Compositio Mathematica

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Compositio Mathematica, tome 10 (1952), p. 241-244
<http://www.numdam.org/item?id=CM_1952__10__241_0>

On a conjecture of Nelder

by

J. M. Hammersley

In a statistical problem connected with the Poisson distribution, J. A. Nelder came to consider the determinant of the information matrix

$$\int_0^\infty \begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} e^{-x}\, dF(x) \tag{1}$$

in which F is a distribution function of a non-negative variable, that is to say a non-decreasing function continuous on the right and satisfying

$$F(x) = 0, \ x < 0; \ \lim_{x \to \infty} F(x) = 1.$$

To solve his problem he had to determine what function (or functions) F would maximise this determinant. He conjectured (a) that a maximum occurred when

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \leqq x < 2, \\ 1, & x \geqq 2 \end{cases} \tag{2}$$

and (b) that (2) was the unique solution. In this note I shall prove conjecture (a) together with a weaker form of (b), namely that (2) is unique amongst the class of distribution functions having commensurable saltuses.

The determinant of (1) is equal to

$$\frac{1}{2} \int_0^\infty \int_0^\infty (x - y)^2 e^{-(x+y)}\, dF(x)\, dF(y). \tag{3}$$

To relate this expression to familiar inequalities, suppose temporarily that F is a step function with saltuses of magnitude F_i at x_i for $i = 1, 2, \ldots$ Writing

$$a_{ij} = (x_i - x_j)^2 \exp(- x_i - x_j)$$

we have to prove a best possible inequality of the type

$$\Sigma_{ij} a_{ij} F_i F_j \leqq M$$

subject to $\Sigma_i F_i = 1$. Inequalities of the type

$$\Sigma_{ij} a_{ij} F_i G_j \leqq M$$

subject to the conditions $\Sigma_i F_i^p = 1$, $\Sigma_j G_j^q = 1$ are known as inequalities of the space $[p, q]$. The inequality theory of the space $[2, 2]$, known as Hilbert space, is well-developed, and other spaces in which p or q exceed unity have received some attention. However, results in the space $[1, 1]$ seem pretty scarce.

To prove conjecture (a) we note that the integrand in (3) is a bounded continuous function, and hence the integral exists as a Cauchy-Stieltjes integral. Consequently if C_n denotes the class of functions

$$E_n(x) = \frac{1}{n} \sum_{i=1}^{n} E(x - x_i); \quad x_i \geqq 0, \quad E(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geqq 0 \end{cases}$$

there exists a sequence of functions $E_n(x)$ belonging to C_n such that

$$\lim_{n \to \infty} E_n(x) = F(x)$$

is a solution which maximises (3).

When F belongs to C_n (with $n \geqq 2$) we can write Q/n^2 for (3) where

$$Q = Q(x_1, x_2, \ldots, x_n) = S_0 S_2 - S_1^2 = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2 \exp(-x_i - x_j), \tag{4}$$

$$S_m = S_m(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^m \exp(-x_i), \quad m = 0, 1, 2. \tag{5}$$

Let us maximise Q subject to $0 \leqq x_i \leqq \infty$. It is easy to see that Q is not a maximum if $x_i = \infty$ for any value of i. So hereafter we confine our attention to finite values of x_i. We take care of the restriction $0 \leqq x_i$ by writing $x_i = \xi_i^2$ and maximising Q with respect to the ξ_i. Consider solutions of

$$\partial Q / \partial \xi_k = -2\xi_k \exp(-\xi_k^2) L(\xi_k^2) = 0, \tag{6}$$

where

$$L(\xi_k^2) = S_0 \xi_k^4 - 2(S_0 + S_1)\xi_k^2 + (2S_1 + S_2). \tag{7}$$

For a graetest maximum (or *peak*) of Q either $\xi_k = 0$ or ξ_k^2 is a solution of $L(\xi_k^2) = 0$. So far everything is straightforward; but now we have to dispose of an unwanted root of $L = 0$, and the way of doing this is by no means obvious. Suppose that at any particular peak exactly ν of the ξ's are zero. Then

$$\begin{aligned} \nu(2S_1 + S_2) &= \sum_{k=1}^{n} \exp(-\xi_k^2) L(\xi_k^2) \\ &= S_0 S_2 - 2(S_0 + S_1)S_1 + (2S_1 + S_2)S_0 = 2Q \geqq 16n^2/9e^2, \end{aligned} \tag{8}$$

since, when $x_i = 2$ for $1 \leqq i \leqq \frac{1}{2}n$ and $x_i = 0$ for $\frac{1}{2}n < i \leqq n$,

$$Q = \begin{cases} n^2/e^2 & \text{for } n \text{ even} \\ (n^2 - 1)/e^2 & \text{for } n \text{ odd} \end{cases}. \tag{9}$$

Next, because $(2x + x^2)e^{-x} \leqq 2(1 + \sqrt{2})e^{-\sqrt{2}}$, we get

$$2\nu(n - \nu)(1 + \sqrt{2})e^{-\sqrt{2}} \geqq 16n^2/9e^2$$

and hence

$$\frac{\nu}{n} \geqq \frac{1}{2}\left\{1 - \sqrt{\left[1 - \frac{32}{9(1 + \sqrt{2})e^{2-\sqrt{2}}}\right]}\right\} \geqq 0.287. \tag{10}$$

Now x^2e^{-x} is an increasing function for $0 < x < 2$: so at any peak of Q at least one of the roots of $L(x_k) = 0$ must satisfy $x_k \geqq 2$ for at least one value of k; for otherwise we could increase Q by multiplying each x_i by some constant greater than unity. When the greater root of (7) satisfies $\xi_k^2 \geqq 2$, we have

$$\sqrt{(S_0^2 + S_1^2 - S_0S_2)} \geqq S_0 - S_1. \tag{11}$$

We now derive a contradiction by supposing that at any peak of Q there is at least one value of k, say $k = l$, such that x_l is strictly positive and is the lesser root of $L(x_l) = 0$. We have

$$S_0x_l = S_0 + S_1 - \sqrt{(S_1^2 + S_0^2 - S_0S_2)} \tag{12}$$

$$\partial^2Q/\partial\xi_l^2 = -8\xi_l^2\exp(-\xi_l^2)\{S_0\xi_l^2 - (S_0 + S_1) + \exp(-\xi_l^2)\}. \tag{13}$$

At a maximum the right-hand side of (13) cannot be positive. Also $(1 - x)e^{-x} \geqq -e^{-2}$ and $xe^{-x} \leqq e^{-1}$. By (11) and (12)

$$\begin{aligned} 0 &\leqq S_0x_l - (S_0 + S_1) + \exp(-x_l) = \exp(-x_l) - \sqrt{(S_0^2 + S_1^2 - S_0S_2)} \\ &\leqq \exp(-x_l) - (S_0 - S_1) \leqq \exp(-x_l) - \exp(-x_l) + \\ &\qquad + x_l\exp(-x_l) - \nu + (n - \nu - 1)e^{-2} \\ &\leqq e^{-1} - \nu + (n - \nu - 1)e^{-2} = \frac{n}{e^2}\left\{1 + \frac{1}{n}(e - 1) - \frac{\nu}{n}(e^2 + 1)\right\} \\ &\leqq \frac{n}{e^2}\left\{1 + \frac{1}{2}(e - 1) - \frac{\nu}{n}(e^2 + 1)\right\} = \frac{n}{e^2}\left\{\frac{1}{2}(e + 1) - \frac{\nu}{n}(e^2 + 1)\right\}. \end{aligned}$$

Hence

$$\frac{\nu}{n} \leqq \frac{(e + 1)}{2(e^2 + 1)} \leqq 0.252$$

which contradicts (10). Hence at any peak any non-zero value of

x_k is equal to the greater root of $L(x_k) = 0$, say $x_k = x_0$. Then

$$Q = \nu(n - \nu)x_0^2 \exp(- x_0)$$

and Q attains its maximum for

$$x_0 = 2,\ \nu = \begin{cases} \tfrac{1}{2}n & \text{if } n \text{ is even} \\ \tfrac{1}{2}(n \pm 1) & \text{if } n \text{ is odd} \end{cases}$$

and then Q satisfies (9). This completes the proof and shows that the least upper bound of (3) is $1/e^2$.

(Oblatum 24-3-52)