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C. T. RAJAGOPAL

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# On Tauberian oscillation theorems

by

C. T. Rajagopal.

Madras

## Introduction.

It is known that certain positive (regular) transforms of a function  $s(t)$ , or of a sequence  $s_n$ , possess the property that the boundedness of the transform implies the boundedness of  $s(t)$ , or of  $s_n$ , when an appropriate Tauberian condition is imposed on  $s(t)$ , or on  $s_n$ . Two transforms with the property stated above, covering many particular cases of summability processes, have been considered by Karamata ([3], Théorème IV) and Hardy ([2], Theorem 238). Theorems A and B of this note show that, in the case of these two transforms, the oscillation of the transform is equal to the oscillation of  $s(t)$ , or  $s_n$ , provided that certain additional conditions are assumed, one of which consists in a refinement of the Tauberian hypothesis on  $s(t)$ , or on  $s_n$ .

## 1. The first theorem.

The following theorem generalizes the essentials of a Tauberian oscillation theorem of V. Ramaswami ([6], Theorem I.2), somewhat like a theorem of H. Delange ([1], Théorème 11), but more completely, since it includes an analogue of Ramaswami's theorem for the Borel transform. It is a refinement of an oscillation theorem of which one version is Lemma 3 A of this note proved by Karamata ([4], Satz IV); and it is on the same lines as a convergence theorem of Karamata ([4], Satz VI; [3], pp. 36—7, § 18).<sup>1)</sup>

**THEOREM A.** *Let the following assumptions be made.*

- (i)  $\psi(x, t) \geq 0$  for all large  $x$  and every  $t \geq 0$ ;
- (ii)  $\varphi(x, y) \equiv \int_y^\infty \psi(x, t) dt \rightarrow 1$  as  $x \rightarrow \infty$ , for every  $y \geq 0$ ;  $\varphi(x, 0) = 1$ .

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<sup>1)</sup> Basing ourselves on Karamata's ideas, we can extend also a theorem of Delange on absolute Tauberian constants ([1], Théorème 3) so that it covers the case of the Borel transform. Such an extension appears elsewhere [5].

(iii)  $y' = y'(x)$ ,  $y'' = y''(x)$  are single-valued, steadily increasing, unbounded functions of  $x$ , and conversely, such that, given any small  $\varepsilon > 0$ , we have, for all large enough  $x$ ,  $y'$ ,  $y''$ ,

$$(1) \quad \varphi(x, y') < \varepsilon, \quad \varphi(x, y'') > 1 - \varepsilon,$$

with the implication, on account of (i), that  $y'' < y'$ .

(iv)  $\Lambda(t)$  is a continuous, strictly increasing, unbounded function satisfying the condition

$$(2) \quad \int_0^{\infty} \psi(x, t) \left| \log \frac{\Lambda(t)}{\Lambda(y)} \right| dt < K \quad (\text{a constant}),$$

where  $y = \text{either } y' \text{ or } y''$ .

(v)  $s(t)$  is a function of bounded variation in every finite interval of  $(0, \infty)$  subject to the conditions:

$$(3) \quad \text{bound} \{s(t') - s(t)\}_{t \leq t' \leq T} = o_L(1) \log \lambda \text{ as } t \rightarrow \infty,$$

for  $\Lambda(T) = \lambda \Lambda(t)$ ,  $\lambda > 1$ ;

$$(4) \quad \Psi(x) \equiv \int_0^{\infty} \psi(x, t) s(t) dt = O(1) \text{ as } x \rightarrow \infty.$$

Then

$$(5) \quad \lim_{t \rightarrow \infty} s(t) = \lim_{x \rightarrow \infty} \Psi(x), \quad \overline{\lim}_{t \rightarrow \infty} s(t) = \overline{\lim}_{x \rightarrow \infty} \Psi(x).$$

## 2. Lemmas.

To prove Theorem A we require the following lemmas.

LEMMA 1A. Hypotheses (i) and (ii) of Theorem A imply

$$\lim_{t \rightarrow \infty} s(t) \leq \overline{\lim}_{x \rightarrow \infty} \Psi(x) \leq \overline{\lim}_{t \rightarrow \infty} s(t), \text{ where } \Psi(x) = \int_0^{\infty} \psi(x, t) s(t) dt,$$

$s(t)$  being a function of bounded variation in every finite interval of  $(0, \infty)$ .

This is a simple Abelian result proved by an argument of the usual type (cf. [2], proof of Theorem 9).

LEMMA 2A. The condition

$$(3') \quad \text{bound} \{s(t') - s(t)\}_{t \leq t' \leq T} > -\omega < 0, \quad \Lambda(T) = \lambda \Lambda(t), \quad \lambda > 1,$$

involves

$$(6) \quad s(u) - s(t) > -\frac{\omega}{\log \lambda} \log \left\{ \lambda \frac{\Lambda(u)}{\Lambda(t)} \right\} \text{ for every } u \geq t > 0.$$

This lemma is due to Karamata ([4], Hilfsatz 1) and may be

readily proved in the form

$$s(V(u)) - s(V(t)) > -\frac{\omega}{\log \lambda} \log \left( \lambda \frac{u}{t} \right), \quad u \geq t > 0,$$

obtained by replacing  $u, t$  in (6) by  $V(u), V(t)$  respectively,  $V(x)$  being the function which is the inverse of  $\Lambda(x)$ . For, assuming that  $\lambda^{r-1}t \leq u < \lambda^r t$  ( $r \geq 1$ ), we have

$$s(V(u)) - s(V(t)) = \{s(V(u)) - s(V(\lambda^{r-1}t))\} \\ + \{s(V(\lambda^{r-1}t)) - s(V(\lambda^{r-2}t))\} + \dots + \{s(V(\lambda t)) - s(V(t))\},$$

and thus

$$(7) \quad s(V(u)) - s(V(t)) > -\omega r,$$

by virtue of (3') in the form

$$s(V(t')) - s(V(t)) > -\omega, \quad t \leq t' \leq \lambda t.$$

Since

$$r - 1 \leq \frac{\log(u/t)}{\log \lambda},$$

it follows from (7) that

$$s(V(u)) - s(V(t)) > -\omega \left\{ 1 + \frac{\log(u/t)}{\log \lambda} \right\}$$

which, as already explained, is equivalent to (6).

LEMMA 3A. *If, in the hypotheses of Theorem A, (1) is replaced by*

$$(1') \quad \varphi(x, y') < c' < c'' < \varphi(x, y'')$$

*and (3) is replaced by (3'), the conclusion of the theorem will assume the form*

$$\Psi(x) = O(1) \text{ as } x \rightarrow \infty \text{ involves } s(t) = O(1) \text{ as } t \rightarrow \infty.$$

*In particular, since (1) implies (1') and (3) implies (3'), conditions (1)—(4) together imply  $|s(t)| < \kappa$  for  $t \geq 0$ .*

This lemma again is due to Karamata ([4], Satz IV; cf. [3], p. 35, Théorème IV).

### 3. Proof of Theorem A.

In the identity

$$(8) \quad \Psi(x) = \int_0^{y''} \psi(x, t) s(t) dt + \int_{y''}^{y'} \dots + \int_{y'}^{\infty} \dots,$$

suppose that  $x \geq X$  and  $\psi(x, t) \geq 0$ . First let  $y''$  assume an

ascending, divergent sequence of values for which

$$s(y'') \rightarrow \bar{s} \equiv \overline{\lim}_{t \rightarrow \infty} s(t),$$

this limit being finite as a result of Lemma 3A. By hypothesis (iii), the corresponding sequence of values of  $x = x(y')$  and the sequence of values of  $y = y'(x)$  are both ascending, divergent and such that, for all large  $y', x, y''$ ,

$$(9) \quad \int_0^{y''} \psi(x, t) dt \equiv \int_0^{\infty} \dots - \varphi(x, y'') < \varepsilon, \quad \int_{y'}^{\infty} \psi(x, t) dt \equiv \varphi(x, y') < \varepsilon.$$

Also, in consequence of (3), we can choose  $t_0$  so that, for  $t \geq t_0$ ,

$$\text{bound}_{t \leq t' \leq T} \{s(t') - s(t)\} > -\varepsilon \log \lambda, \quad \Lambda(T) = \lambda \Lambda(t),$$

and hence, by Lemma 2A,

$$(10) \quad s(u) - s(t) > -\varepsilon \log \left\{ \lambda \frac{\Lambda(u)}{\Lambda(t)} \right\}, \quad u \geq t \geq t_0.$$

In (8) we can confine ourselves to values of  $y'' \geq t_0$  for which  $x \geq X$ , and use (10), obtaining

$$\begin{aligned} \Psi(x) &> \int_0^{y''} \psi(x, t) s(t) dt + s(y'') \int_{y''}^{y'} \psi(x, t) dt \\ &- \varepsilon \int_{y''}^{y'} \psi(x, t) \log \left\{ \lambda \frac{\Lambda(t)}{\Lambda(y'')} \right\} dt + \int_{y'}^{\infty} \psi(x, t) s(t) dt \\ &> -\varkappa \int_0^{y''} \psi(x, t) dt + s(y'') \left[ \int_0^{\infty} \psi(x, t) dt - \int_0^{y''} \dots - \int_{y'}^{\infty} \dots \right] \\ &- \varepsilon \int_{y''}^{y'} \psi(x, t) \log \left| \lambda \frac{\Lambda(t)}{\Lambda(y'')} \right| dt - \varkappa \int_{y'}^{\infty} \psi(x, t) dt \end{aligned}$$

since  $s(t) > -\varkappa$  by Lemma 3A. From the last step, letting  $x, y', y'' \rightarrow \infty$ , and using (9) and (2), we get

$$(11) \quad \bar{\Psi} \equiv \overline{\lim}_{x \rightarrow \infty} \Psi(x) \geq \bar{s} - 2(\bar{s} + \varkappa)\varepsilon - K\varepsilon.$$

Hence,  $\varepsilon$  being arbitrary,

$$\bar{\Psi} \geq \bar{s}.$$

Next we let  $y'$  assume, an ascending divergent sequence of values such that

$$s(y') \rightarrow \underline{s} \equiv \underline{\lim}_{t \rightarrow \infty} s(t)$$

where  $\underline{s}$  is finite, by Lemma 3A. Hypothesis (iii) shows that the corresponding sequences of values of  $x = x(y')$ ,  $y'' = y''(x)$  are

also ascending, divergent and conditioned by (9). If we restrict ourselves to values of  $x \geq X$  and of  $y'' \geq t_0$ , (8) and (10) together give

$$\begin{aligned} \Psi(x) &< \int_0^{y''} \psi(x, t) s(t) dt + s(y') \int_{y''}^{y'} \psi(x, t) dt \\ &+ \varepsilon \int_{y''}^{y'} \psi(x, t) \log \left\{ \lambda \frac{\Lambda(y')}{\Lambda(t)} \right\} dt + \int_{y'}^{\infty} \psi(x, t) s(t) dt \\ &< \kappa \int_0^{y''} \psi(x, t) dt + s(y') \left[ \int_0^{\infty} \psi(x, t) dt - \int_0^{y''} \dots - \int_{y'}^{\infty} \dots \right] \\ &+ \varepsilon \int_{y''}^{y'} \psi(x, t) \log \left| \lambda \frac{\Lambda(t)}{\Lambda(y')} \right| dt + \kappa \int_{y'}^{\infty} \psi(x, t) dt \end{aligned}$$

since  $s(t) < \kappa$  by Lemma 3A. Letting  $x, y', y'' \rightarrow \infty$  in the last step, appealing to (9) and (2), and remembering that  $\varepsilon$  can be chosen arbitrarily small, we conclude that

$$\underline{\Psi} \equiv \lim_{x \rightarrow \infty} \Psi(x) \leq \underline{s}.$$

From the results proved above, we have

$$\underline{\Psi} \leq \bar{s} \leq \bar{\Psi},$$

whence the desired conclusion follows by an appeal to Lemma 1A.

#### 4. Deductions from Theorem A.

(i) Following Karamata ([4], p. 6), we shall define the Borel transform of a sequence  $s_n$  as

$$B(x) = \int_0^{\infty} \varphi(x, t) d\{s(t)\},$$

where

$$s(t) = s_n \text{ for } n \leq t < n + 1, \quad n = 0, 1, 2, \dots,$$

$$\varphi(x, y) = \frac{1}{\Gamma(y)} \int_0^x e^{-t} t^{y-1} dt,$$

and  $\psi(x, t) \equiv -\partial\varphi(x, t)/\partial t$  satisfies the initial conditions in hypotheses (i), (ii), of Theorem A.

Now, in the case of the Borel transform, it is known ([4], p. 7) that

$$\varphi(x, y) \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-t^2/a^2} dt \text{ when } x = y - 1 + a\sqrt{y - 1}, \quad y \rightarrow \infty.$$

Consequently we can choose  $a'' > 0$  and  $a' < 0$  so that

$$\varphi(x, y'') \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a''} e^{-t^2} dt > 1 - \varepsilon/2, \quad x = y'' - 1 + a''\sqrt{y'' - 1}, \quad y'' \rightarrow \infty,$$

$$\varphi(x, y') \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a'} e^{-t^2} dt < \varepsilon/2, \quad x = y' - 1 + a'\sqrt{y' - 1}, \quad y' \rightarrow \infty,$$

where the sign before each square root is positive. Therefore  $\varphi(x, y'') > 1 - \varepsilon$ ,  $\varphi(x, y') < \varepsilon$ , for  $x > x_0$ ,  $y' > y'_0$ ,  $y'' > y''_0$ ; i.e. (1) is satisfied.

In the case of the Borel transform, it is also known ([4], pp. 7—8) that, with  $\Lambda(t) = e^{\sqrt{t}}$  and any given real  $a$ ,

$$(12) \quad \int_0^\infty \psi(x, t) \left| \log \frac{\Lambda(t)}{\Lambda(y)} \right| dt < 2 \frac{x^n \sqrt{n}}{e^x n!} + \frac{|x - n|}{\sqrt{n}} + o(1) < K(a)$$

where

$$n = [y], \quad x = y - 1 + a\sqrt{y - 1} \rightarrow \infty.$$

Hence, taking successively  $a = a'$ ,  $y = y'$  and  $a = a''$ ,  $y = y''$  in (12), we see that (2) holds. The fact that  $K$  depends on  $a'$  in the first case and on  $a''$  in the second case does not vitiate the conclusion drawn from a step like (11) since  $a'$ ,  $a''$  are kept fixed when  $x, y, y'' \rightarrow \infty$ .

Lastly, with our choice of  $\Lambda(t) = e^{\sqrt{t}}$ , we find that  $T$  in (3) is given by

$$T = V\{\lambda\Lambda(t)\} = (\sqrt{t} + \log \lambda)^2,$$

$$\text{or} \quad T/t = 1 + O(\log \lambda/\sqrt{t}), \quad t \rightarrow \infty.$$

Combining the results of the last three paragraphs, we see that Theorem A contains the following as a particular case.

**COROLLARY 1A.** *The conditions*

$$B(x) = e^{-x} \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} = O(1), \quad \text{as } x \rightarrow \infty,$$

$$(T_1) \quad \min_{n \leq n' \leq n + \delta\sqrt{n}} \{s_{n'} - s_n\} = o_L(1)\delta, \quad \text{as } n \rightarrow \infty,$$

together imply

$$\operatorname{osc}_{n \rightarrow \infty} s_n = \operatorname{osc}_{x \rightarrow \infty} B(x).$$

Condition  $(T_1)$  can be replaced by  $(T_1^*)$  below, by an argument as in § 5.

$$(T_1^*) \quad \lim_{n \rightarrow \infty} \min_{n \leq n' \leq n + \delta\sqrt{n}} \{s_{n'} - s_n\} = o_L(\delta), \quad \text{as } \delta \rightarrow 0.$$

(ii) A version of Ramaswami's oscillation theorem referred to at the outset is as follows.

COROLLARY 2A. *The condition*

$$\Psi^*(x) = \frac{1}{x} \int_0^\infty \psi^*\left(\frac{t}{x}\right) s(t) dt = O(1), \text{ as } x \rightarrow \infty,$$

where

$$\psi^*(t) \geq 0, \quad \int_0^\infty \psi^*(t) dt = 1, \quad \int_0^\infty \psi^*(t) |\log t| dt \text{ exists,}$$

in conjunction with the condition

$$(T_2) \quad \text{bound}_{t \leq t' \leq \lambda t} \{s(t') - s(t)\} = o_L(1) \log \lambda, \text{ as } t \rightarrow \infty,$$

$$\text{implies} \quad \text{osc}_{t \rightarrow \infty} s(t) = \text{osc}_{x \rightarrow \infty} \Psi^*(x).$$

We can replace condition  $(T_2)$  by  $(T_2^*)$  below, arguing as in § 5.

$$(T_2^*) \quad \lim_{t \rightarrow \infty} \text{bound}_{t \leq t' \leq \lambda t} \{s(t') - s(t)\} = o_L(\log \lambda), \text{ as } \lambda \rightarrow 1.$$

To prove Corollary 2A, we take

$$\psi(x, t) = \frac{1}{x} \psi^*\left(\frac{t}{x}\right)$$

so that

$$\varphi(x, y) = \int_y^\infty \frac{1}{x} \psi^*\left(\frac{t}{x}\right) dt = \int_a^\infty \psi^*(t) dt, \quad a = \frac{y}{x};$$

and we choose  $a'$ ,  $a''$  so as to satisfy condition (1):

$$\varphi(x, y') \equiv \varphi(x, a'x) = \int_{a'}^\infty \psi^*(t) dt < \varepsilon,$$

$$\varphi(x, y'') \equiv \varphi(x, a''x) = \int_{a''}^\infty \psi^*(t) dt > 1 - \varepsilon.$$

Also, with  $\psi(x, t) = \frac{1}{x} \psi^*\left(\frac{t}{x}\right)$ ,  $\Lambda(t) = t$ ,  $y' = a'x$ ,  $y'' = a''x$ , we find that condition (2) reduces to

$$\int_0^\infty \psi^*(t) \log \left| \frac{t}{a} \right| dt < K(a), \quad a = \text{either } a' \text{ or } a'',$$

and is ensured by the restrictions on  $\psi^*$  in Corollary 2A. The proof of Corollary 2A is thus complete.

The well-known particular cases of Corollary 2A, as of Ramas-



wami's theorem, are given by:

$$(a) \quad \psi^*(u) = -\frac{d}{du} e^{-u}, \quad (b) \quad \psi^*(u) = -\frac{d}{du} (1+u)^{-\varrho}, \quad \varrho > 0,$$

$$(c) \quad \psi^*(u) = -\frac{d}{du} \left( \frac{u}{e^u - 1} \right).$$

### 5. Remark on hypothesis (3) of Theorem A.

This hypothesis may be replaced by the apparently milder one that there exists a sequence  $\{\lambda_p\}$  such that  $1 < \lambda_p \rightarrow 1$  as  $p \rightarrow \infty$  and

$$(3^*) \quad \lim_{t \rightarrow \infty} \text{bound}_{t \leq t' \leq T} \{s(t') - s(t)\} = o_L(\log \lambda_p) \text{ as } p \rightarrow \infty,$$

for  $\Lambda(T) = \lambda_p \Lambda(t)$ .

To justify the replacement of (3) by (3\*) we take (3), (3\*) in the forms

$$(13) \quad \text{bound}_{t \leq t' \leq \lambda t} \{s(V(t')) - s(V(t))\} = o_L(1) \log \lambda, \text{ as } t \rightarrow \infty,$$

$$(13^*) \quad \lim_{t \rightarrow \infty} \text{bound}_{t \leq t' \leq \lambda_p t} \{s(V(t')) - s(V(t))\} = o_L(\log \lambda_p), \text{ as } p \rightarrow \infty,$$

respectively, and argue, as in the proof of Lemma 2A, that (13\*) implies (13). The actual argument is as follows.

(13\*) shows that, corresponding to any  $\lambda > 1$ , we can find  $\lambda_p < \lambda$  and such that, for all large  $t$ ,

$$(14) \quad s(V(t')) - s(V(t)) > -\frac{\varepsilon}{2} \log \lambda_p, \quad t \leq t' \leq \lambda_p t.$$

There is evidently a positive integer  $r \geq 2$  such that  $\lambda_p^{r-1} < \lambda \leq \lambda_p^r$ . Hence (14) gives, for all large  $t$  and  $t \leq t' \leq \lambda t$ ,

$$\begin{aligned} s(V(t')) - s(V(t)) &= \{s(V(t') - s(V(\lambda_p^{r-1}t))\} \\ &+ \{s(V(\lambda_p^{r-1}t)) - s(V(\lambda_p^{r-2}t))\} + \dots + \{s(V(\lambda_p t)) - s(V(t))\} \\ &> -\frac{\varepsilon}{2} r \log \lambda_p > -\frac{\varepsilon}{2} \frac{r}{r-1} \log \lambda \geq -\varepsilon \log \lambda. \end{aligned}$$

The conclusion reached above leads at once to (13) and shows that (3\*) implies (3) and so may replace (3) in the enunciation of Theorem A.

### 6. A supplementary theorem.

**THEOREM B.** *Let the following assumptions be made.*

$$(i) \quad c_n(x) \geq 0 \text{ for } n = 0, 1, 2, \dots \text{ and } x > 0;$$

- (ii)  $c_n(x) \rightarrow 0$  as  $x \rightarrow \infty$ ,  $\Sigma c_n(x) = 1$ .
- (iii)  $f(u)$  is positive and differentiable for  $u \geq 1$ ;

$$f \rightarrow \infty, \quad 0 < f' < k = a \text{ constant,}$$

$$F(u) = \int_1^u \frac{dt}{f(t)} \text{ (so that } F \rightarrow \infty \text{ with } u).$$

- (iv)  $x$  and positive integers  $M, N$  are defined by the relations

$$F(x) - F(M) = \mu, \quad F(N) - F(x) = \nu,$$

with the condition that, given any small  $\varepsilon > 0$ , we have, for all sufficiently large  $x, M, N, \mu, \nu$ ,

$$(15) \quad \sum_{n=0}^M c_n(x) < \varepsilon, \quad \sum_{n=N}^{\infty} c_n(x) < \varepsilon, \quad \sum_{n=N}^{\infty} c_n(x) \{F(n) - F(N)\} < \varepsilon,$$

while, for large enough fixed  $\mu, \nu$  and all sufficiently large  $x, M, N$ , we have, in addition to (15),

$$(16) \quad \sum_{n=M}^{\infty} c_n(x) |F(n) - F(p)| < K(\mu, \nu), \text{ where } p = \text{either } M \text{ or } N.$$

- (v)  $s(t) = s_n$  for  $n \leq t < n + 1$  and satisfies the conditions:

$$(17) \quad \text{bound}_{i \leq i' \leq T} \{s(i') - s(i)\} = o_L(1)\delta, \text{ as } t \rightarrow \infty, \text{ for } T = t + \delta f(t), \delta > 0;$$

$$(18) \quad \tau(x) \equiv \Sigma c_n(x) s_n = O(1) \text{ as } x \rightarrow \infty.$$

Then

$$(19) \quad \lim_{n \rightarrow \infty} s_n = \lim_{x \rightarrow \infty} \tau(x), \quad \overline{\lim}_{n \rightarrow \infty} s_n = \overline{\lim}_{x \rightarrow \infty} \tau(x).$$

### 7. Further Lemmas.

In the proof of Theorem B we require the following lemmas which are similar to Lemmas 1A, 2A, 3A.

LEMMA 1B. Hypotheses (i), (ii) of Theorem B make

$$\lim_{n \rightarrow \infty} s_n \leq \overline{\lim}_{x \rightarrow \infty} \tau(x) \leq \overline{\lim}_{n \rightarrow \infty} s_n, \text{ where } \tau(x) = \Sigma c_n(x) s_n.$$

This lemma is established like Lemma 1A.

LEMMA 2B. The condition

$$(17') \quad \text{bound}_{i \leq i' \leq T} \{s(i') - s(i)\} > -\omega < 0, \quad T = t + \delta f(t), \quad \delta > 0,$$

where  $s(t)$  is defined as in hypothesis (v) of Theorem B and  $f(u)$  is as in hypothesis (iii) of Theorem B, implies

$$(20) \quad s(u) - s(t) > -\omega \left( \frac{1}{\delta} + k \right) \{F(u) - F(t)\} - \omega \text{ for } u \geq t \geq 1.$$

This is a known result ([2], Theorem 239).

**LEMMA 3B.** *If, in the hypotheses of Theorem B, (17) is replaced by condition (17') of Lemma 2B, and (16) is dropped, then the conclusion of Theorem B will assume the form*

$$\tau(x) = O(1) \text{ as } x \rightarrow \infty \text{ involves } s_n = O(1) \text{ as } n \rightarrow \infty.$$

Since (17) implies (17'), conditions (15), (17), (18) together imply  $|s_n| < \kappa$  for  $n \geq 0$ .

This is a theorem of Vijayaraghavan and Hardy ([2], Theorem 238).

### 8. Proof of Theorem B.

The proof may be modelled on that of Theorem A and divided into two parts which separately lead us to infer that

$$(21) \quad \bar{\tau} \equiv \overline{\lim}_{x \rightarrow \infty} \tau(x) \geq \overline{\lim}_{n \rightarrow \infty} s_n \equiv \bar{s}, \quad \underline{\tau} \equiv \underline{\lim}_{x \rightarrow \infty} \tau(x) \leq \underline{\lim}_{n \rightarrow \infty} s_n \equiv \underline{s}.$$

To justify the first inference of (21), we begin by fixing  $\mu, \nu, x_0, M_0, N_0$  so that, for  $x \geq x_0, M \geq M_0, N \geq N_0$ , (15) and (16) hold. We then find  $M_1 \geq M_0$  (and correspondingly  $x_1 \geq x_0, N_1 \geq N_0$ ) so that

$$\min_{M \leq n \leq M + \delta \varphi(M)} (s_n - s_M) > -\varepsilon \delta \text{ for } M \geq M_1.$$

This is possible by hypothesis (17) of Theorem B, and it ensures, as a result of Lemma 2B,

$$(22) \quad s_n - s_M > -\varepsilon(1 + k\delta)\{F(n) - F(M)\} - \varepsilon \delta \text{ for } n \geq M \geq M_1.$$

Next we write

$$(23) \quad \tau(x) = \sum_{n=0}^{M-1} c_n(x)s_n + \sum_{n=M}^N c_n \dots + \sum_{n=N+1}^{\infty} \dots = \tau_1(x) + \tau_2(x) + \tau_3(x)$$

and choose  $M$  to be one of an ascending, divergent sequence of integers such that

$$s_M \rightarrow \bar{s}$$

where  $\bar{s}$  is finite since  $|s_n| < \kappa$  by Lemma 3B. Then, using (15) in  $\tau_1(x)$  and  $\tau_3(x)$ , and using (22) in  $\tau_2(x)$ , we obtain from (23):

$$\begin{aligned} \tau(x) &> -\kappa \sum_{n=0}^{M-1} c_n + \sum_{n=M}^N c_n (s_M - \varepsilon \delta) \\ &\quad - \sum_{n=M}^N c_n \varepsilon (1 + k\delta) \{F(n) - F(M)\} - \kappa \sum_{n=N+1}^{\infty} c_n \end{aligned}$$

$$\begin{aligned}
&= - (s_M - \varepsilon\delta + \kappa) \left( \sum_{n=0}^{M-1} c_n + \sum_{n=N+1}^{\infty} c_n \right) + s_M - \varepsilon\delta \\
&\quad - \varepsilon(1 + k\delta) \sum_{n=M}^N c_n \{F(n) - F(M)\} \\
(24) \quad &> - 2(s_M - \varepsilon\delta + \kappa)\varepsilon + s_M - \varepsilon\delta - \varepsilon(1 + k\delta)K,
\end{aligned}$$

if we suppose (as we may) that  $s_M > -\kappa + \varepsilon\delta$  and use (16). Letting  $M \rightarrow \infty$  in (24), and remembering that  $\varepsilon$  is arbitrary and  $K$  fixed (on account of  $\mu, \nu$  being fixed), we obtain

$$\bar{\tau} \geq \bar{s}.$$

The second inference of (21) is justified in the same way as the first, and the two inferences taken along with Lemma 1B yield conclusion (19).

## 9. Deductions from Theorem B.

We can deduce Corollary 1A from Theorem B, taking

$$c_n(x) = e^{-x} x^n / n!, \quad f(u) = 2\sqrt{u},$$

and using well-known properties of the Borel transform ([2], p. 313, § 12.15; [4], pp. 7—8).

We may also take

$$c_n(x) = \frac{1}{x} g\left(\frac{n}{x}\right), \quad g(t) = \left(\frac{\sin t}{t}\right)^2, \quad f(u) = u,$$

and deduce

**COROLLARY 1B.** *The condition*

$$\tau(x) = \frac{2x}{\pi} \sum \frac{\sin^2(n/x)}{n^2} s_n = O(1) \text{ as } x \rightarrow \infty,$$

*along with either (T<sub>2</sub>) or (T<sub>2</sub><sup>\*</sup>) of Corollary 2A, ensures*

$$\operatorname{osc}_{n \rightarrow \infty} s_n = \operatorname{osc}_{x \rightarrow \infty} \tau(x).$$

The deduction of Corollary 1B from Theorem B requires us to verify that conditions (15) and (16) of the theorem are fulfilled for the particular choices of  $c_n$  and  $f$  in the corollary. That (15) is fulfilled is known ([2], proof of Theorem 240). That (16) is fulfilled follows from the facts:

$$x \sum_M^{\infty} \frac{\sin^2(n/x)}{n^2} \log \frac{n}{M} = O\left(\frac{x}{M} \int_1^{\infty} \frac{\log u}{u^2} du\right) < K$$

when  $x, M \rightarrow \infty$ ,  $\log x - \log M = \mu$  (fixed); and

$$\begin{aligned} x \sum_M^{\infty} \frac{\sin^2(n/x)}{n^2} \left| \log \frac{n}{N} \right| &= x \sum_M^N \frac{\sin^2(n/x)}{n^2} \log \frac{N}{n} + x \sum_{N+1}^{\infty} \frac{\sin^2(n/x)}{n^2} \log \frac{n}{N} \\ &< O\left(x \int_{M-1}^N \frac{\log(N/u)}{u^2} du\right) + K' \\ &= O\left(\frac{x}{N} \int_1^{N/(M-1)} \log u du\right) + K' < K'' + K' \end{aligned}$$

when  $x, M, N \rightarrow \infty$ ,  $\log N - \log x = \nu$  (fixed),  $\log N - \log M = \mu + \nu$  (fixed).

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Ramanujan Institute of Mathematics,  
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