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A minimum-problem on areas of inscribed and circumscribed polygons of a circle.

by

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The present paper is concerned with the following problem of P. Szász [1]).

In the unit circle let us consider a convex inscribed polygon containing the centre of the circle as interior point, then take the circumscribed polygon whose points of contact with the circle coincide with the vertices of the inscribed polygon. The problem consists in finding the inscribed polygon for which the sum of the areas of these two polygons is *as small as possible.*

P. Szász conjectured that the *square* will be obtained as solution. Our aim is to verify this conjecture.

1. The inscribed polygons will be characterized by the halves of the central angles belonging to their sides. By these angles α_i the area-sum is uniquely determined [2]), namely

$$T = T(\alpha_1, \ldots, \alpha_n) = \tfrac{1}{2}\sum_{i=1}^{n} (\sin 2\alpha_i + 2 \operatorname{tg} \alpha_i)$$

where $\alpha_1, \ldots, \alpha_n$ are subject to the conditions

$$\sum_{i=1}^{n} \alpha_i = \pi \quad \text{and} \quad 0 \leqq \alpha_i < \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad (1)$$

When dealing with the stated problem it will be convenient to consider T as the sum of the values of the function

$$f(x) = \tfrac{1}{2}(\sin 2x + 2 \operatorname{tg} x)$$

taken in n places α_i with the restrictions (1).

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[1]) We are indebted to our professor P. Szász for calling our attention to this problem.

[2]) For convenience we shall speak about the angles of the inscribed polygon meaning the half central angles α_i.

The first derivative of $f(x)$,

$$f'(x) = \cos 2x + \frac{1}{\cos^2 x}$$

being positive between 0 and $\frac{\pi}{2}$, $f(x)$ is a strictly increasing function in this interval. Since further

$$f''(x) = \sin 2x \left(\frac{1}{\cos^4 x} - 2\right)$$

changes sign only in $x = \vartheta$ [3]), namely it goes over from negative in positive, $f(x)$ is concave for $0 < x < \vartheta$ and convex for $\vartheta < x < \frac{\pi}{2}$.

We shall make use of the following two properties of a convex function.

I. Jensen's inequality: If $f(x)$ is convex, then

$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) \leqq \frac{f(x_1) + \ldots + f(x_n)}{n} \quad \ldots \quad (2)$$

II. An inequality of Hardy-Littlewood-Pólya [4]): If $f(x)$ is convex and

$$a_1 \geqq \ldots \geqq a_n, \; b_1 \geqq \ldots \geqq b_n, \; \sum_{k=1}^{r} a_k \leqq \sum_{k=1}^{r} b_k \;\; (1 \leqq r \leqq n-1), \; \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k,$$

then

$$\sum_{k=1}^{n} f(a_k) \leqq \sum_{k=1}^{n} f(b_k) \quad \ldots \ldots \ldots \quad (3)$$

If $f(x)$ has no linear part, i.e., is convex in stricter sense, then in (2) and (3) there can not stand the sign of equality provided x_i are not all equal resp. at least for one r, $\sum_{k=1}^{r} a_k < \sum_{k=1}^{r} b_k$ is valid.

Both inequalities, (2) and (3), are valid with the sign $\geqq$ if $f(x)$ is a concave function.

Now we are going to simplify our problem by finding a special class of polygons to which we can restrict ourselves. This essential reduction needs at most three steps.

A. If a polygon has angles greater than $\frac{\pi}{4}$, then take the

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[3]) ϑ is defined by the equation $\cos \vartheta = \frac{1}{\sqrt[4]{2}}$. The approximative value of ϑ is 32°44′05″.

[4]) Inequalities, (Cambridge, 1934), p. 89.

function

$$\varphi(x) = f\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right)$$

which is for $\frac{\pi}{4} \leqq x < \frac{\pi}{2}$ like $f(x)$ in $0 \leqq x < \frac{\pi}{4}$. At $x = \frac{\pi}{4}$, $\varphi(x)$ and $f(x)$ have a common tangent, because

$$\varphi\left(\frac{\pi}{4}\right) = f(0) + f\left(\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$$

and

$$\varphi'\left(\frac{\pi}{4}\right) = f'(0) = 2 = f'\left(\frac{\pi}{4}\right).$$

This common tangent separates $\varphi(x)$ and $f(x)$ in the interval $\frac{\pi}{4} < x < \frac{\pi}{2}$, for there $f(x)$ is evidently above and $\varphi(x)$ below it [5]). Thus we have

$$\varphi(x) = f\left(\frac{\pi}{4}\right) + f\left(x - \frac{\pi}{4}\right) < f(x) \qquad \left(\frac{\pi}{4} < x < \frac{\pi}{2}\right). \quad (4)$$

From the inequality just obtained it is easy to conclude that the polygon got by dividing the angles $\alpha^* > \frac{\pi}{4}$ of the original polygon (provided it has such angles) into two parts, namely into $\frac{\pi}{4}$ and $\alpha^* - \frac{\pi}{4}$, has a smaller area-sum.

B. If the angles $\beta_1, \ldots, \beta_m$ lie in the interval $0 < x < \vartheta$, then $\beta_1^*, \ldots, \beta_m^*$ can obviously be chosen in such a way that they all except possibly one are equal to 0 or to ϑ, further they satisfy the equation

$$\beta_1^* + \ldots + \beta_m^* = \beta_1 + \ldots + \beta_m \, . \quad . \quad . \quad . \quad . \quad (5)$$

There is no loss of generality in assuming that both systems, $\beta_1, \ldots, \beta_m$ and $\beta_1^*, \ldots, \beta_m^*$ are arranged in decreasing order. Then we have obviously the inequalities

$$\sum_{\mu=1}^{k} \beta_\mu < \sum_{\mu=1}^{k} \beta_\mu^* \quad (1 \leqq k < m, \ m > 1) \ . \quad . \quad . \quad (6)$$

(5) and (6) imply together, by the quoted inequality of Hardy-

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[5]) For every point of the interval $0 < x < \frac{\pi}{4}$ the inequality $f'(x) < f'(0) = f'\left(\frac{\pi}{4}\right)$ holds, as it follows at once from the behaviour of $f''(x)$.

Littlewood-Pólya, that (in the case $m > 1$) we have

$$\sum_{\mu=1}^{m} f(\beta_\mu^*) < \sum_{\mu=1}^{m} f(\beta_\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

because $f(x)$ is concave in $0 < x < \vartheta$. Hence it follows immediately, if a polygon has at least two angles β_μ in $0 < x < \vartheta$, then the polygon got by replacing β_μ by β_μ^* is of a smaller area-sum.

C. If, finally, a polygon has angles $\gamma_1, \ldots, \gamma_s$ lying in the interval $\vartheta \leqq x \leqq \frac{\pi}{4}$, then we put

$$\gamma^* = \frac{\gamma_1 + \ldots + \gamma_s}{s}.$$

As $f(x)$ is convex for $\vartheta \leqq x \leqq \frac{\pi}{4}$, by Jensen's inequality we get

$$s \cdot f(\gamma^*) < \sum_{\sigma=1}^{s} f(\gamma_\sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

provided that γ_σ are not all equal.

Hence it is clear that if a polygon has different angles in $\vartheta \leqq x \leqq \frac{\pi}{4}$, then the polygon obtained by making all the angles between ϑ and $\frac{\pi}{4}$ equal to their arithmetical mean has surely a smaller area-sum.

2. In conclusion of 1*A*, *B*, *C* we may infer that in solving our problem instead of the consideration of all possible inscribed polygons it is sufficient to restrict ourselves only to the class P of polygons consisting of all the polygons which have one angle β in the interval $0 \leqq x < \vartheta$ and the other k angles α being equal in the interval $\vartheta \leqq x \leqq \frac{\pi}{4}$. (The meaning of k shows that only $k = 4$ and 5 are possible.) Indeed, to any given polygon which is not contained in P there exists a polygon in P with a strictly smaller T. In order to find such a polygon in P, starting from the given polygon we must proceed by the methods of *A*, then *B*, finally *C*.

The area-sum for a polygon of P is furnished by the function

$$\Phi(\alpha) = T(\alpha, \ldots, \alpha, \beta) = kf(\alpha) + f(\beta)$$
$$(\beta = \pi - k\alpha).$$

Next we are going to inquire the behaviour of $\Phi(\alpha)$ as function of α if

$$\vartheta \leqq \alpha \leqq \frac{\pi}{k} \quad \text{and} \quad 0 \leqq \beta < \vartheta.$$

The first derivative of $\Phi(\alpha)$,

$$\Phi'(\alpha) = k\,f'(\alpha) - k\,f'(\beta) \gtreqless 0$$

(the dash indicating the derivative of the function with respect to the argument) according as

$$\cos 2\beta - \cos 2\alpha \lesseqgtr \frac{1}{\cos^2 \alpha} - \frac{1}{\cos^2 \beta} = \frac{1}{2}\,\frac{\cos 2\beta - \cos 2\alpha}{\cos^2 \alpha \cos^2 \beta}.$$

Here we have $\cos 2\beta - \cos 2\alpha > 0$ because the cosine function decreases in $0 \leqq x \leqq \frac{\pi}{4}$ and β is always less than α. Therefore, $\Phi'(\alpha) \gtreqless 0$ according as

$$\cos \alpha \cos \beta \lesseqgtr \frac{1}{\sqrt{2}}.$$

First we consider the case $k = 5$ and prove that here the lowest sign stands. In fact, $\alpha \leqq 36^\circ$ and $\beta \leqq 180^\circ - 5\vartheta < 20^\circ$ involve that

$$\cos \alpha \,.\, \cos \beta > \cos 36^\circ \,.\, \cos 20^\circ > 0{,}8 \,.\, 0{,}9 > \frac{1}{\sqrt{2}}.$$

Thus $\Phi(\alpha)$ decreases if α increases, consequently, among the polygons of P in the case $k = 5$ the regular polygon with five sides has the minimal T.

We pass now to the examination of the case $k = 4$. Let us consider the function

$$\psi(\alpha) = \cos \alpha \cos \beta - \frac{1}{\sqrt{2}} = -\cos \alpha \cos 4\alpha - \frac{1}{\sqrt{2}},$$

or, in other form,

$$\psi(\alpha) = -8 \cos^5 \alpha + 8 \cos^3 \alpha - \cos \alpha - \frac{1}{\sqrt{2}}$$

and take its first derivative,

$$\psi'(\alpha) = \sin \alpha \,(40 \cos^4 \alpha - 24 \cos^2 \alpha + 1).$$

For the sake of convenience we shall allow α to vary between $\frac{\pi}{5}$ and $\frac{\pi}{4}$. In this interval $\psi'(\alpha)$ is distinct from 0 except one place $\bar{\alpha}$. Since $\psi(\alpha)$ is negative at $\alpha = \frac{\pi}{5}$, further it vanishes at $\alpha = \frac{\pi}{4}$ such that $\psi'\left(\frac{\pi}{4}\right) < 0$, we may establish that $\psi(\alpha)$ starting at $\frac{\pi}{5}$ from a negative value, increases until $\bar{\alpha}$, here has a positive value, then decreases being steadily positive until $\alpha = \frac{\pi}{4}$, where it vanishes. Between $\frac{\pi}{5}$ and $\bar{\alpha}$ $\psi(\alpha)$ vanishes in one place denoted by ξ; here it becomes positive from negative.

Now returning to the function $\Phi(\alpha)$, we have the result: if α increases from $\frac{\pi}{5}$ until ξ, the area-sum also increases, then decreases, if α varies until $\frac{\pi}{4}$. This means, in the case $k = 4$ only the regular polygons with four or five sides may possess the minimal T, any other polygon of P has a greater area-sum than the less of the area-sums for these two regular polygons.

Thus we are led only to examine two polygons, namely the regular polygons with four and five sides. A simple calculation shows that the area-sum is in the case of four sides somewhat less than in the case of five sides [6]).

This completes the solution of our problem.

3. It is worth examining what happens if the angles of the polygons decrease to 0, i.e., the polygon converges to the circle. It is clearly sufficient to consider the case if no angle of the inscribed polygon exceeds ϑ. Then $f(x)$ is concave in the interval considered.

In some cases the inequality of Hardy-Littlewood-Pólya used formerly states the possibility of comparing the area-sums for two inscribed polygons. In order to do this, let us inscribe in the circle the sides of two polygons in decreasing order starting from a common point of the circle. If these two polygons possess the property that the vertices of one of them are preceding on the circle the respective vertices of the other, then the comparison may be carried out. As result we get: the area-sum for the former polygon is smaller than that for the latter polygon. This result is also true if the numbers of the sides are different in the two polygons.

Hence it is easy to infer, if the polygons converge to the circle in such a manner that any two of them are comparable in the sense of the Hardy-Littlewood-Pólya inequality, then the area-sum converges increasingly to the double area of the circle.

If instead of the sum of the areas of the inscribed and circumscribed polygons we take their arithmetical mean, then our results show the remarkable fact that the mean of the areas takes up its smallest value, 3, for the pair of squares and by diminishing

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[6]) We get $T = 6$ and $T \approx 6{,}01035$.

the length of the sides the area-mean tends increasingly to $\pi = 3, 14 \ldots$, i.e., during the process of increasing the number of the sides from 4 to ∞ the mean of the areas changes less than 5%.

From the foregoing remark it is obvious that the area-sum has for the square a very *flat* minimum. This seems to be the reason why we needed not quite elementary methods and also some computation in solving this problem.