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# The total length of the edges of a polyhedron

by

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Fejes Tóth, in a paper <sup>1)</sup> to which I have not had access, has conjectured that  $L$ , the sum of the lengths of the edges of a convex polyhedron containing a sphere of unit diameter, satisfies  $L \geq 12$ ; and he has proved that  $L > 10$  for all such polyhedra, and  $L > 14$  for polyhedra with triangular faces only. In this note I prove that, if no face is a polygon of more than  $n$  sides, then

$$L > \frac{10}{3} \sqrt{\left(\pi n \tan \frac{\pi}{n}\right)}. \quad (1)$$

For triangular faces only, this is weaker ( $L > 13.47 \dots$ ) than Fejes Tóth's result; for triangular and/or quadrilateral faces it gives  $L > 11.82 \dots$ ; and for faces with any number of sides it gives

$$L \geq 10\pi/3 = 10.47 \dots, \quad (2)$$

which is slightly stronger than Fejes Tóth's result.

Let  $S$  be the surface and the area of the sphere, centre  $O$ , radius  $\frac{1}{2}$ . Let  $P$  be the plane containing any face of the polyhedron. Let  $p$  denote the perimeter of this face and its length. The area  $A'$  of this face cannot exceed that of a regular polygon of  $n$  sides with perimeter  $p$ ; so

$$A' \leq \frac{p^2}{4n} \cot \frac{\pi}{n}. \quad (3)$$

Let  $A$  denote the projection (and its area) from  $O$  of the interior of  $p$  upon  $S$ . Define  $\theta$  by

$$A = \frac{1}{2}\pi(1 - \cos \theta), \quad (0 \leq \theta \leq \frac{1}{2}\pi). \quad (4)$$

Let  $C$  be the cone of semi-vertical angle  $\theta$ , with vertex at  $O$  and axis normal to  $P$ . Let  $B$  and  $B'$  be the areas cut off by  $C$  upon  $S$  and  $P$  respectively. Since  $B = A$ , we have

$$A' > B' \geq \frac{1}{4}\pi \tan^2 \theta. \quad (5)$$

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<sup>1)</sup> *Norske. Vid. Selsk. Forh., Trondhjem* (1948) **21**, 32—4. See *Math. Rev.* (1950) **11**, 386.

Strict inequality holds in (5) because  $A$  has not got a circular boundary. From (3) and (5)

$$p > \left( n\pi \tan \frac{\pi}{n} \right)^{\frac{1}{2}} \tan \theta. \quad (6)$$

Use the suffix  $i = 1, 2, \dots$  for the various faces of the polyhedron. Summing we have

$$\sum_i A_i = S = \pi = \frac{1}{2}\pi \sum_i (1 - \cos \theta_i), \quad (0 \leq \theta_i \leq \frac{1}{2}\pi). \quad (7)$$

$$2L = \sum_i p_i > \left( n\pi \tan \frac{\pi}{n} \right)^{\frac{1}{2}} \sum_i \tan \theta_i = T. \quad (8)$$

A minimum of  $T$  cannot occur unless either

$$\sec^2 \theta_i + \lambda \sin \theta_i = 0 \quad (9)$$

where  $\lambda$  is a Lagrangian undetermined multiplier, or  $\theta_i$  is an end-point of the interval  $0 \leq \theta_i \leq \frac{1}{2}\pi$ . If  $\theta_i = \frac{1}{2}\pi$ ,  $T$  is infinite. Suppose that exactly  $N$  of the  $\theta_i$  are not zero. Since these values must then satisfy (9), they are equal; whence, from (7), for these  $\theta_i$

$$\cos \theta_i = 1 - \frac{2}{N}, \quad \tan \theta_i = \frac{2(N-1)^{\frac{1}{2}}}{(N-2)}, \quad (10)$$

the first of the relations (10) implying  $N \geq 2$ . Then

$$T \geq \left( n\pi \tan \frac{\pi}{n} \right)^{\frac{1}{2}} \frac{2N(N-1)^{\frac{1}{2}}}{(N-2)} = \left( n\pi \tan \frac{\pi}{n} \right)^{\frac{1}{2}} U(N). \quad (11)$$

When  $N \geq 2$  ranges over the positive integers,  $U(N)$  attains its minimum for  $N = 5$ ; whereupon (8) and (11) yield (1).

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