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JAKOB LEVITZKI

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by

Jakob Levitzki

Jerusalem

It has been recently proved [1]) that each nil-ring in a ring which satisfies the minimum condition for the right ideals, is nilpotent. In the present note it is shown that this result holds for the wider class of rings in which merely a certain maximum condition is satisfied (see § 2, III and IV). In particular follows the theorem (which solves a problem raised by Köthe [2])): Each right or left nil-ideal of a ring which satisfies the minimum or the *maximum* condition for right (left) ideals is nilpotent. The results obtained are based upon two general theorems (which are possibly of independent interest) on multiplicative nil-systems.

§ 1. On multiplicative nil-systems.

Definition 1. A set A is called a multiplicative system, in short: M-system if in A an operation, called multiplication, is defined satisfying the conditions:

α) If $a \in A$, $b \in A$, then the product ab is a uniquely defined element of A.

β) If $a \in A, b \in A, c \in A$, then $(ab)c = a(bc)$.

Definition 2. The set of all products $a_1 a_2 \cdots a_n$, where the a_i are arbitrary elements of a M-system A, and n is a fixed positive integer, is denoted by A^n. Obviously A^n is also a M-system. If $n < m$ then clearly $A^n \supseteqq A^m$.

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[1]) Ch. Hopkins, Nil-rings with minimal condition for admissible left ideals [Duke Math. Journal 4 (1938), 664—667]. Referred to as H. Also J. Levitzki, On rings which satisfy the minimum condition for the right hand ideals [Compositio Math. 7 (1939), 214—222]. Referred to as L.

[2]) G. Köthe, Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist [Math. Zeitschrift 32 (1930), 161—186]. Referred to as K. In H and in L the first part of the problem (concerning the minimum condition) was already solved.

DEFINITION 3. The M-system A is called a nil-M-system if

α) A contains a zero 0 (i.e. $a0 = 0a = 0$ for each $a \in A$. As easily seen, 0 is uniquely defined).

β) Each element of A is nilpotent.

DEFINITION 4. A M-system A is called nilpotent, if for a certain positive integer n the relation $A^n = 0$ holds (here 0 denotes the M-system containing the zero only). Otherwise A is said to be potent.

DEFINITION 5. The M-system A is said to be generated by the finite set of elements $a_1, a_2, \ldots, a_n$ if each element a of A has the form $a = b_1 b_2 \cdots b_m$, where each b_i is a certain a_j and m depends on a.

DEFINITION 6. If A^* is an arbitrary subset of a M-system A, then we denote by $Z(A^*)$ the right annihilator of A^* in A, i.e. the set of all elements a of A satisfying the relation $a^*a = 0$ for each a^* of A^*. Evidently $Z(A^*)$ is also a M-system. Similarly the left annihilator is defined (see H, 665).

THEOREM 1. *If A is a potent nil-M-system generated by the finite set $a_1, a_2, \ldots, a_n$ then A contains a proper potent nil-M-subsystem A^* which is generated by a finite set $b_1, b_2, \ldots, b_m$ having the form $b_i = a_s^{r_i} c_i$, where s is suitabely fixed, r_i is a positive integer smaller than the index of the nilpotent element a_s, and the c_i are elements of the M-subsystem of A generated by the set $a_1, a_2, \ldots, a_{s-1}, a_{s+1}, \ldots a_n$.*

Proof. Let $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_t$ be a subset of the a_i so that the M-system $\bar{A}$ generated by the $\bar{a}_j$ is still potent, and t is of the least possible value. By the definition of t it is clear that $t \geqq 2$ and that the M-system A_1 generated by $\bar{a}_2, \ldots, \bar{a}_t$ is nilpotent. Let now u be the index of the nilpotent element $\bar{a}_1$ and v the index of the nilpotent M-system A_1. Let further $b_1, b_2, \ldots, b_m$ denote the finite set of all elements of the form

(1) $\bar{a}_1^r \bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_p}$, where $i_j \neq 1$, $j = 1, \ldots, p$; $1 \leqq r < u$; $1 \leqq s < v$.

Finally let A^* denote the M-system generated by the b_i. The theorem will be proved if we show that A^* is a proper potent M-subsystem of A. Now, since $\bar{A}$ is potent, it follows that for each positieve integer x, the elements $d_1, d_2, \ldots, d_x$ can be found so that $d_1 d_2 \cdots d_x \neq 0$, where each d_i is a certain $\bar{a}_j$. From the definition of u and v it follows that if $x \geqq u$, $x \geqq v$ then at least one of the d_i is different from $\bar{a}_1$, and at least one of the d_i is equal to $\bar{a}_1$. Hence by choosing an arbitrary integer y and fixing x so that $x > (u + v)(y + 2)$ we have $d_1 d_2 \cdots d_x = f g_1 g_2 \cdots g_y h$, where f and h are either certain powers of $\bar{a}_1$ or certain elements of A_1, while the g_i are elements of A^*. Since $g_1 g_2 \cdots g_v \neq 0$ it follows that A^* is potent. Since further $A^* \subseteqq \bar{a}_1 A \subset A$ (otherwise $\bar{a}_1$ would be potent), we have $A^* \subset A$, which completes the proof of the theorem.

Corollary. As a consequence follows the existence of an infinite chain

$$A \supset A^* \supset A^{**} \supset \ldots \tag{2}$$

where each term of the chain is a potent nil-M-system, standing to the preceding term in the similar relation as A^* to A.

Remark. The preceding theorem and corollary remain true, if the elements $b_i = a_s^{r_i} c_i$ are replaced by $\bar{b}_i = c_i a_s^{r_i}$.

Theorem 2. *Let A and A^* be the same as in theorem* 1. *Let further B be a M-system containing both A and A^*. Finally let $Z(A)$ and $Z(A^*)$ denote the left annihilators* (*see definition* 6) *of A, resp. A^* in B. Then $Z(A) \subset Z(A^*)$.*

Proof. Let $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_t$ and $\bar{A}$ be the same as in the proof of theorem 1, and let $Z(\bar{A})$ be the left annihilator of $\bar{A}$ in B. Since $A \supseteqq \bar{A}$, we obviously have $Z(A) \subseteqq Z(\bar{A})$, and hence it remains to prove that $Z(\bar{A}) \subset Z(A^*)$. Now, since u is the index of $\bar{a}_1$, it follows that $\bar{a}_1^{u-1} b_i = 0$, $i = 1, \ldots, m$, i.e. $\bar{a}_1^{u-1} \in Z(A^*)$. Now two cases are possible: either $\bar{a}_1^{u-1} \notin Z(\bar{A})$, i.e. $Z(\bar{A}) \subset Z(A^*)$ in which case the theorem is proved; or $\bar{a}_1^{u-1} \in Z(\bar{A})$, i.e. $\bar{a}_1^{u-1} \bar{a}_j = 0$, $j = 1, \ldots, t$, then obviously $u - 1 > 1$ and $\bar{a}_1^{u-2} b_i = 0$ for $i = 1, \ldots, m$; hence $\bar{a}_1^{u-2} \in Z(A^*)$. Since on the other hand $\bar{a}_1^{u-2} \notin Z(\bar{A})$ on account of $\bar{a}_1^{u-2} \bar{a}_1 \neq 0$, we have also in this case $Z(\bar{A}) \subset Z(A^*)$, q.e.d.

Corollary Applying theorem 2 to the infinite descending chain (2), we obtain the following infinite ascending chain.

$$Z(A) \subset Z(A^*) \subset Z(A^{**}) \subset \ldots \tag{3}$$

Remark. The preceding theorem and corollary remain true if the elements $b_i = a_s^{r_i} c_i$ are replaced by $\bar{b}_i = c_i a_s^{r_i}$, and the left annihilators by the right annihilators.

§ 2. Applications.

I. *Rings with minimum condition for potent right ideals.* If S is a ring[3]) which satisfies the minimum condition for the potent right ideals, then we have

[3]) To avoid confusion we recall that if K is a subset of S than KS denotes the right ideal generated by the totality of products ks, $k \in K$, $s \in S$.

Theorem 3. *Each nil-M-system (in particular: each nil-ring) of S which is generated by a finite set is nilpotent.*

Proof. In fact, suppose S contains a potent nil-M-system A which is generated by the finite set $a_1, a_2, \ldots, a_n$. If now $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m$ and $\bar{A}$ are the same as in the proof of theorem 1, then the potent right ideals AS and A^*S evidently satisfy the relation $A^*S \subseteqq \bar{a}_1 AS \subset AS$ (otherwise we would have $\bar{a}_1 AS = AS$, and hence $\bar{a}_1^l AS = AS$ for each positive integer l, which is not true since $\bar{a}_1$ is nilpotent). Applying this result to (2) we obtain the infinite chain of potent right ideals

$$AS \supset A^*S \supset A^{**}S \supset \ldots \tag{4}$$

which contradicts the minimum assumption.

Remark. If the minimum condition is assumed for all the right ideals of the ring, then the above theorem can be extended as follows: Each nil M-system of S (in particular: Each nil-subring) is nilpotent (see L, theorem 11).

II. *Rings with maximum condition for right ideals.* If S is a ring which satisfies the maximum condition for the right ideals, then we prove

Theorem 4. *Each nil-M-system A in S (in particular, each nil-subring) which is generated by a finite set is nilpotent.*

Proof. In fact, suppose S contains a potent nil-M-system A which is generated by the finite set $a_1, a_2, \ldots, a_n$. Let A^* be the M-system defined as in theorem 1 where the b_i are replaced by the $\bar{b}_i$ (see remark to theorem 1). If the infinite chain $A \supset A^* \supset A^{**} \supset \ldots$ is defined according to the corollary to theorem 1, and if $Z(A)$, $Z(A^*)$ etc. are right annihilators in S, then these annihilators are obviously right ideals in S, forming according to theorem 2 the infinite chain $Z(A) \subset Z(A^*) \subset Z(A^{**}) \subset \ldots$ which contradicts the maximum assumption.

Theorem 5. *Each right nil-ideal R in S is nilpotent.*

Proof. From the maximum condition follows the existence of a finite set of elements $a_1, a_2, \ldots, a_n$ in R so that $R^2 = (a_1R, a_2R, \ldots, a_nR)$; if then A is the ring generated by the a_i, we have $R^2 = AR$, hence $R^3 = AR^2 = A^2R$, and in general: $R^s = A^{s-1}R$ for each positive integer s. Since (by theorem 4) A is nilpotent, it follows that also R is nilpotent.

Remark. It follows now easily that also each left nil-ideal in S is nilpotent, and hence that the generalized radical (see K, 169) of S coincides with the nilpotent radical.

III. *Rings with maximum condition for right as well as for left ideals.* If S is a ring which satisfies the maximum condition for the right as well as for the left ideals, then theorems 4 and 5 can be extended as follows:

THEOREM 6. *Each nil-M-system A (in particular each nil-subring) in S is nilpotent.*

Proof. From the maximum condition for left ideals follows for an arbitrary positive integer l the existence of the finite sets $a_1, a_2, \ldots, a_n$ and $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m$ in A^l so that $(A^l, SA^l) = (\ldots, a_i, \ldots, Sa_j, \ldots)$ and $(A^{2l}, SA^{2l}) = (\ldots, a_i\bar{a}_k, \ldots, Sa_t\bar{a}_k, \ldots)$. If now $\bar{A}$ denotes the M-system generated by the a_i and the $\bar{a}_j$, then clearly $(SA^l, A^l) = (S\bar{A}, \bar{A})$ and $(SA^{2l}, A^{2l}) = (S\bar{A}^2, \bar{A}^2)$. Now either $\bar{A}^2 = 0$ for a certain l, in which case the theorem is proved, since then $A^{2l} = 0$; or $\bar{A}^2 \neq 0$ for each l. In this case we consider the right annihilators $Z(A^l, SA^l) = Z(\bar{A}, S\bar{A})$ and $Z(A^{2l}, SA^{2l}) = Z(\bar{A}^2, S\bar{A}^2)$. Since by theorem 4 the M-system $\bar{A}$ is nilpotent for each l, it follows easily that $Z(\bar{A}, S\bar{A}) \subset Z(\bar{A}^2, S\bar{A}^2)$ [4]. Hence supposing that A is potent we obtain the infinite chain of right ideals

$$(5) \qquad Z(A, SA) \subset Z(A^2, SA^2) \subset Z(A^4, SA^4) \subset \ldots$$

which contradicts the maximum condition for right ideals.

IV. *A generalisation.* In H and L it was proved that a ring S which satisfies the minimum condition for the right ideals possesses a nilpotent radical R, that the ring S/R is semi simple, and that theorem 6 holds in S. Since S/R satisfies the minimum and the maximum condition for the right as well as for the left ideals, we easily obtain by theorem 6 the following generalisation:

THEOREM 7. *If S is a ring which contains a nilpotent ideal T so that the ring S/T satisfies the maximum condition for the right as well as for the left ideals, then each nil-M-system (in particular, each nil-ring) in S is nilpotent.*

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[4]) If namely $\bar{A}^t = 0$, $\bar{A}^{t-1} \neq 0$, then $t > 2$ and $\bar{A}\,\bar{A}^{t-2} \neq 0$, while $\bar{A}^2\bar{A}^{t-2} = 0$, i.e. $Z(S\bar{A}, \bar{A})$ does not contain $\bar{A}^{t-2}$ while $\bar{A}^{t-2} \subseteqq Z(S\bar{A}^2, \bar{A}^2)$.