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On exceptional values of entire functions

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On Exceptional Values of Entire Functions

by

S. M. Shah.

1. Let \( f(z) \) be an entire function of finite order \( \varrho \). A value \( \alpha \) is said to be an exceptional value (e.v.) \( B \) if \(^1\)
\[
\limsup_{r \to \infty} \frac{\log n(r, \alpha)}{\log r} = \varrho_1(\alpha) < \varrho
\]
e.v. \( N \) if \([1,78—107; 2, 254—269]\)
\[
\delta(\alpha) = 1 - \limsup_{r \to \infty} \frac{N(r, \alpha)}{T(r)} > 0,
\]
and e.v. \( V \) (in the sense of Valiron \(^2\)) if
\[
\Delta(\alpha) = 1 - \liminf_{r \to \infty} \frac{N(r, \alpha)}{T(r)} > 0.
\]

2. Let \( E \) denote the set of positive non-decreasing functions \( \varphi(x) \) such that \(^3\)
\[
\int_{A}^{\infty} \frac{dx}{x \varphi(x)}
\]
is convergent. It is known that for functions of non-integral and zero order and for a class of functions of integral order, including all functions of maximum or minimum type, we have \([4 (i), (ii)]\)
\[
\liminf_{r \to \infty} \frac{\log M(r)}{n(r, \alpha) \varphi(r)} = 0
\]
where \( \varphi(x) \) is any function of \( E \), for every \( \alpha \). Hence it is natural to define a value \( \alpha \) \((0 \leq | \alpha | < \infty) \) e. \( E \) for \( f(z) \) if
\[
(1) \quad \liminf_{r \to \infty} \frac{\log M(r)}{n(r, \alpha) \varphi(r)} > 0
\]

\(^1\) For notations see [1] chapter 1.

\(^2\) See [9] where further references will be found. It is known that \( \delta(\alpha) \) is not invariant with respect to a change of the origin [12]. To overcome this difficulty Valiron has suggested another definition for \( \delta(\alpha) [13] \).

\(^3\) In what follows, \( A \) denotes a positive constant not necessarily the same at each occurrence.
for some \( \varphi \subset E \). Let \( n_1(r, \alpha) \) denote the number of simple zeros of \( f - \alpha \) in \( |z| \leq r \). We define \( \alpha \) to be an e.v. \( E \) for simple zeros if

\[
R_1(\alpha) = \liminf_{r \to \infty} \frac{\log M(r)}{n_1(r, \alpha)\varphi(r)} > 0
\]

for some \( \varphi \subset E \), and normal \( E \) for simple zeros if \( R_1(\alpha) = 0 \) for every \( \varphi \subset E \).

3. We prove the following results. Theorem 1 generalises a well known result of Borel [5,279]. Theorems 2,3 and 4 give results analogous to those [3,75—78] for a v.e. \( B \) for simple zeros. We note however that a v.e. \( B \) for simple zeros may not be a v.e. \( E \) for simple zeros 1).

**Theorem 1.** (i) If \( \alpha \) is a v.e. \( B \) then it is also a v.e. \( E \) but the converse is not true.

(ii) If \( \alpha \) is a v.e. \( E \), then it is also a v.e. \( N \) but the converse is not true.

(iii) If \( f(z) \) has a v.e. \( E \), then \( q \) is necessarily an integer and \( f(z) \) is of perfectly regular growth order \( q \); also \( f(z) \) can have no other v.e. \( E \) or \( N \).

**Theorem 2.** If for a function, the deficiency sum (excluding \( \alpha = \infty \)) \( \sum \delta(\alpha) = 1 \), then there cannot be two values \( e. E \) for simple zeros.

**Corollary.** If a function has a v.e. \( E \) for the whole aggregate of zeros, then there can be no other v.e. \( E \) for simple zeros.

**Theorem 3.** Let \( f(z) \) be of order \( q \) and suppose that either \( q \) is non-integer, or when \( q \) is integer or zero then \( f(z) \) satisfies the condition

\[
(2) \quad \limsup_{r \to \infty} \frac{\log M(r)}{r^{\varphi L(r)}} = 1,
\]

where \( L(r) \) is any positive continuous and monotone function for all large \( r \) and satisfies the condition \( L(kr) \sim L(r) \), as \( r \to \infty \), for any fixed positive \( k \). If \( q = 0 \), suppose further that \( \log r = o(L(r)) \).

Then there cannot be more than two values \( e. E \) for simple zeros.

**Theorem 4.** Let \( f(z) \) satisfy the conditions of Theorem 3. Then there cannot be more than one v.e. \( E \) for the joint sequence of simple and double zeros and if such a value exists, the sequence of simple zeros is normal \( E \) for every other value.

It is known that a v.e. \( N \) for a proper meromorphic function \( f(z) \) (that is, \( n(r, \infty) > 0 \)) may not be an asymptotic value of

1) See § 7 below.
f(z) [14]. If f(z) be an entire function and δ(α) > 0 for f(z), then it is not known whether α is necessarily an asymptotic value of f(z). For a v.e. E we have

**Theorem 5.** (i) If α is a v.e. E for an entire function f(z), then it is also an asymptotic value but the converse is not true.

(ii) A v.e. E is 'invariant' with respect to the displacement of the origin; that is, if

\[ \lim_{r \to \infty} \inf \frac{\log M(r)}{n(r, \alpha)\varphi(r)} > 0 \]

for some \( \varphi \subset E \), and if \( M_A(r), n_A(r, \alpha) \) refer to another 'origin' \( A \) then

\[ \lim_{r \to \infty} \inf \frac{\log M_A(r)}{n_A(r, \alpha)\varphi(r)} > 0. \]

We now give two theorems of a different type. Theorem 6 extends a result of Polya and Pfluger [7].

**Theorem 6.** If a function of finite order \( \phi \) has a v.e. E, its power series has a density equal to one of the fractions

\[ \frac{1}{\phi}, \frac{2}{\phi}, \ldots, \frac{\phi}{\phi}. \]

**Theorem 7.** Suppose f(z) is of order 1 and has a v.e. E. If \( \lim \log M(r)/r = T \) and if f(z) has an asymptotic period \( \beta \) then

\[ |\beta| \geq 2\pi/T. \]

We suppose here \( \beta \) to be Whittaker [6,84] period. If we follow the definition of an asymptotic period as given by S. S. Macintyre [15] then \( |\beta| \geq \pi/T. \)

4. **Proof of Theorem 1.** (i) If \( \beta \) is a v.e. B then \( \log M(r) \sim Tr^\phi \) \((0 < T < \infty)\) and \( \varepsilon_1(\alpha) < \phi \). Hence

\[ \lim_{r \to \infty} \frac{\log M(r)}{n(r, \alpha)r^\beta} = \infty, \quad 0 < \beta < \phi - \varepsilon_1(\alpha). \]

To show that the converse is not true, we consider

(3) \[ f(z) = e^z P(z) = e^z \prod_{n=2}^{\infty} \left( 1 + \frac{z}{n (\log n)^2} \right); \quad \alpha = 0. \]

(ii) If \( \phi > 0 \) is non-integer then \([3,69]\]

\[ \lim_{r \to \infty} \inf \frac{\log M(r)}{n(r, \alpha)} < A \]

1) This limit exists. See Theorem 1 (iii) above.
for every $\alpha$. If $\varphi = 0$ then \([4(i), 29-30]\)

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{n(r, 0)\varphi(r)} = 0$$

and hence

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{n(r, \alpha)\varphi(r)} = 0.$$ 

Hence if $\alpha$ is a v.e. $E$, $\varphi$ must be integer and we will have

\[(4) \quad f(z) = z^n \exp \{Q(z, \alpha)\} P(z, \alpha)\]

where $Q(z, \alpha)$ is a polynomial of degree $q(\alpha)$ (say) and $P(z, \alpha)$ is the canonical product (c.p.) of genus $p(\alpha)$ (say). We have either \([4 (ii) 186-187]\) $q_1(\alpha) < q$ or $q_1(\alpha) = q(\alpha) = q; p(\alpha) = q - 1$. In either case we have log $M(r) \sim T r^q$ ($0 < T < \infty$) for we have

**LEMA.** If $f(z) = z^N e^{q(z)} P(z)$ is of order $q$, $\varphi$ integer and $q = q$, $p \leq q - 1$, then

\[(5) \quad \log M(r, f) \sim T r^q \quad (0 < T < \infty)\]

**PROOF.** Let $Q(z) = a z^q + \ldots, | a | = T$. Then

$$\log M(r, f) < O(\log r) + (T + O(1))r^q + O(r^q) \sim T r^q.$$ 

Suppose if possible

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{r^q} = l < T$$

and let $l < L < T$, $1 < k < (T/L)^{1/q}$. Then for a sequence of values of $r = r_n$ ($n = 1, 2, \ldots$), $\log M(r, f) < L r^q$.

For any $R_n$ ($n = 1, 2, \ldots$) such that $r_n/k \leq R_n \leq r_n$ we have

$$\log M(R_n, f) \leq \log M(r_n, f) < L r^q \leq L k^q R_n^q.$$  

Further $\log M(r, P) = 0(r^q)$ and there is always a circle $| z | = R_n$ in the annulus $r_n/k \leq | z | \leq r_n$ on which $[3,89]

$$| P(z) | > \{M(k r, P)\}^{-H}.$$ 

Hence for $r = R_n$ ($n > N_0$)

$$\log \left| \frac{1}{P(z)} \right| < H \log M(k r, P) < H k^q r^q$$

$$e^{(R^q(z))} = \left| \frac{f(z)}{z^N P(z)} \right| < \exp \{L k^q R_n^q + 2H k^q R_n^q\}.$$
Hence
\[ \max \frac{R(Q(z))}{\liminf_{\mathbb{R}_n \to \infty} \frac{|z| = R_n}{R_n^k}} \leq Lk^q. \]

But the left hand expression has the limit \( T > kL \).

Hence we have a contradiction and so \( l = T \) which proves the lemma.

If \( \alpha \) is a v.e. \( E \), then \( \log M(r) \sim Tr^q \) and so
\[ T(r) > A \log M(r) > An(\alpha) \varphi(r) \]
and since \( \log r = o(\varphi(r)) \) we have
\[ (6) \quad \lim_{r \to \infty} \frac{T(r)}{N(r, \alpha)} = \infty; \quad \delta(\alpha) = \Delta(\alpha) = 1. \]

To show that \( \delta(\alpha) \) may be equal to unity but \( \alpha \) may not be a v.e. \( E \) we need consider the c.p. \( P(z) \) defined in (3). For this c.p. \( \delta(0) = 1 \) and
\[ \lim_{r \to \infty} \frac{\log M(r)}{n(\alpha, 0) \varphi(r)} = 0. \]

(iii) To complete the proof of (iii) we note that \( \sum \delta(\alpha) \leq 1 \) the summation being over all finite values of \( \alpha \). Since \( \delta(\alpha) = 1 \) there can be no other v.e. \( N \) and a fortiori e. \( E \).

5. PROOF OF THEOREM 2. Since \( \sum \delta(\alpha) = 1 \), \( q \) is integer \([8,92-94]\)

Let \( q(r) \) be a proximate order. Then
\[ \lim_{r \to \infty} q(r) = q, \quad \lim_{r \to \infty} q(r) \log r = 0 \]
\[ \log M(r) \leq r^q(r) \text{ for all } r > r_0 \]
\[ = r^q(r) \text{ for an infinity of } r. \]

Further \([8,94]\) \( N(r, f') = o(r^q(r)) \).

If \( \alpha \) and \( \beta \) (\( \alpha \neq \beta \)) be v.e. \( E \) for simple zeros then
\[ N(r, \alpha) + N(r, \beta) > A(k) \log M(r/k) > Ar^q(r) \]
for an infinity of \( r \), say \( r = r_n \). Also if \( N_1 \) refers to simple zeros then
\[ N_1(r, \alpha) + N_1(r, \beta) + 4N(r, f') + O(\log r) > N(r, \alpha) + N(r, \beta) \]
and so for \( r = r_n \) (\( n > n_0 \))
\[ N_1(r, \alpha) + N_1(r, \beta) > Ar^q(r). \]

Now
\[ \log M(r) > An_1(r, \alpha) \varphi_1(r), \quad \varphi_1(r) \subset E, \ r > R_0 \]
\[ \log M(r) > An_1(r, \beta) \varphi_2(r), \quad \varphi_2(r) \subset E, \ r > R_0. \]
Let \( \varphi(x) = \min \{ \varphi_1(x), \varphi_2(x) \} \). Then it is easily seen that \( \varphi(x) \subset E \) and we have for \( r > R_0 \)

\[
\log M(r) > A \{ n_1(r, \alpha) + n_1(r, \beta) \} \varphi(r).
\]

Hence for \( r = r_n \) \((n > N_2 > n_0)\)

\[
r^{q(r)} \geq \log M(r) > A \{ n_1(r, \alpha) + n_1(r, \beta) \} \varphi(r)
\]

\[
> \frac{A \varphi(r)}{\log r} \{ N_1(r, \alpha) + N_1(r, \beta) \} > \frac{A \varphi(r)}{\log r} r^{q(r)}.
\]

Hence we have a contradiction and so the theorem is proved.

**Proof of Corollary.** Let \( \alpha \) be a value exceptional \( E \) for the whole aggregate of zeros. Then \( \delta(\alpha) = 1 \) and so by the theorem there cannot be two values \( e. E \) for simple zeros. Since \( \alpha \) is a fortiori a v.e. \( E \) for simple zeros, there can be no other v.e. \( E \) for simple zeros.

6. **Proof of Theorem 3.** Suppose if possible there are three such values \( a, b, c \) \((a \neq b \neq c)\). Let \( P(z, a) = P_a \) denote the c.p. formed with the simple zeros of \( f(z) - a \) and denote by \( p_1(a) \) its genus and by \( q_{11}(a) \) its order. Similarly for \( P(z, b) \) and \( P(z, c) \).

Then

\[
\theta(z) = \frac{P(z, a)P(z, b)P(z, c)\{f'(z)\}^2}{\{f(z) - a\}\{f(z) - b\}\{f(z) - c\}}
\]

is an entire function. \([3,76]\).

(i) Consider first when \( \varrho > 0 \) is non-integer. We have

\[
n_1(r, a) < A \frac{\log M(r, f)}{\varphi(r)} \leq \frac{A r^{q(r)}}{\varphi(r)}, \quad r > r_0.
\]

We prove that

\[
(8) \quad \log M(r, P_a) = o(r^{q(r)}).
\]

If \( q_{11}(a) < \varrho \) then \( (8) \) follows. Suppose therefore \( q_{11}(a) = \varrho, \) \( p_1(a) < \varrho < 1 + p_1(a) \). Writing \( p_1(a) = p \) and \( n_1(x, a) = n(x) \) we have

\[
(9) \quad \log M(r, P_a) < A \left\{ r^p \int_0^r \frac{n(t) dt}{t^{p+1}} + r^{p+1} \int_r^{\infty} \frac{n(t) dt}{t^{p+2}} \right\}.
\]

Now for all \( x > x_0, p < q(x) < 1 + p \) and so \( x^{q(x) - p} \) is increasing and \( x^{q(x) - p-1} \) is decreasing for \( x > x_1 \). Hence from \( (7) \) and \( (9) \) we obtain \( (8) \). Similarly for \( P_b \) and \( P_c \). Let the zeros of \( f - a, f - b, f - c \) be respectively

\[
(a_n)^\infty, (b_n)^\infty, (c_n)^\infty;
\]
and denote by $S$ the set of circles
\[ |z - a_n| = |a_n|^{-h}, \quad |z - b_n| = |b_n|^{-h}, \quad |z - c_n| = |c_n|^{-h}; \]
\[ (|a_n| \geq 1, \quad |b_n| \geq 1, \quad |c_n| \geq 1, \quad h > \varrho) \]
Then in the domain $D$ exterior to the circles $S$ we have \[3, 74\] for $r > r_0$
\[ \left| \frac{f'(z)}{f(z) - a} \right| < r^{2K} \]
and hence in $D$
\[
\log M(r, (f - c)\theta) = o(r^{\varrho(r)}) + O(\log r) = o(r^{\varrho(r)}).
\]
Similarly for $(f - b)\theta$ and hence in $D$
\[
\log M(r, \theta) = o(r^{\varrho(r)}).
\]
Now
\[
\log M(r, f - c) > Ar^{\varrho(r)}
\]
for a sequence of values of $r = r_n \to \infty$. Let $k > 1$ be a fixed positive constant and let $r_n \leq r \leq kr_n$. Then for $n > n_0$
\[
\log M(r, f - c) \geq \log M(r_n, f - c) > A_1 r_n^{\varrho(r_n)} > Ar^{\varrho(r)}.
\]
Further
\[
\log M(r, f - c) \leq \lambda T(2r, f - c)
\]
\[
\leq A \left[ T\{2r, (f - c)\theta\} + T\left\{2r, \frac{1}{\theta}\right\}\right]
\]
\[
\leq A \left[ \log M\{2r, (f - c)\theta\} + \log M(2r, \theta) + O(1)\right]
\]
\[
< \epsilon r^{\varrho(r)}
\]
for all $r > R_1$, such that $2r \subset D$. Let $r_n > R_1, n > n_0$. Since we can always draw a circle $|z| = r$ in the annulus $r_n \leq |z| \leq kr_n$ such that $2r \subset D$, we have for a sequence of values of $r \to \infty$,
\[
Ar^{\varrho(r)} < \log M(r, f - c) < \epsilon r^{\varrho(r)}
\]
which leads to a contradiction and so the theorem is proved.

(ii) $\varrho$ integer. We prove first that

(10) \[
\log M(r, P_n) = o(r^{\varrho L(r)}).
\]

We have
\[
n_1(r, a) = n(r) \quad (\text{say}) \quad < \frac{Ar^{\varrho L(r)}}{\varphi(r)}.
\]
It is known that \( [10] r^2L(r) \to \infty, r^{-c}L(r) \to 0 \), for every constant \( c > 0 \), as \( r \to \infty \). Further
\[
\int_1^r L(t)dt \sim rL(r), \quad \int_r^\infty \gamma^{-2}L(t)dt \sim r^{-1}L(r).
\]

If \( \varepsilon_{11}(a) < \varepsilon \) then (10) is obvious. Suppose therefore
\[
\varepsilon_{11}(a) = \varepsilon, \quad p_1(a) = p \text{ (say)} = \varepsilon - 1 \text{ or } \varepsilon.
\]

(a) Consider first when \( p = \varepsilon - 1 \) and \( L(r) \downarrow \). We divide the interval of integration \((0, r)\) of the first integral on the right hand side of (9) in the intervals \((0, \sqrt{r}), (\sqrt{r}, r)\). Then each of these three integrals is \( o(r^2L(r)) \).

(b) \( p = \varepsilon - 1, \quad L(r) \uparrow \).

Here \( \log M(r, P_a) = o(r^2) = o(r^2L(r)) \)

(c) \( p = \varepsilon, \quad L(r) \uparrow \). We choose \( \lambda = \lambda(r), \quad (0 < \lambda < r) \) tending to infinity with \( r \) so slowly that \( L(\lambda(r)) = o(L(r)) \) and divide the interval of integration \((0, r)\) in the intervals \((0, \lambda) (\lambda, r)\). Then each of these three integrals is \( o(r^2L(r)) \).

(d) \( p = \varepsilon, \quad L(r) \uparrow \) or \( \downarrow \). This alternative is not possible since it would make the integral \( \int_1^\infty \{ n(x/x^{p+1}) \} dx \) convergent.

Hence in all cases (10) holds and the rest of the argument is similar to that given in (i).

(iii) \( \varepsilon = 0 \). The proof is similar to that given in (i). The proof of Theorem 4 is similar to that of Theorem 3.

7. Example. Let \( G(z) \) be any entire function of order \( \varepsilon > 1 \) and lower order \( \lambda < 1 \) and let
\[
f(z) = \{G(z)\}^aP(z)
\]
where \( P(z) \) is c.p. defined in (3). Then it is easily seen that \( 0 \) is a v.e. \( B \) for the simple zeros of \( f(z) \). But
\[
n_1(r, 0) \sim r/\log^2 r
\]
\[
\log M(r, f) \leq 2 \log M(r, G) + \log M(r, P).
\]
Hence for a sequence of values of \( r \) tending to infinity we have
\[
\log M(r, f) < A r/\log r,
\]
\[
\liminf_{r \to \infty} \frac{\log M(r, f)}{n_1(r, 0)\phi(r)} = 0.
\]
Hence 0 is not a v.e. E for simple zeros of f(z).

8. PROOF OF THEOREM 5. (i) From (4) we have

$$|f(z) - \alpha| = r^ne^{RQ(z, \alpha)}|P(z, \alpha)|$$

Let $Q(z, \alpha) = az^p + Q_1(z); a = Te^{i\theta}$, $Q_1(z)$ a polynomial of degree $\leq e - 1$. Then

$$\log |f(z) - \alpha| = Tr^e \cos(\theta \theta_0 + \beta) + RQ_1(z) + \log |P(z, \alpha)|$$

Let $0 < \delta < \pi/10$ and $\theta_0$ be such that

$$\frac{\pi}{2} + \delta \leq \theta \theta_0 + \beta + 2k\pi \leq \pi + \frac{\pi}{2} - \delta$$

($k$ integer or zero); and let $0 < \epsilon < -(T/4) \cos(\theta \theta_0 + \beta)$, $z = re^{i\theta_0}$. Choose $r_0$ so large that for all $r > r_0$ and all $\theta$

$$RQ_1(z) < \epsilon r^e, n \log r < \epsilon r^e, \log |P(z, \alpha)| < \epsilon r^e.$$ 

Then for $z = re^{i\theta_0}$, $r > r_0$.

$$\log |f(z) - \alpha| < r^e\{T \cos(\theta \theta_0 + \beta) + 3\epsilon\} \to -\infty \text{ as } r \to \infty.$$ 

Hence $f(z) \to \alpha$ as $z = re^{i\theta_0} \to \infty$; that is $\alpha$ is an asymptotic value.

To show that the converse is not true, we consider [2,160—161]

$$f(z) = \int_0^\infty e^{-r^e} \varrho \text{ integer, } 2 \leq \varrho < \infty.$$ 

Let

$$a_\mu = \exp \left( \frac{2\mu\pi i}{\varrho} \right) \int_0^\infty e^{-r^e} dr, \mu = 0, 1, 2, \ldots, \varrho - 1.$$ 

Then for $a = a_0, a_1, \ldots, a_{\varrho-1}$.

$$T(r) \sim \frac{r^e}{\pi}; n(r, a) > \frac{A_1r^e}{\log r}; \lim_{r \to \infty} \frac{\log M(r)}{n(r, a)\varphi(r)} = 0.$$ 

Hence each of these numbers $a_0, a_1, \ldots, a_{\varrho-1}$ is an asymptotic value but not a value exceptional E.

(ii) We may suppose that the new 'origin' $A$ is on the real positive axis at a distance $h$ from 0. Then since

$$\log M(r) \sim Tr^e \quad (0 < T < \infty)$$

$$M(r - h) \leq M_A(r) \leq M(r + h)$$

it follows that $\log M_A(r)$ lies between $A_1r^e$ and $A_2r^e$ for all
Further
\[ n(r - h, \alpha) \leq n_A(r, \alpha) \leq n(r + h, \alpha) \]
\[ n(r + h, \alpha)\varphi(r + h) < A_3r^\varrho. \]

Hence
\[ n_A(r, \alpha)\varphi(r) \leq n(r + h, \alpha)\varphi(r + h) < A_3r^\varrho < A_4 \log M_A(r). \]
\[ \liminf_{r \to \infty} \frac{\log M_A(r)}{n_A(r, \alpha)\varphi(r)} > 0. \]

We omit the proofs of Theorems 6 and 7 which can be proved by following the argument given by Whittaker [6,61—62; 84—87].

9. Meromorphic Functions. Let \( F(z) \) be a meromorphic function of finite order \( \varrho \). We define a number \( \alpha \) (\( 0 \leq |\alpha| \leq \infty \)) to be an e.v. \( E \) for \( F(z) \) if

\[
\text{for some } \rho \subset E. \]

It is easily seen that the two definitions of values e. \( E \) for entire functions are equivalent. Obviously \( \infty \) is a v.e. \( E \) for entire functions according to (1) or (11). We can also prove that if \( \alpha \) is a v.e. \( E \) for a meromorphic function \( F(z) \) then it is a v.e. \( N \), with deficiency \( \delta(\alpha) = 1 \) and \( A(\alpha) = 1 \). To see that the converse is not true we consider the meromorphic function [1,91—93]

\[
f_\lambda(z) = \sum_{\nu=0}^{2\lambda-1} \eta^\nu f(\eta^\nu z)
\]

where \( \lambda > 1 \) is an odd integer, \( \eta = \exp (\pi i/\lambda) \) and \( f(z) = e^z/(e^z-1) \). This function \( f_\lambda(z) \) is a meromorphic function of order 1 and has \( 2\lambda \) values e. \( N \); \( \eta^\nu \alpha \) (\( \nu = 0, 1, 2, \ldots, 2\lambda - 1 \)) each with deficiency

\[
\frac{1}{\lambda} \left( 1 - \cos \frac{\pi}{2\lambda} \right) < 1.
\]

Hence none of these \( 2\lambda \) values can be a v.e. \( E \).

We note also that if \( \alpha \) be a v.e. \( B \) for a meromorphic function \( F(z) \) then it may not be a v.e. \( E \). In fact Valiron has shown that [13] a value \( \alpha \) e. \( B \) may have deficiency \( \delta(\alpha) = 0 \).

**Theorem 8.** If \( F(z) \) is a meromorphic function of finite order \( \varrho \), then there cannot be more than two values e. \( E \) for \( F(z) \) and if \( F(z) \) has two values e. \( E \) then \( \varrho \) is necessarily an integer and \( T(r, F)/r^\varrho \) tends to a finite non-zero limit as \( r \) tends to infinity.

**Proof.** If \( \alpha \) be a v.e. \( E \) then \( \delta(\alpha) = 1 \). Since \( \sum \delta(\alpha) \leq 2 \) there cannot be more than two values e. \( E \). Suppose then \( \alpha, \beta \) (\( \alpha \neq \beta \),
0 \leq |\alpha| \leq \infty, 0 \leq |\beta| \leq \infty) be two values e. E. Then for all 
r > r_0
\[ T(r) > \delta \{n(r, \alpha) + n(r, \beta)\} \varphi(r) > \delta_1 \{N(r, \alpha) + N(r, \beta)\} \frac{\varphi(r)}{\log r}, \]
\[ \frac{N(r, \alpha) + N(r, \beta)}{T(r)} < \frac{\log r}{\delta_1 \varphi(r)}. \]

But if \( \varrho > 0 \) is non-integer then [1,51—54]
\[ \limsup_{r \to \infty} \frac{N(r, \alpha) + N(r, \beta)}{T(r)} > 0 \]
and if \( \varrho = 0 \) then [11,67—69]
\[ \limsup_{r \to \infty} \frac{N(r, \alpha) + N(r, \beta)}{T(r)} \geq 1. \]

Hence \( \varrho \) must be integer. Further since
\[ T\left(r, \frac{AF + B}{CF + D}\right) = T(r) + O(1), \]
we may suppose that 0 and \( \infty \) are values e.e. Write
\[ F(z) = z^a e^{\varphi(z)} P_1(z)/P_2(z) \]
where \( P_1 \) is c.p. of genus \( p_1 \) (say) formed with zeros \( a_n(\varphi a_n > 0) \) of \( F(z) \) and \( P_2 \) is c.p. of genus \( p_2 \) (say) formed with poles \( b_n(\varphi b_n > 0) \) of \( F(z) \). \( Q(z) \) is a polynomial of degree \( q \) (say). We know that [4 (ii) 188]
\[ \liminf_{r \to \infty} \frac{T(r, F)}{\{n(r, 0) + n(r, \infty)\} \varphi(r)} = 0 \]
for every \( \varphi \subset E \), except when \( q > \max (p_1, p_2) \). Hence \( q = \varrho, p_1 < \varrho, p_2 < \varrho \). So
\[ \limsup_{r \to \infty} \frac{T(r, P_a)}{r^{\varrho}} \leq \limsup_{r \to \infty} \frac{\log M(r, P_a)}{r^{\varrho}} = 0; \ a = 1,2. \]
\[ \lim_{r \to \infty} \frac{T(r, F)}{r^{\varrho}} = \lim_{r \to \infty} \frac{T(r, e^{\varphi})}{r^{\varrho}}. \]

Now \( T(r) \sim \max_a N(r, a) \). Hence if \( Q(z) = b z^\varrho + \ldots \)
then
\[ T(r, e^{\varphi}) \sim \frac{r^\varrho |b|}{\pi}; \ \lim_{r \to \infty} \frac{T(r, F)}{r^{\varrho}} = \frac{|b|}{\pi} \]
and the theorem is proved.
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