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A new proof of the Existence of the minimum for a Classical Integral*)

by

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The present paper treats of the establishment of the minimum of a classical integral (studied more or less completely in any treatise on the Calculus of the Variations [1])) making use of a procedure derived from certain theorems indicated recently by M. Picone [2]).

Let us consider an *almost elementary* set X (of a Hausdorff space) — that is, a set containing a sequence $\{X_n\}$ of closed and compact (*elementary*) sets by which it is invaded (such that, in other words, $X_1 \subset X_2 \subset X_3 \subset \ldots \subset X = \sum_{n=1}^{\infty} X_n$); obviously, given a function $f(x)$ which is real valued and lower semi-continuous in each X_n, then the greatest lower bound of $f(x)$ in X is equal to the limit of the non-increasing sequence $\{m_n\}$ of the minima of $f(x)$ in each X_n, and in order that $f(x)$ shall have a minimum in X it is necessary and sufficient that for n sufficiently large, $m_n = m_{n+1} = m_{n+2} = \ldots$: it is just this value which represents the minimum of $f(x)$ in X. From the theorems XIII and XIV of the work cited in note [2]), it results in fact that the totality of continuous and rectifiable curves (of the plane, or of the space, etc.) makes up, with a certain metric, an almost elementary space, and also almost elementary are all the open or closed sets of curves, or the sets of curves obtained from them by means of a finite number of sum and product operations.

From this there is deduced, for the finding of a greatest lower bound (possibly a minimum) of a lower semi-continuous line function, a general method of the same character as those used in

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*) Work done at the Istituto Nazionale per le Applicazione del Calcolo, Rome.

1) See, for example, L. Tonelli, *Fondamenti del Calcolo delle Variazioni*, vol. II.

2) M. Picone, *Due conferenze sui fondamenti del Calcolo delle Variazioni*, „Giornale di Matematiche" di Battaglini, IV, vol. 80 (1950—1951). pp. 50—79.

finding a greatest lower bound of an ordinary real valued point function (continuous) in a set of a Euclidean space.

In the present case the work is accomplished with great simplicity and rapidity; the notions used can well be extended to include integrals much more general than that here treated [3]).

1. Let us consider the integral $F(L) = \int_L y^{1/\nu}\, ds$ (with ν real $\neq 0$), in the class Γ of continuous and rectifiable curves of the (x, y) plane for which the integral itself is defined, which connect the two given points $A \equiv (a, c)$ and $B \equiv (b, d)$. For this integral the following is to be noted: [4])

a) for $\nu > 0$, the integral $F(L)$ is defined for all the curves (continuous and rectifiable) of the *closed* half-plane $y \geqq 0$;

for $\nu < -1$, the integral $F(L)$ is defined for all the curves of the *open* half-plane $y > 0$, and for some of the closed half-plane $y \geqq 0$;

if $0 > \nu \geqq -1$, the integral $F(L)$ is defined only for the curves of the *open* half-plane $y > 0$;

in the first two cases we suppose $c \geqq 0$, $d \geqq 0$, and in the third $c > 0$, $d > 0$;

b) the integral $F(L)$ is positively regular in the half-plane $y > 0$, and hence possible minimal arcs traced in this half-plane are extremal arcs of class C'' [5]);

c) the extremals of the integral $F(L)$ are: the segments $x =$ const., and the Ribaucour curves of the parameter ν, which can be represented by the parametric equations

$$x = x_0 + y_0\, \nu \int_0^\theta \frac{dt}{\cos^\nu t}, \quad y = \frac{y_0}{\cos^\nu \theta} \quad \begin{array}{ll} -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \text{if } \nu > 0 \\ -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2} & \text{if } \nu < 0 \end{array};$$

note that $dy/dx = \tan\theta$, or that the parameter θ to which the curve is referred represents the direction angle of the curve at any general point;

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[3]) It is my intention to devote a paper soon to such extensions.

[4]) Everything in sec. 1 is classical or provable by elementary means; cf. for example Goursat, *Cours d'Analyse*, M. Picone, *Introduzione al Calcolo delle Variazioni*, and so on. I am working on an article on the variational properties of Ribaucour curves to be published soon, in which the statements of sec. 1 are again proved.

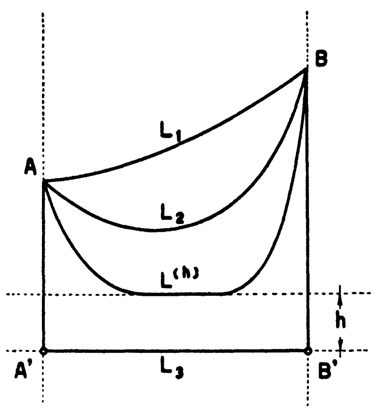
[5]) Let the curve L be represented with the parametric equation $z = z_L(t)$, $0 \leqq t \leqq 1$ as in sec. 2; if the derivatives dz/dt, $d^2z/dt^2, \ldots, d^nz/dt^n$ exist and are continuous, the curve L will be defined „of class $C^{(n)}$".

d) if $\nu > 0$, through the points A and B — supposed of different abscissa — two extremal curves at most can pass, which I will call L_1 and L_2: neither of them has points on the x-axis;

if $\nu < 0$, there is one and only one extremal curve in Γ, which I will call L_1, which connects A and B — supposed of different abscissa — and has no point on the x-axis;

in any case, if the two points A and B have the same abscissa, the only extremal which connects them is the segment AB;

e) if $\nu > 0$, supposing $a < b$, $c > 0$, $d > 0$, for every positive number h *sufficiently small* (let us say, smaller than a certain $h' > 0$, which in turn is chosen smaller than the smallest ordinate of possible extremals L_1 and L_2) there is one and only one curve $L^{(h)}$ of the class C', belonging to Γ, having the smallest ordinate h (tangent to the straight line $y = h$) and whose arcs of ordinate greater than h are extremals; this curve is composed of two extremal arcs from A and B, which are joined by a segment $y = h$, and is concave in the positive y- direction;



if $\nu < 0$, supposing $a < b$, for every h *sufficiently large* (greater, let us say, than a certain $h' > 0$, which in turn is chosen greater than the greatest ordinate of the extremal L_1) there exists no curve of the class C', belonging to Γ, with the greatest ordinate h (tangent to the straight line $y = h$) whose arcs of ordinate less than h are extremals;

f) under the hypothesis of the first clause of e), if A' and B' are the respective projections of the points A and B on the x-axis, the broken line $L_3 \equiv AA'B'B$ (belonging to Γ) has a length greater than all the curves $L^{(h)}$, as well as possible extremals L_1 and L_2.

2. On any curve L of Γ, directed from A to B, let us put A as origin of the curvilinear abscissas s, which increase from zero to $\lambda(L)$, the length of the curve; with L referred to the parameter $t = \dfrac{s}{\lambda(L)}$, its parametric equations can be written as

$$z = z_L(t), \quad 0 \leqq t \leqq 1,$$

where z is the vector with the components (x, y). It is well known that the vector valued function $z_L(t)$ satisfies the Lipschitz condition, having a Lipschitz number $\lambda(L)$.

Now let us consider the totality Γ^* of the vector valued functions $z(t) \equiv [x(t), y(t)]$ which are continuous and of bounded variation within the interval $(0, 1)$; Γ^* can be considered a metric space, with the distance between two of its elements $z'(t)$ and $z''(t)$ defined as $\max\,[|z'(t) - z''(t)|]$. The class Γ of curves (where each curve L is identified with its corresponding function $z_L(t)$ or with the right hand side of any other of its parametric equations of base on the interval $(0,1)$) is thus a set of elements of the metric space Γ^*, in which, as classical results have shown, the integral $F(L)$ is lower semi-continuous.[6])

3. Let us now study the case when $\nu > 0$. If the points A and B have the same abscissa, the inequalities

$$\int\limits_L y^{1/\nu}\, ds \geqq \int\limits_L y^{1/\nu} \left|\frac{dy}{ds}\right| ds \geqq \left|\int_c^d y^{1/\nu}\, dy\right|$$

show that the integral $F(L)$ allows for a minimum in Γ, and that the segment AB is the only minimal curve. If both the points A and B have ordinate zero, every curve traced on the x-axis, and only those curves, make the integral zero giving it a minimum value: among these we will admit only the segment AB.

If one of the two points, for example B, has a zero ordinate, and the other point has a positive one, the preceding considerations show that the integral has a minimum and that the unique [7]) minimizing curve is the broken line $AA'B'B$.

Apart from these trivial cases, let us consider the case of sec. 1. e): here, the possible minimizing curve can only be the broken line L_3 and (if they exist) the extremals L_1, L_2. Let us consider the following classes of curves in Γ:

a) the class $\Gamma^{(h)}$ of all the curves of Γ traced in the half-plane $y \geqq h$ (with $0 < h < h'$);

b) the class Γ_n of all the curves of Γ having a length not greater than n (where $n = \lambda(L_3) + 1,\ \lambda(L_3) + 2,\ \lambda(L_3) + 3, \ldots$);

c) the class $\Gamma_n^{(h)}$, the intersection of the two classes Γ_n and $\Gamma^{(h)}$.

In the class $\Gamma^{(h)}$ is found the curve $L^{(h)}$ and (if they exist) the

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[6]) Cf. op. cit. in note [2]), theorem XV.

[7]) Here as elsewhere I exclude from consideration curves with arcs analytically distinct (or corresponding to different intervals of the t-axis), but geometrically coincident in the same segment of the x-axis.

extremals L_1 L_2; in the class L_n is found the broken line L_3, and hence (by sec. 1. *e*) the infinite curves $L^{(h)}$ as well as (if they exist) the extremals L_1 and L_2.

Obviously, for every fixed h, the sequences $\{\Gamma_n^{(h)}\}$ and $\{\Gamma_n\}$ invade respectively $\Gamma^{(h)}$ and Γ; and it is well known that the sets $\Gamma_n^{(h)}$ and Γ_n are closed and compact in Γ^*[8]), and hence the integral $F(L)$ has a minimum in them, equal respectively to $m_n^{(h)}$ and m_n.

We can now apply the procedure indicated in the first paragraph.

3.1. Consider the integral $F(L)$ in the class $\Gamma^{(h)}$; it is well known that it has a minimum $m^{(h)}$, which can also be seen from the following. The curves of $\Gamma^{(h)}$, which give a value not greater than $F(L^{(h)})$ to the integral, make up a subclass of $\Gamma^{(h)}$ which is contained in Γ_n, for every n greater than $F(L^{(h)}).\, h^{-1/\nu}$: in fact in such a subclass we have

$$F(L^{(h)}) \geqq \int_L y^{1/\nu}\, ds \geqq h^{1/\nu}.\, \lambda(L).$$

Hence, for a certain n, we have $m_n^{(h)} = m_{n+1}^{(h)} = \ldots .$

The minimum $m^{(h)}$ of the integral in $\Gamma^{(h)}$ can be assumed only corresponding to one of the curves $L^{(h)}$ and (if they exist) L_1 and L_2: these last are the only possible extremals of the class $\Gamma^{(h)}$; the first is the only *boundary curve* which could satisfy the Weierstrass boundary conditions (necessary for the minimum [9])) and in which the arcs *internal* to $\Gamma^{(h)}$ are extremals.

Since the curves L_1, L_2, $L^{(h)}$ belong to all the sets $\Gamma_n^{(h)}$ only in correspondence to one of these curves $F(L)$ can assume its minimum value in $\Gamma_n^{(h)}$; we would have then $m_n^{(h)} = m^{(h)}$ in any case.

3.2. Now consider the integral $F(L)$ in the class Γ_n. We will show that in this class the only admissible minimal arcs are the broken line L_3 and (if they exist) the extremals L_1 and L_2. Let L' be a minimal curve in the class Γ_n.

If L' has points on the x-axis, it must coincide with L_3, otherwise we would have $F(L') > F(L_3)$, for the reasons stated in sec. 3. [10])

If L' has a smallest positive ordinate y', then it must belong to

[8]) Cf. for example, op. cit. in [2]), no. 7, or also M. Picone, *Lezioni di Analisi funzionale*, Rome 1946, nos. 79—80.

[9]) Cf. for example, op. cit. in [1]), p. 127.

[10]) See note [7]).

the class $\Gamma_n^{(h)}$, with $h > 0$, less than y' and h', and must also be a minimal curve of the integral in $\Gamma_n^{(h)}$. But it cannot coincide with the curve $L^{(h)}$, which has a smallest ordinate h; it must hence coincide with L_1 or else L_2. The proposition is then proved.

In conclusion, the minimum of $F(L)$ in Γ_n does not depend on n, and hence the integral $F(L)$ allows a minimum in Γ, where the only minimal curves can be L_1, L_2, and L_3.

4. Consider now the case $\nu < 0$, always excluding the case that A and B have the same abscissa, because then there exist a trivial minimum and the only minimal curve is the segment AB. We also suppose here that $cd > 0$. In this case, too, the only admissible regular minimal curve is the extremal L_1 (cfr. sec. 1. d). Consider the following classes of curves in Γ:

a) the class $\Gamma^{(h)}$ of all the curves of Γ traced in the strip $1/h \leqq y \leqq h$ (h greater than the three numbers h', $1/c$, $1/d$);

b) the class $\Gamma_n^{(h)}$ of all the curves of $\Gamma^{(h)}$ having a length not greater than n (n greater than $\lambda(L_1)$).

Evidently the sequence $\{\Gamma_n^{(h)}\}$ invades $\Gamma^{(h)}$: the sequence $\{\Gamma^{(h)}\}$ invades Γ only if $-1 \geqq \nu > 0$, while if $\nu < -1$ it invades the set of curves of Γ which have no points on the x-axis, a set which I will call Γ'. That the integral $F(L)$ has a minimum in $\Gamma^{(h)}$ and that this minimum is given only by the extremal L_1, can be shown in a way similar to that used in sec. 3. Since there are no *boundary curves* in $\Gamma^{(h)}$ satisfying the necessary Weierstrass conditions — it follows that the integral $F(L)$ has a minimum also in Γ' and that the only minimal curve is the extremal L_1 [11]).

In the case that Γ' does not coincide with Γ, it easy to see from simple limit considerations that $F(L_1)$ represents the greatest lower bound of the integral in Γ, and hence the minimum. Other minimals distinct from L_1 are not possible.

5. Finally, suppose that one (at least) of the two points A and B lies on the x-axis (still with $\nu < -1$): for example, let $c = 0 < d$, $a < b$.

Compare the extremal L_1 with an arbitrary curve L of Γ. It is well known that if L is not concave in the negative y-direction, there is a curve L' in Γ concave in the negative y-direction, such that $F(L') \leqq F(L)$ [12]). We can suppose that L has such a property.

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[11]) See note [7]).

[12]) This elementary property is proved in the article cited in note [4]).

Consider the straight line $y = h$, with $0 < h < d$: it intersects the extremal L_1 in a point P_1 with abscissa x_1, and the curve L in a point P on it with the abscissa x. We can evidently write

$$(L) \int_A^B y^{1/\nu} ds - (L_1) \int_A^B y^{1/\nu}\, ds = (L) \int_P^B y^{1/\nu}\, ds - (L_1) \int_{P_1}^B y^{1/\nu}\, ds +$$

$$+ h^{1/\nu} \mid x_1 - x \mid - h^{1/\nu} \mid x_1 - x \mid + (L) \int_A^P y^{1/\nu}\, ds - (L_1) \int_A^{P_1} y^{1/\nu}\, ds;$$

the last two integrals written tend to zero when h is infinitesimal; because of the concavity of the two curves L_1 and L we have

$$\mid x_1 - x \mid \leqq \frac{h \,.\, (b - a)}{d}$$

and hence also $h^{1/\nu} \mid x_1 - x \mid$ is infinitesimal; on the other hand, from the preceding results,

$$(L) \int_P^B y^{1/\nu}\, ds - (L_1) \int_P^B y^{1/\nu}\, ds + h^{1/\nu} \mid x_1 - x \mid \geqq 0,$$

and hence $F(L_1) \leqq F(L)$. In this last relationship the equal sign can be suppressed if the two curves L_1 and L do not coincide, for the reason that L_1 is the only possible minimal curve in Γ.

In a similar way this result can be extended to include the case when $c = d = 0$.

(Oblatum 26-5-52).