two vectors x, y, as given e.g. in H. L. Hamburger and M. E. Grimshaw [4], p. 153. Such products (x, y) have also recently been used by W. Givens [2] to obtain theorems on the fields of values of a square matrix, which considerably extend the well known results due to O. Toeplitz [12] and F. Hausdorff [5].

I am very much indebted to Prof. Dr. A. Ostrowski for having most kindly allowed me to see through the manuscript of the yet unpublished book [10] from which I received many suggestions. In particular a chapter of [10] on the bounds (1) was the starting point of our investigations, which rather closely follow the disposition of this chapter.

2. *Notations and definitions.* Let $A = (a_{\mu\nu})$ $(\mu = 1, \ldots, m;$ $\nu = 1, \ldots, n)$ be an $m \times n$ matrix with real or complex elements $a_{\mu\nu}$. By $A^* = \bar{A}'$ we denote its conjugate-transpose and by $A^{(p)}$ $(p = 1, \ldots, \mathrm{Min}\,(m, n))$ its pth compound matrix, i.e. the $\binom{m}{p} \times \binom{n}{p}$ matrix consisting of all minors of A of order p. The groups of p rows and columns which form the minors are supposed to be arranged in lexicographical order. We have to use the following rules concerning $A^{(p)}$:

$$(AB)^{(p)} = A^{(p)}B^{(p)}, \quad (A^*)^{(p)} = (A^{(p)})^*, \tag{2}$$

if the product AB exists (cf. e.g. [1], p. 90ff). The first relation in (2) (the so called *Binet-Cauchy* theorem) is readily extended to more than two factors. Further if $m = n$ and A^{-1} exists, from (2) (with $B = A^{-1}$) it follows that

$$(A^{-1})^{(p)} = (A^{(p)})^{-1}. \tag{3}$$

tr A will denote the trace $\sum_\nu a_{\nu\nu}$ of a square matrix A, λ_A an eigenvalue of A and $|\lambda_A|^{\max}$, $|\lambda_A|^{\min}$ respectively the maximal, minimal modulus of the eigenvalues of A.

By x, y etc. we denote *column*-vectors of a k-dimensional complex Euclidean space, by x^* the conjugate-transposed row-vector

$\bar{x}'$ and by $|x|$ the Euclidean length of x. In order to introduce an Hermitian metric we define the scalar product (x, y) of two vectors x, y by

$$(x, y) \equiv y^* H x \quad (H > 0), \tag{4}$$

where H is an Hermitian matrix of order k; the meaning of the relation $H > 0$ is that H is positive definite. In particular (x, x) is real and $\geqq 0$ with $(x, x) = 0$ only when $x = 0$. We therefore define

$$\| x \| \equiv \sqrt{(x, x)} \tag{5}$$

as the *norm of x with regard to H*. Sometimes we add the subscript H and write $\| x \|_H$ instead of $\| x \|$. By routine arguments (cf. e.g. [11], p. 5, [3], p. 90—92, or [4], p. 4—5) the following three properties of $\| x \|$ are obtained:

$$\left.\begin{array}{l} \| x \| \geqq 0 \text{ with equality if and only if } x = 0 \\ \| \gamma x \| = | \gamma | \; \| x \| \; (\gamma \text{ any complex scalar}) \\ \| x + y \| \leqq \| x \| + \| y \|. \end{array}\right\} \tag{6}$$

Now let A be an $m \times n$ matrix and $H > 0, K > 0$ be Hermitian matrices of orders m, n respectively; we then define the upper and lower bounds $\Omega_{H,K}(A)$, $\omega_{H,K}(A)$ of A by

$$\left.\begin{array}{l} \Omega_{H,K}(A) = \underset{\| x \|_K = 1}{\text{Max}} \| A x \|_H = \left(\underset{\| x \|_K = 1}{\text{Max}} x^* A^* H A x \right)^{\frac{1}{2}}, \\ \omega_{H,K}(A) = \underset{\| x \|_K = 1}{\text{Min}} \| A x \|_H = \left(\underset{\| x \|_K = 1}{\text{Min}} x^* A^* H A x \right)^{\frac{1}{2}}. \end{array}\right\} \tag{7}$$

If in particular $m = n$ and $H = K$ we write $\Omega_{H,H} \equiv \Omega_H$, $\omega_{H,H} \equiv \omega_H$. The definition (7) can also (partly) be expressed in terms of Euclidean lengths: Let K be transformed to a diagonal matrix by the unitary matrix U:

$$D = U^* K U = \text{Diag}\,(k_1, \ldots, k_n), \; U^* U = I_n, \tag{8}$$

where I_n is the $n \times n$ unity matrix. Since $k_\nu > 0$ $(\nu = 1, \ldots, n)$ D can further be reduced to I_n by multiplying on the right and left by $\Delta = \text{Diag}\left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right)$:

$$\Delta U^* K U \Delta = I_n, \; \Delta = \text{Diag}\left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right). \tag{9}$$

If we now apply to x the substitution $x = U \Delta y$ we get

$$\| A x \|_H^2 = x^* A^* H A x = y^* \Delta U^* A^* H A U \Delta y \; (x = U \Delta y)$$

and by (9)

$$x^*Kx = y^*\Delta U^*KU\Delta y = y^*y.$$

Hence $\| x \|_K = 1$ implies $| y | = 1$ and viceversa; we therefore have

$$\mathfrak{F}_{\|x\|_K=1} \| Ax \|_H^2 = \mathfrak{F}_{|y|=1} y^*By, \quad B = \Delta U^*A^*HAU\Delta, \tag{10}$$

where $\mathfrak{F}_{\|x\|_K=1}$, $\mathfrak{F}_{|y|=1}$ denote the fields of values over the sets of vectors x, y with $\| x \|_K = 1$, $| y | = 1$ respectively. Since B is non-negative definite, from (10) we see that both Max, Min in (7) actually exist and

$$\Omega^2_{H,K}(A) = \lambda_B^{\max}, \quad \omega^2_{H,K}(A) = \lambda_B^{\min}, \quad B = \Delta U^*A^*HAU\Delta. \tag{11}$$

If in particular we take $H = I_m$, $K = I_n$, so that clearly $U = \Delta = I_n$, we obtain the bounds defined in (1).

Throughout this paper we denote respectively by $h_1, \ldots, h_m > 0$, $k_1, \ldots, k_n > 0$ the eigenvalues (not necessarily distinct and arranged in any order) of H, K and we put $h' = \operatorname{Max}_{\mu=1,\ldots,m} h_\mu$, $h'' = \operatorname{Min}_{\mu=1,\ldots,m} h_\mu$; $k' = \operatorname{Max}_{\nu=1,\ldots,n} k_\nu$, $k'' = \operatorname{Min}_{\nu=1,\ldots,n} k_\nu$.

3. *Elementary properties of* $\Omega_{H,K}$, $\omega_{H,K}$. If not otherwise stated in this section A, $H > 0$, $K > 0$ are respectively $m \times n$, $m \times m$, $n \times n$ matrices.

(i) The following properties of $\Omega_{H,K}$, $\omega_{H,K}$ are immediate consequences of (6) and (7):

$$\begin{gathered}\Omega_{H,K}(\gamma A) = |\gamma| \Omega_{H,K}(A), \omega_{H,K}(\gamma A) = |\gamma| \omega_{H,K}(A) \ (\gamma \text{ any complex scalar}) \\ \Omega_{H,K}(A+B) \leqq \Omega_{H,K}(A) + \Omega_{H,K}(B), \quad \omega_{H,K}(A+B) \geqq \omega_{H,K}(A) - \Omega_{H,K}(B),\end{gathered} \tag{12}$$

$$\begin{gathered}\Omega_{H,K}(A) = 0 \text{ if and only if } A = 0 \\ \omega_{H,K}(A) = 0 \text{ if and only if the rank of } A \text{ is } < n.\end{gathered} \tag{13}$$

(ii) Obviously we can also write

$$\Omega_{H,K}(A) = \operatorname*{Max}_{x \neq 0} \frac{\| Ax \|_H}{\| x \|_K}, \quad \omega_{H,K}(A) = \operatorname*{Min}_{x \neq 0} \frac{\| Ax \|_H}{\| x \|_K}, \tag{14}$$

so that for any $m \times n$ matrix C:

$\| Cx \|_H \leqq \Omega_{H,K}(C) \| x \|_K$, $\| Cx \|_H \geqq \omega_{H,K}(C) \| x \|_K$. Hence, if A, B, $L > 0$ respectively are $m \times l$, $l \times n$, $l \times l$ matrices, we have

$$\Omega_{H,K}(AB) \leqq \Omega_{H,L}(A)\Omega_{L,K}(B), \quad \omega_{H,K}(AB) \geqq \omega_{H,L}(A)\omega_{L,K}(B). \tag{15}$$

On the other hand, for any vector x with $Bx \neq 0$

$$\frac{\| ABx \|_H}{\| x \|_K} = \frac{\| A(Bx) \|_H}{\| Bx \|_L} \frac{\| Bx \|_L}{\| x \|_K} \; (Bx \neq 0). \tag{16}$$

Suppose now that for the vector x: $\dfrac{\| Bx \|_L}{\| x \|_K} = \Omega_{L,K}(B)$. Then by (16) and (14)

$$\Omega_{H,K}(AB) \geqq \frac{\| ABx \|_H}{\| x \|_K} = \frac{\| A(Bx) \|_H}{\| Bx \|_L} \Omega_{L,K}(B) \geqq \omega_{H,L}(A)\Omega_{L,K}(B).$$

Similarly, if B is of rank n, from (16) we deduce $\omega_{H,K}(AB) \leqq \Omega_{H,L}(A)\omega_{L,K}(B)$. If B is of rank $< n$, then the same holds for AB (cf. e.g. [1], p. 96—97) and therefore $\omega_{H,K}(AB) = \omega_{L,K}(B) = 0$. Thus we can extend (15) as follows:

$$\Omega_{H,K}(AB) \geqq \omega_{H,L}(A)\Omega_{L,K}(B), \; \omega_{H,K}(AB) \leqq \Omega_{H,L}(A)\omega_{L,K}(B). \tag{17}$$

(iii) Suppose that $m = n$ and A^{-1} exists; then putting $x = A^{-1}y$ we see that

$$\mathop{\mathfrak{F}}_{x \neq 0} \frac{\| Ax \|_H}{\| x \|_K} = \mathop{\mathfrak{F}}_{y \neq 0} \frac{\| y \|_H}{\| A^{-1}y \|_K} = \mathop{\mathfrak{F}}_{y \neq 0} \left(\frac{\| A^{-1}y \|_K}{\| y \|_H} \right)^{-1} \; (x = A^{-1}y).$$

Hence in using (14) we get

$$\Omega_{H,K}(A) = \frac{1}{\omega_{K,H}(A^{-1})}, \quad \omega_{H,K}(A) = \frac{1}{\Omega_{K,H}(A^{-1})}. \tag{18}$$

(iv) *Let S, T be two nonsingular matrices of orders m, n respectively; then we have*

$$\Omega_{H,K}(A) = \Omega_{S^*HS,\, T^*KT}(S^{-1}AT), \; \omega_{H,K}(A) = \omega_{S^*HS,\, T^*KT}(S^{-1}AT). \tag{19}$$

If in particular $m = n$, $\Omega_{H,K}$, $\omega_{H,K}$ do not change, if a unitary transformation S is applied both to A, H and K.

Indeed, putting $x = Ty$ we see that the field of values x^*A^*HAx over the set of vectors x with $x^*Kx = 1$ coincides with the field of values

$$y^*T^*A^*HATy = y^*(T^*A^*(S^*)^{-1})(S^*HS)(S^{-1}AT)y$$

taken for all vectors y with $y^*T^*KTy = 1$. Hence (19) follows at once from the definition (7).

(v) *Suppose that both A, H and K are respectively the "direct sums" of $A_1, \ldots, A_s$, $H_1, \ldots, H_s$ and $K_1, \ldots, K_s$, i.e. that in an obvious notation*

$$A = \mathrm{Diag}(A_1, \ldots, A_s), \; H = \mathrm{Diag}(H_1, \ldots, H_s), \; K = \mathrm{Diag}(K_1, \ldots, K_s),$$

where A_σ is an $m_\sigma \times n_\sigma$, $H_\sigma > 0$ an $m_\sigma \times m_\sigma$ and $K_\sigma > 0$ an $n_\sigma \times n_\sigma$ matrix ($\sigma = 1, \ldots, s$). Then we have

$$\Omega_{H,K}(A) = \operatorname*{Max}_{\sigma=1,\ldots,s} \Omega_{H_\sigma, K_\sigma}(A_\sigma), \omega_{H,K}(A) = \operatorname*{Min}_{\sigma=1,\ldots,s} \omega_{H_\sigma, K_\sigma}(A_\sigma). \quad (20)$$

In fact, let K_σ be transformed to a diagonal matrix by the unitary matrix U_σ ($\sigma = 1, \ldots, s$) and put $U = \operatorname{Diag}(U_1, \ldots, U_s)$, so that clearly (8) holds. Put $\Delta = \operatorname{Diag}\left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right) = \operatorname{Diag}(\Delta_1, \ldots, \Delta_s)$, Δ_σ being of the same order as K_σ ($\sigma = 1, \ldots, s$). Then obviously the matrix B in (10) is the direct sum of $\Delta_\sigma U_\sigma^* A_\sigma^* H_\sigma A_\sigma U_\sigma \Delta_\sigma$ ($\sigma = 1, \ldots, s$), whence (20) follows from (11).

(vi) *For every $H > 0$, $K > 0$ we have*

$$\sqrt{\frac{h''}{k'}}\, \omega(A) \leqq \omega_{H,K}(A) \leqq \Omega_{H,K}(A) \leqq \sqrt{\frac{h'}{k''}}\, \Omega(A), \quad (21)$$

where $\omega(A)$, $\Omega(A)$ are the bounds defined in (1).

Indeed, (21) follows from (14) by putting $y = Ax$ in

$$h'' \mid y \mid^2 \leqq \| y \|_H^2 \leqq h' \mid y \mid^2, \; k'' \mid x \mid^2 \leqq \| x \|_K^2 \leqq k' \mid x \mid^2.$$

(vii) *We have for any eigenvalue λ_A of a square matrix A:*

$$\omega_H(A) \leqq \mid \lambda_A \mid \leqq \Omega_H(A). \quad (22)$$

In fact, let x be an eigenvector corresponding to λ_A with $\| x \|_H = 1$. Then $Ax = \lambda_A x$, $\| Ax \|_H = \mid \lambda_A \mid$, whence (22) follows directly from (7).

4. For the proof of our first theorem we need the following

LEMMA 1. *Let S be an $n \times m$ matrix and T an $m \times n$ matrix. Then, if $m < n$, we have*

$$(ST)^{(p)} = 0 \quad (p > m). \quad (23)$$

PROOF. Put $S_0 = (SO_1)$, $T_0 = \begin{pmatrix} T \\ O_2 \end{pmatrix}$, where O_1, O_2 are $n \times (n-m)$, $(n-m) \times n$ zero-matrices respectively. Obviously both S_0 and T_0 are $n \times n$ matrices and $ST = S_0 T_0$. Hence by (2) $(ST)^{(p)} = S_0^{(p)} T_0^{(p)}$, and (23) follows from $S_0^{(p)} = T_0^{(p)} = 0$ ($p > m$).

LEMMA 2. *Suppose that $D = \operatorname{Diag}(k_1, \ldots, k_n)$ ($k_\nu > 0$),*

$$\Delta = \operatorname{Diag}\left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right) \text{ and } G = \operatorname{Diag}(h_1, \ldots, h_m) \; (h_\mu > 0),$$

$$\Gamma = \operatorname{Diag}\left(\frac{1}{\sqrt{h_1}}, \ldots, \frac{1}{\sqrt{h_m}}\right).$$ *Further let $R = (r_{\mu\nu})$ be an $m \times n$*

matrix and put $B = \Gamma R D R^* \Gamma$, $C = \Delta^{-1} R^* G^{-1} R \Delta^{-1}$. *Then, if* $\varphi(\lambda) = |\lambda I_m - B|$, $\psi(\lambda) = |\lambda I_n - C|$ *are the characteristic polynomials of* B, C, *we have*

$$\psi(\lambda) = \lambda^{n-m}\varphi(\lambda).$$

PROOF. Without loss of generality we may assume $m \leqq n$. Put $B = (b_{\mu\nu})$ $(\mu, \nu = 1, \ldots, m)$, $C = (c_{\mu\nu})$ $(\mu, \nu = 1, \ldots, n)$; by direct multiplication we get

$$b_{\mu\nu} = \frac{1}{\sqrt{h_\mu}} \frac{1}{\sqrt{h_\nu}} \sum_{\sigma=1}^{n} k_\sigma r_{\mu\sigma} \bar{r}_{\nu\sigma}$$

$$c_{\mu\nu} = \sqrt{k_\mu} \sqrt{k_\nu} \sum_{\tau=1}^{m} \frac{1}{h_\tau} \bar{r}_{\tau\mu} r_{\tau\nu}.$$

Hence

$$\operatorname{tr} B = \sum_{\mu=1}^{m} b_{\mu\mu} = \sum_{\mu=1}^{m} \sum_{\sigma=1}^{n} \frac{k_\sigma}{h_\mu} |r_{\mu\sigma}|^2 = \sum_{\nu=1}^{n} \sum_{\tau=1}^{m} \frac{k_\nu}{h_\tau} |r_{\tau\nu}|^2 = \sum_{\nu=1}^{n} c_{\nu\nu} = \operatorname{tr} C.$$

We now form the pth compound matrices $B^{(p)}$, $C^{(p)}$ of B, C; from (2), (3) it follows that

$$\begin{aligned} B^{(p)} &= \Gamma^{(p)} R^{(p)} D^{(p)} (R^{(p)})^* \Gamma^{(p)} \\ C^{(p)} &= (\Delta^{(p)})^{-1} (R^{(p)})^* (G^{(p)})^{-1} R^{(p)} (\Delta^{(p)})^{-1} \quad (p = 1, \ldots, m). \end{aligned}$$

Evidently $B^{(p)}$, $C^{(p)}$ are built analogously to B, C. Therefore our first conclusion again is applicable and we get

$$\operatorname{tr} B^{(p)} = \operatorname{tr} C^{(p)} \quad (p = 1, \ldots, m). \tag{24}$$

If $m < n$ by the lemma 1 with $S = \Delta^{-1} R^*$, $T = G^{-1} R \Delta^{-1}$ we have

$$C^{(p)} = 0 \quad (p > m). \tag{25}$$

Since generally $(-1)^p \operatorname{tr} A^{(p)}$ is the coefficient of λ^{n-p} in the characteristic polynomial $|\lambda I_n - A|$ of an $n \times n$ matrix A, (cf. e.g. [1], p. 88), our assertion now follows immediately from (24) and (25).

THEOREM 1. *Let* A *be an* $m \times n$ *matrix and* $H > 0$, $K > 0$ *be respectively of orders* m, n; *then we have*

$$\Omega_{K,H}(A^*) = \Omega_{H^{-1},K^{-1}}(A), \tag{26}$$

and, if $m = n$,

$$\omega_{K,H}(A^*) = \omega_{H^{-1},K^{-1}}(A). \tag{27}$$

PROOF. Let

$$G = V^* H V = \operatorname{Diag}(h_1, \ldots, h_m), \quad V^* V = I_m, \tag{28}$$

$$\Gamma V^* H V \Gamma = I_m, \quad \Gamma = \operatorname{Diag}\left(\frac{1}{\sqrt{h_1}}, \ldots, \frac{1}{\sqrt{h_m}}\right) \tag{29}$$

be the equations corresponding to (8), (9), applied to the matrix H. Then in using (8), (28) we have

$$D = U^*KU,\ D^{-1} = U^*K^{-1}U;\ \ G = V^*HV,\ G^{-1} = V^*H^{-1}V, \tag{30}$$
$$K = UDU^*,\ K^{-1} = UD^{-1}U^*;\ \ H = VGV^*,\ H^{-1} = VG^{-1}V^*. \tag{31}$$

According to (11) and (28)—(30) we have to examine the eigenvalues of

$$B = \Gamma V^*AKA^*V\Gamma,\quad C = \Delta^{-1}U^*A^*H^{-1}AU\Delta^{-1}.$$

By means of (31) we can write

$$B = \Gamma(V^*AU)D(U^*A^*V)\Gamma = \Gamma RDR^*\Gamma$$
$$C = \Delta^{-1}(U^*A^*V)G^{-1}(V^*AU)\Delta^{-1} = \Delta^{-1}R^*G^{-1}R\Delta^{-1},$$

putting $R = V^*AU$. If we now apply the lemma 2, our assertion follows at once.

COROLLARY 1. *For any square matrix A and $H > 0$ we have*

$$\Omega_H(A^*) = \Omega_{H^{-1}}(A),\ \omega_H(A^*) = \omega_{H^{-1}}(A). \tag{32}$$

COROLLARY 2. *If A is an Hermitian matrix, then for any $H > 0$*

$$\Omega_H(A) = \Omega_{H^{-1}}(A),\ \omega_H(A) = \omega_{H^{-1}}(A). \tag{33}$$

§ 2. Examples.

For the sake of simplicity in this section we only consider square $n \times n$ matrices A and we take $H = K$. As to the selection of examples we follow very closely the arrangement given by A. Ostrowski in [10].

(i) Let $A = (a_{\mu\nu})$ be a matrix the only non-vanishing element of which is $a_{ik} \equiv a$. Put $H = (h_{\mu\nu})$, $B = (b_{\mu\nu})$, $U = (u_{\mu\nu})$, where U satisfies (8) and B is the matrix defined in (10). By direct multiplication we get

$$b_{\mu\nu} = |a|^2 h_{ii}\bar{u}_{k\mu}u_{k\nu}\frac{1}{\sqrt{h_\mu}}\frac{1}{\sqrt{h_\nu}}.$$

If by v we denote the row-vector $\left(\frac{u_{k1}}{\sqrt{h_1}}, \ldots, \frac{u_{kn}}{\sqrt{h_n}}\right)$, B can be considered as the product $|a|^2 h_{ii} v^* v$ and is therefore of rank 1. Hence by (11) we have

$$\Omega_H^2(A) = \overset{\max}{\lambda_B} = \operatorname{tr} B = |a|^2 h_{ii}\sum_{\nu=1}^{n}\frac{|u_{k\nu}|^2}{h_\nu}.$$

On the other hand, by (31), $h_{ii} = \sum_{\nu=1}^{n} h_\nu \mid u_{i\nu} \mid^2$ and so

$$\Omega_H^2(A) = \mid a \mid^2 \sum_{\nu=1}^{n} h_\nu \mid u_{i\nu} \mid^2 \sum_{\nu=1}^{n} \frac{\mid u_{k\nu} \mid^2}{h_\nu}. \tag{34}$$

If in particular H is a diagonal matrix, and therefore $U = I_n$, we get

$$\Omega_H(A) = \mid a \mid \sqrt{\frac{h_i}{h_k}}, \quad H = \mathrm{Diag}\,(h_1, \ldots, h_n). \tag{35}$$

Let us in this example discuss, to what extent $\Omega_H(A)$ is determined by the eigenvalues of H. Clearly all Hermitian matrices having the *fixed* eigenvalues $h_1, \ldots, h_n > 0$ are obtained by letting U in $H = UDU^*$, $D = \mathrm{Diag}\,(h_1, \ldots, h_n)$, run through *all unitary* $n \times n$ matrices. In the case $i \neq k$, from (34) we can derive the following bounds, between which $\Omega_H^2(A)$ varies:

$$\frac{h''}{h'} \leqq \frac{1}{\mid a \mid^2} \Omega_H^2(A) \leqq \frac{h'}{h''},$$

where the upper and lower bounds are attained by taking in (34) for $(u_{i1}, \ldots, u_{in})$, $(u_{k1}, \ldots, u_{kn})$ suitable unit vectors. Similarly, if $i = k$, from (34) we see that $\frac{1}{\mid a \mid^2} \Omega_H^2(A)$ takes values in a certain closed interval, the left-hand end point of which by (22) is equal to 1.

If on the other hand we let H run through all diagonal matrices, (35) shows that in the case $i \neq k$ the range of $\Omega_H(A)$ is the whole interval $(0, \infty)$, while $\Omega_H(A)$ for $i = k$ is always equal to $\mid a \mid$.

Evidently in this example $\omega_H(A) = 0$ by (13).

(ii) Let $A = (a_{\mu\nu})$ be a matrix all elements of which are zero except those lying in the ith row, and put $(a_{i1}, \ldots, a_{in}) = (a_1, \ldots, a_n) = \alpha$. We suppose that $H = \mathrm{Diag}\,(h_1, \ldots, h_n)$, i.e. $U = I_n$. Then for the matrix B in (10) we have $B = \Delta A^* H A \Delta$, $b_{\mu\nu} = h_i \bar{a}_\mu a_\nu \frac{1}{\sqrt{h_\mu}} \frac{1}{\sqrt{h_\nu}}$ and as in our example (i) the rank of B is equal to 1, so that

$$\Omega_H^2(A) = \mathrm{tr}\, B = h_i \sum_{\nu=1}^{n} \frac{\mid a_\nu \mid^2}{h_\nu} = \mid a_i \mid^2 + h_i \sum_{\substack{\nu=1 \\ \nu \neq i}}^{n} \frac{\mid a_\nu \mid^2}{h_\nu}. \tag{36}$$

Clearly we have always $\omega_H(A) = 0$, and, by a suitable choice of H, $\Omega_H(A)$ can take values arbitrarily near to $\mid a_i \mid = \mid \lambda_A \mid^{\max}$.

(iii) If all elements of the matrix A are equal to $a \neq 0$ and if we take $H = \mathrm{Diag}\,(h_1, \ldots, h_n)$, we have $b_{\mu\nu} = \frac{1}{\sqrt{h_\mu}} \frac{1}{\sqrt{h_\nu}} \mid a \mid^2 \sum_{\varkappa=1}^{n} h_\varkappa$ and therefore by the Cauchy-Schwarz inequality

$$\Omega_H^2(A) = \mathrm{tr}\, B = \mid a \mid^2 \left(\sum_{\nu=1}^{n} h_\nu\right)\left(\sum_{\nu=1}^{n} \frac{1}{h_\nu}\right) \geqq \mid a \mid^2 n^2. \tag{37}$$

The lower bound for $\Omega_H(A)$, $\mid a \mid n = \mid \lambda_A \mid^{\max}$, is attained for $H = I_n$, while $\Omega_H(A)$ is not bounded at all from above. On the other hand $\omega_H(A) = 0$.

(iv) Let $A = \mathrm{Diag}\,(a_1, \ldots, a_n)$, $H = \mathrm{Diag}\,(h_1, \ldots, h_n)$. Then $B = \mathrm{Diag}\,(\mid a_1 \mid^2, \ldots, \mid a_n \mid^2)$, so that

$$\Omega_H(A) = \operatorname*{Max}_{\nu} \mid a_\nu \mid = \mid \lambda_A \mid^{\max}, \quad \omega_H(A) = \operatorname*{Min}_{\nu} \mid a_\nu \mid = \mid \lambda_A \mid^{\min}. \tag{38}$$

(v) Let $A = (a_{\mu\nu})$ be a matrix all elements of which are zero except those lying in the ith row and k^{th} column, while we have also $a_{ik} = 0$. We further assume H to be a diagonal matrix. In applying to both A and H the same permutation to the rows and columns, whereby in virtue of § 1, section 3(iv), $\Omega_H(A)$, $\omega_H(A)$ are not changed, we can make $k = 1$. Having carried through this transformation we denote by $\alpha = (a_2, \ldots, a_n)$ the ith $(n-1)$-dimensional row-vector of A (without its first element), by $\beta = (b_1, \ldots, b_n)$ the first (n-dimensional) column-vector of A (where $b_i = 0$) and we put $H = \mathrm{Diag}\,(h_1, \ldots, h_n)$, $\Delta_1 = \mathrm{Diag}\left(\frac{1}{\sqrt{h_2}}, \ldots, \frac{1}{\sqrt{h_n}}\right)$. For the matrix B of (10) we then obtain by direct multiplication (observing that $U = I_n$)

$$B = \begin{pmatrix} \frac{1}{h_1}\beta^* H \beta & 0 \\ 0 & h_i \Delta_1 \alpha^* \alpha \Delta_1 \end{pmatrix}.$$

Since again the $(n-1) \times (n-1)$ matrix in the lower right-hand corner of B is of rank 1, it follows from (11) that

$$\Omega_H(A) = \mathrm{Max}\,(\Omega_1, \Omega_2), \text{ where } \begin{cases} \Omega_1 = \left(\frac{1}{h_1} \sum\limits_{\substack{\nu=1 \\ \nu \neq i}}^{n} h_\nu \mid b_\nu^2 \mid\right)^{\frac{1}{2}} \\ \Omega_2 = \left(h_i \sum\limits_{\nu=2}^{n} \frac{1}{h_\nu} \mid a_\nu \mid^2\right)^{\frac{1}{2}} \end{cases}. \tag{39}$$

$\omega_H(A) = 0 \quad (n > 2)$.

§ 3. A generalization of Ledermann's theorem and the determination of $\operatorname{Inf}_{H>0} \Omega_H$, $\operatorname{Sup}_{H>0} \omega_H$.

1. The reason we succeeded to calculate directly Ω_H, ω_H in the examples given in § 2 was that the matrices A contained a sufficiently large number of zeros. We now prove a general theorem which in similar cases always yields an upper bound for $\Omega_{H,K}(A)$ and which is a generalization of a theorem due to W. Ledermann [7]. More precisely:

THEOREM 2. *Let $A = (a_{\mu\nu})$ be an $m \times n$ matrix and denote by α_μ its μth row-vector; then, if $H = \operatorname{Diag}(h_1, \ldots, h_m)$ $(h_\mu > 0)$, $K = \operatorname{Diag}(k_1, \ldots, k_n)$ $(k_\nu > 0)$ and if every column-vector of A contains at most s non-vanishing elements, we have*

$$\Omega^2_{H,K}(A) \leqq \sum_{\sigma=1}^{s} h_{\mu_\sigma} \| \alpha_{\mu_\sigma} \|^2_{K^{-1}}, \tag{40}$$

where the sum on the right-hand side has to be taken over the s largest numbers $h_{\mu_\sigma} \| \alpha_{\mu_\sigma} \|^2_{K^{-1}}$ $(\sigma = 1, \ldots, s)$ among $h_\mu \| \alpha_\mu \|^2_{K^{-1}}$ $(\mu = 1, \ldots, m)$.

PROOF. Our proof is essentially the same as that given for the case $H = I_m$, $K = I_n$ by A. Ostrowski in [10].

Without loss of generality we may assume that

$$h_1 \| \alpha_1 \|^2_{K^{-1}} \geqq h_2 \| \alpha_2 \|^2_{K^{-1}} \geqq \ldots \geqq h_m \| \alpha_m \|^2_{K^{-1}}. \tag{41}$$

Indeed, let a permutation P be applied to the rows of A; if we further permute the rows *and the columns* of H according to P, by (19) (with $T = I_n$) $\Omega_{H,K}(A)$ does not change and the numbers $h_\mu \| \alpha_\mu \|^2_{K^{-1}}$ $(\mu = 1, \ldots, m)$ are arranged as required.

Let $\operatorname{Max}_{\| x \|_{K^{-1}}} \| Ax \|^2_H$ be attained for the vector $x = (x_1, \ldots, x_n)$ and put $y = Ax$, $y = (y_1, \ldots, y_m)$. For every μ $(\mu = 1, \ldots, m)$ replace the coordinates x_ν of x for which $a_{\mu\nu} = 0$ by zeros and denote the vector so obtained by $x^{(\mu)}$. Then we have

$$\Omega^2_{H,K}(A) = \| y \|^2_H = \sum_{\mu=1}^{m} h_\mu | y_\mu |^2 = \sum_{\mu=1}^{m} h_\mu | \alpha_\mu x^{(\mu)} |^2. \tag{42}$$

We further put $x^{(\mu)} = (x_1^{(\mu)}, \ldots, x_n^{(\mu)})$ $(\mu = 1, \ldots, m)$. Since by the Cauchy-Schwarz inequality

$$| \alpha_\mu x^{(\mu)} |^2 = \left| \sum_{\nu=1}^{n} a_{\mu\nu} x_\nu^{(\mu)} \right|^2 \leqq \left(\sum_{\nu=1}^{n} \frac{1}{\sqrt{k_\nu}} | a_{\mu\nu} | \sqrt{k_\nu} \, | x_\nu^{(\mu)} | \right)^2 \leqq$$

$$\leqq \left(\sum_{\nu=1}^{n} \frac{1}{k_\nu} | a_{\mu\nu} |^2 \right) \left(\sum_{\nu=1}^{n} k_\nu | x_\nu^{(\mu)} |^2 \right) = \| \alpha_\mu \|^2_{K^{-1}} \sum_{\nu=1}^{n} k_\nu | x_\nu^{(\mu)} |^2,$$

from (42) we get

$$\Omega^2_{H,K}(A) \leqq \sum_{\mu=1}^{m} h_\mu \,||\, \alpha_\mu \,||^2_{K^{-1}} \sum_{\nu=1}^{n} k_\nu \,|\, x_\nu^{(\mu)} \,|^2 =$$

$$= \sum_{\nu=1}^{n} k_\nu \left[\sum_{\mu=1}^{m} |\, x_\nu^{(\mu)} \,|^2 h_\mu \,||\, \alpha_\mu \,||^2_{K^{-1}} \right]. \quad (43)$$

If $x_\nu^{(\mu)} \neq 0$ then

$$x_\nu^{(\mu)} = x_\nu \quad (x_\nu^{(\mu)} \neq 0) \quad (44)$$

and $a_{\mu\nu} \neq 0$. From this and the hypothesis it follows that for any fixed ν at most s of the $x_\nu^{(\mu)}$ are $\neq 0$. Therefore taking the sum in brackets on the right-hand side of (43) only over the terms with $x_\nu^{(\mu)} \neq 0$ and using (44), (41) we see that

$$\sum_{\mu=1}^{m} |\, x_\nu^{(\mu)} \,|^2 h_\mu \,||\, \alpha_\mu \,||^2_{K^{-1}} \leqq |\, x_\nu \,|^2 \sum_{\sigma=1}^{s} h_\sigma \,||\, \alpha_\sigma \,||^2_{K^{-1}} \quad (\nu = 1, \ldots, n),$$

whence by (43)

$$\Omega^2_{H,K}(A) \leqq \left(\sum_{\nu=1}^{n} k_\nu \,|\, x_\nu \,|^2 \right) \left(\sum_{\sigma=1}^{s} h_\sigma \,||\, \alpha_\sigma \,||^2_{K^{-1}} \right).$$

This proves our assertion, since $\sum_{\nu=1}^{n} k_\nu \,|\, x_\nu \,|^2 = ||\, x \,||^2_K = 1$.

Remarks. The theorem of Ledermann is obtained by taking $H = I_m$, $K = I_n$. If in particular we apply (40) with $H = K$ to our examples (i), (ii) and (iv) we obtain respectively as upper bounds $\frac{h_i}{h_k} |\, a \,|^2$, $h_i \,||\, \alpha \,||^2_{H^{-1}}$, $\operatorname{Max}_\nu |\, a_\nu \,|^2$, which all coincide with the corresponding Ω^2_H.

Even if $s = m$ the theorem 2 is often useful. Take e.g.

$$A = \begin{pmatrix} 2 & 1 & 8 \\ 7 & 0 & 5 \\ 0 & 1 & 3 \end{pmatrix},$$

where the elements of the second column are comparatively small. In order to get favourable bounds for $\Omega_H(A)$ in applying (40), we choose h_2 relatively small. With $H = \operatorname{Diag}(4, 1, 20)$ we obtain $\Omega^2_H(A) \lneqq 63{,}3$, while $H = I$ gives $\Omega^2(A) \leqq 153$.

2. We now use our theorem 2 to give a refinement of (22):

Theorem 3. *For any $n \times n$ matrix A we have*

$$\operatorname{Inf}_{H>0} \Omega_H(A) = |\, \lambda_A \,|^{\max}, \quad \operatorname{Sup}_{H>0} \omega_H(A) = |\, \lambda_A \,|^{\min}. \quad (45)$$

If in particular A has only simple elementary divisors both Inf and Sup in (45) are attained for suitable matrices $H > 0$.

PROOF. Since for a nonsingular matrix $|\lambda_{A^{-1}}|^{\max} = 1/|\lambda_A|^{\min}$ and by (18) $\omega_H(A) = \dfrac{1}{\Omega_H(A^{-1})}$, it is sufficient to prove the relation concerning $\operatorname{Inf}\Omega_H$. Let A be transformed to Jordan's canonical form by the nonsingular matrix S:

$$S^{-1}AS = \Lambda + C,$$

where Λ is a diagonal matrix the elements of which are the eigenvalues of A, and C denotes a matrix consisting of zeros except possibly some elements $c_{\mu\nu} = 1$ with $\nu = \mu + 1$. If in (19) we take $T = S$ we have by (22), (12)

$$|\lambda_A|^{\max} \leqq \Omega_H(A) = \Omega_K(\Lambda + C) \leqq \Omega_K(\Lambda) + \Omega_K(C), \quad (46)$$

where $K = S^*HS$. It suffices to show that for a suitable choice of K the sum on the right-hand side of (46) is arbitrarily near to $|\lambda_A|^{\max}$. Take $K = \operatorname{Diag}(k_1, \ldots, k_n)$; then by (38) $\Omega_K(\Lambda) = |\lambda_A|^{\max}$; if on the other hand γ_ν is the νth row-vector of C, then $\|\gamma_\nu\|^2_{K^{-1}} = \begin{cases} 0 (\gamma_\nu = 0) \\ 1/k_{\nu+1} (\gamma_\nu \neq 0) \end{cases}$. Hence by the theorem 2

$$\Omega_K^2(C) \leqq \operatorname*{Max}_{\nu=1,\ldots,n-1} \frac{k_\nu}{k_{\nu+1}},$$

which obviously can be made as small as we please.

If all elementary divisors of A are simple we have in (46) $C = 0$, $\Omega_K(C) = 0$, so that $\Omega_H(A)$ attains the value $|\lambda_A|^{\max}$ for a suitable matrix $H > 0$.

It is natural to ask whether we could in (45) take Inf, Sup only over the set of all *diagonal* matrices $H > 0$. This is however not true as the following example shows: Take

$$A = \begin{pmatrix} 0 & i & 1 \\ i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (i^2 = 1),$$

where $|\lambda_A|^{\max} = 0$. If $H = \operatorname{Diag}(h_1, h_2, h_3)$, it follows from § 2, Ex. (v), that

$$\Omega_H(A) = \operatorname{Max}(\Omega_1, \Omega_2), \quad \text{where} \begin{cases} \Omega_1 = \sqrt{\dfrac{h_2 + h_3}{h_1}} \\ \Omega_2 = \sqrt{h_1\left(\dfrac{1}{h_2} + \dfrac{1}{h_3}\right)} = \sqrt{\dfrac{h_1(h_2 + h_3)}{h_2 h_3}}. \end{cases}$$

But by the inequality of the arithmetic and geometric mean

$$\Omega_1\Omega_2 = \frac{h_2 + h_3}{\sqrt{h_2 h_3}} \geqq 2,$$

so that certainly $\Omega_H(A) \geqq \sqrt{2}$ for all diagonal matrices $H > 0$.

The second statement in theorem 3 can be made more precise by the following

THEOREM 4. *In order that for some matrix* $H > 0$

$$\Omega_H(A) = |\lambda_A|^{\max} \tag{47}$$

it is necessary and sufficient that the elementary divisors corresponding to the eigenvalues of A *with maximal modulus are simple. Similarly, if* A *is a non-singular matrix, we have*

$$\omega_H(A) = |\lambda_A|^{\min} \quad (|\lambda_A|^{\min} > 0) \tag{48}$$

for some $H > 0$, *if and only if all elementary divisors associated with the eigenvalues of* A *of minimal modulus are simple.*

PROOF. *Necessity*: let λ be an eigenvalue of A of either maximal or minimal modulus having multiple elementary divisors. It then suffices to show that, given a matrix $H > 0$, there always exists a vector x for which

$$\frac{\|Ax\|_H}{\|x\|_H} \begin{cases} < |\lambda_A|^{\min}, & \text{if } |\lambda| = |\lambda_A|^{\min} \\ > |\lambda_A|^{\max}, & \text{if } |\lambda| = |\lambda_A|^{\max}. \end{cases} \tag{49}$$

From Jordan's canonical form of A it is easily seen that under our hypothesis on λ there exist two linearly independent vectors u_1, u_2 such that $Au_1 = \lambda u_1$, $Au_2 = \lambda u_2 + u_1$. Put $v_1 = u_1$, $v_2 = \alpha u_1 + u_2$; in order to make v_1, v_2 orthogonal with respect to H, using the notation (4) we must have

$$(v_2, v_1) = (\alpha u_1 + u_2, u_1) = \alpha(u_1, u_1) + (u_2, u_1) = 0,$$
$$\alpha = -(u_2, u_1)/(u_1, u_1).$$

Clearly $Av_1 = \lambda v_1$, $Av_2 = \alpha\lambda u_1 + \lambda u_2 + u_1 = \lambda v_2 + v_1$, and so the vectors $w_1 = v_1/\|v_1\|_H$, $w_2 = v_2/\|v_2\|_H$ satisfy

$$\begin{aligned} Aw_1 &= \lambda w_1 \\ Aw_2 &= \lambda w_2 + \beta w_1 \end{aligned} \quad (\|w_1\|_H = \|w_2\|_H = 1,\ (w_2, w_1) = 0), \tag{50}$$

where $\beta = \|v_1\|_H/\|v_2\|_H > 0$. We now take

$$x = \gamma w_1 + w_2 \tag{51}$$

and determine the scalar γ in such a way that (49) holds. In fact, by (50)

$$Ax = \gamma\lambda w_1 + \lambda w_2 + \beta w_1 = \lambda x + \beta w_1,$$
$$\|Ax\|_H^2 = x^*A^*HAx = (\bar{\lambda}x^* + \beta w_1^*)(\lambda Hx + \beta Hw_1) =$$
$$= |\lambda|^2 \|x\|_H^2 + 2\Re(\beta\lambda w_1^* Hx) + \beta^2.$$

Substituting the expression (51) for x in w_1^*Hx we obtain

$$\frac{\|Ax\|_H^2}{\|x\|_H^2} = |\lambda|^2 + \frac{\beta}{\|x\|_H^2}\left\{2\Re(\gamma\lambda) + \beta\right\} \qquad (\beta > 0).$$

Now (49) certainly holds, if in the case $|\lambda| = |\lambda_A|^{\min}$ we choose γ such that $\Re(\gamma\lambda) < \frac{-\beta}{2}$ and $\gamma = 0$ if $|\lambda| = |\lambda_A|^{\max}$.

Sufficiency: suppose that all eigenvalues with maximal modulus have simple elementary divisors. Let A be transformed to Jordan's canonical form $S^{-1} A S = J = \mathrm{Diag}(J_1, J_2)$, where J_1 is a *diagonal* matrix containing the eigenvalues λ with $|\lambda| = |\lambda_A|^{\max}$. By (19) we have $\Omega_H(A) = \Omega_K(J)$ $(K = S^*HS)$. To show that for a suitable matrix $K > 0$: $\Omega_K(J) = |\lambda_A|^{\max}$, take $K = \mathrm{Diag}(K_1, K_2)$, where $K_1, K_2 > 0$ are matrices of the same order as J_1, J_2 respectively and K_1 is a unity matrix. Since $\Omega_{K_1}(J_1) = |\lambda_A|^{\max}$, from (20) we get

$$\Omega_K(J) = \mathrm{Max}\{|\lambda_A|^{\max}, \Omega_{K_2}(J_2)\}.$$

On the other hand $|\lambda_{J_2}|^{\max} < |\lambda_A|^{\max}$, whence, by theorem 3, K_2 can be chosen such that $\Omega_{K_2}(J_2) < |\lambda_A|^{\max}$.

A similar argument shows that (48) holds for some $H > 0$, if all eigenvalues of A with minimal modulus are simple.

REFERENCES.

A. C. Aitken.

[1] Determinants and matrices, University Math. Texts, vol. 1, 7th edition. Edinburgh & London 1951.

[2] W. Givens.
Fields of values of a matrix, Bull. Amer. Math. Soc., vol. 58 (1952), p. 53.

P. R. Halmos.

[3] Finite dimensional vector spaces, Annals of Mathematics Studies, nr. 7, Princeton 1942.
(P. 91, line 10: instead of $|y|^2$ read $\|y\|^2$
p. 91, line 15: instead of $R[\alpha\beta(x, y)]$ read $R[\alpha\bar{\beta}(x, y)]$.)

H. L. Hamburger and M. E. Grimshaw.

[4] Linear Transformations in n-dimensional vector space, Cambridge University Press, Cambridge 1951.

F. Hausdorff.

[5] Der Wertevorrat einer Bilinearform, Math. Z., Bd. 3 (1919), pp. 314—316.

M. R. Hestenes and M. L. Stein.

[6] The Solution of Linear Equations by Minimization, NBS-NAML Report 52—45, Washington, D.C., 1951.

W. Ledermann.

[7] On an upper limit for the latent roots of a certain class of matrices, J. Lond. Math. Soc., vol. 12 (1937), pp. 12—18.

J. von Neumann and H. H. Goldstine.

[8] Numerical inverting of matrices of high order, Bull. Amer. Math. Soc., vol. 53 (1947), pp. 1021—1099.

A. Ostrowski.

[9] Un nouveau théorème d'existence pour les systèmes d'équations, C. R. Acad. Sci. Paris, t. 232 (1951), pp. 786—788.

[10] Leçons sur la résolution des systèmes d'équations, to appear in the series of the cahiers scientifiques at Gauthier-Villars, Paris.

M. H. Stone.

[11] Linear Transformations in Hilbert Space and their applications to Analysis, Amer. Math. Soc. Coll. Publ., vol. XV (1932).

O. Toeplitz.

[12] Das algebraische Analogon zu einem Satze von Fejér, Math. Z., Bd. 2 (1918), pp. 187—197.

(Oblatum 10-2-53).
University of Basle, Switzerland.