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Chain transformations in Mayer chain complexes

by

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1. Introduction[1])

The development of topology in the last two decades has been marked by the important rôle played by the obstruction methods in the theory of continuous maps and that of fibre bundles. Owing to some intrinsic difficulties of both theories, the obstruction methods have been successfully applied so far only to yield various particular results. There are drawbacks to the general solution of the extension and classification problem in both theories named above.

In a recent paper [2] [2]) Eilenberg and MacLane have used the obstruction method to study the chain transformations of abstract complexes with operators, whence they deduced their celebrated determination of the n-th cohomology group of a space X by means of homotopy invariants under the hypothesis that $\pi_q(X) = 0$ $(1 < q < n)$. For chain transformations, the drawbacks mentioned disappear and the obstruction method works satisfactorily to produce general theorems with proofs much simpler than the usual homotopy arguments which are familiar to topologists in this field. This confirms the old belief that homology arguments should be easier than homotopy ones and hence more elegant.

The object of the present paper is to establish a general theory of extension and homotopy classification of chain transformations by means of the obstruction methods. To make our investigations as general as possible, we consider Mayer chain complexes with a group W as operators and their equivariant chain transformations. By taking W to be the group which consists of a single element, one can easily deduce as corollaries the corresponding assertions

[1]) This work was done under contract N7-onr-434, Task Order III, Navy Department, Office of Naval Research, U.S.A.

[2]) Numbers in brackets refer to the bibliography.

for complexes without operators. Both materials and arguments are purely algebraic throughout the paper. No topology will be considered in the sequel.

The sections 2—7 are preliminary. A brief account of the definitions and constructions for Mayer chain complexes analogous to those of closure finite abstract complexes given by Eilenberg [1] will be sketched.

The main part of the theory is stated in §§ 8—14 with Theorems 11.7 and 14.7 as our main theorems. Most of the notions and results are suggested by the analogous ones given by the author in his work on continuous maps in topological groups, [4].

In § 15, a number of special cases are given. The most interesting particular case might be the analogue of Hopf's classification theorem [3; 7] and the generalization of Steenrod [6]. In § 16, the induced homomorphisms on the groups defined in § 9 are defined for a given equivariant chain transformation. These will be found useful when one considers the topological invariance of the groups and invariants introduced here if the Mayer chain complex consists of the chain groups of a finite polyhedron. In the last section we take up the most important case of an augmentable abstract complex.

2. Mayer chain complexes with operators

First of all, let us recall briefly the definition of a Mayer chain complex, [5, p. 684], as follows.

A *Mayer chain complex* $M = \{C_q, \partial\}$ consists of the following two entities:— (i) a sequence of abelian additive groups C_q, $(q = \ldots, -1, 0, 1, \ldots)$; and (ii), for each C_q, a homomorphism

(2.1) $$\partial : C_q \to C_{q-1}$$

satisfying

(2.2) $$\partial\partial = 0.$$

The groups $C_q = C_q(M)$ are called the (integral) *chain groups* and their elements, the (integral) *chains*. The homomorphisms ∂ are called the *boundary homomorphisms*. The index q is called the *dimension*.

When there is no danger of ambiguity, we shall use the same symbol ∂ to denote the boundary homomorphisms in various Mayer chain complexes.

Let M_1 and M_2 be two Mayer chain complexes and let τ be a collection of homomorphisms, one for each dimension q

(2.3) $$\tau : C_q(M_1) \to C_q(M_2).$$

τ is called a *chain transformation* $\tau : M_1 \to M_2$ if

(2.4) $$\tau\partial = \partial\tau.$$

More precisely, consider the following diagram

$$\begin{array}{ccc} C_q(M_1) & \xrightarrow[\partial]{} & C_{q-1}(M_1) \\ \downarrow \tau & & \downarrow \tau \\ C_q(M_2) & \xrightarrow[\partial]{} & C_{q-1}(M_2) \end{array}$$

Condition (2.4) means the two homomorphisms of $C_q(M_1)$ into $C_{q-1}(M_2)$ that can be derived from the diagram are equal.

Given chain transformations $\tau_1 : M_1 \to M_2$ and $\tau_2 : M_2 \to M_3$, the composite transformation $\tau_2\tau_1 : M_1 \to M_3$ is readily defined.

Let W be an abstract group written multiplicatively. Following S. Eilenberg [1, p. 380], we shall say that W *operates* on the Mayer chain complex M (on the left) or that M is a *Mayer chain complex with operators* W, provided for every $w \in W$ a chain transformation $w : M \to M$ is given such that for every chain $c_q \in C_q(M)$

(2.5) $$w_2(w_1c_q) = (w_2w_1)c_q,$$

(2.6) $$1c_q = c_q.$$

Conditions (2.5) and (2.6) imply that for every $w \in W$ and each dimension q, the homomorphism w is an automorphism of $C_q(M)$ onto itself, i.e.

(2.7) $$w : C_q(M) \approx C_q(M).$$

If $wc_q = c_q$ for all $w \in W$ and all chains c_q, then we say that W operates *simply* on M.

3. The integral homology groups

For the sake of completeness, we shall recall the usual definition of the (integral) homology groups of a Mayer chain complex $M = \{C_q, \partial\}$.

A chain $c_q \in C_q(M)$ is called a *cycle* if $\partial c_q = 0$. The set of all q-dimensional cycles of M is denoted by

$$Z_q = Z_q(M).$$

Being the kernel of boundary homomorphism $\partial : C_q \to C_{q-1}$, Z_q is a subgroup of C_q.

A chain $c_q \in C_q(M)$ is called a *boundary* if there exists a chain

$c_{q+1} \in C_{q+1}(M)$ such that $c_q = \partial c_{q+1}$. The set of all q-dimensional boundaries is denoted by

$$B_q = B_q(M).$$

Being the image of C_{q+1} under the boundary homomorphism ∂, B_q is a subgroup of C_q.

The following inclusion is an immediate consequence of (2.2):

$$B_q(M) \subset Z_q(M) \tag{3.1}$$

The quotient group

$$H_q = H_q(M) = Z_q(M)/B_q(M)$$

is defined to be the *q-dimensional* (integral) *homology group* of the Mayer chain complex M.

4. Segregated subcomplexes

Let $M = \{C_q(M), \partial\}$ be a Mayer chain complex with operators W.

A Mayer chain complex $M_0 = \{C_q(M_0), \partial_0\}$ is called a *subcomplex* of M if for each dimension q

$$C_q(M_0) \subset C_q(M). \tag{4.1}$$

$$\partial_0 = \partial \mid C_q(M). \tag{4.2}$$

Conditions (4.1) and (4.2) imply the following inclusion:

$$\partial(C_q(M_0)) \subset C_{q-1}(M_0). \tag{4.3}$$

A subcomplex M_0 of M is said to be *segregated* (with respect to W) if, for each $w \in W$ and each dimension q, the automorphism w: $C_q(M) \approx C_q(M)$ maps $C_q(M_0)$ into itself, i.e.

$$w(C_q(M_0)) = C_q(M_0). \tag{4.4}$$

For a segregated subcomplex M_0 of M, we shall also consider W as a group of operators of M_0.

There are two trivial segregated subcomplexes of M; namely, M itself and the subcomplex 0 defined by

$$C_q(0) = 0.$$

The following special segregated subcomplexes of M are of importance in the sequel.

For any given integer n, let us denote by M^n the subcomplex of M defined by

$$C_q(M^n) = \begin{cases} C_q(M), & (q \leqq n), \\ 0, & (q > n). \end{cases} \tag{4.5}$$

M^n is obviously segregated and will be called the *n-dimensional skeleton* of M.

Now let M_0 be a given segregated subcomplex of M. Let us denote by $\overline{M}^n$ the subcomplex of M defined by

$$C_q(\overline{M}^n) = \begin{cases} C_q(M), & (q \leqq n), \\ C_q(M_0), & (q > n). \end{cases} \tag{4.6}$$

The segregatedness of M_0 implies that of $\overline{M}^n$. Since obviously we have for each dimension q

$$C_q(\overline{M}^n) = C_q(M_0) \cup C_q(M^n), \tag{4.7}$$

we shall use the following notation:

$$\overline{M}^n = M_0 \cup M^n. \tag{4.8}$$

5. The cohomology groups

For the sake of convenience, it seems better to define in sufficient details some of the cohomology groups of a Mayer chain complex M with operators W modulo a segregated subcomplex M_0.

Let G be an abelian additive (discrete [3])) group with W as a group of operators (on the left), that is, for every $g \in G$ and $w \in W$ the element $wg \in G$ is defined and

$$w(g_1 + g_2) = wg_1 + wg_2, \; w_2(w_1 g) = (w_2 w_1)g, \; 1g = g.$$

The group $C^q(M, G)$ of *q-dimensional cochains of M over G* is defined to be the group of homomorphisms of $C_q(M)$ into G, i.e.

$$C^q(M, G) = \text{Hom}\,(C_q(M), G).$$

The coboundary homomorphism

$$\delta : C^q(M, G) \to C^{q+1}(M, G) \tag{5.1}$$

is defined by taking, for every $f \in C^q(M, G)$ and each $c \in C_{q+1}(M)$,

$$(\delta f)c = f(\partial c). \tag{5.2}$$

Clearly, (2.2) implies

$$\delta\delta = 0. \tag{5.3}$$

Remembering that $C_q(M_0)$ is a subgroup of $C_q(M)$, we define the group $C^q(M, M_0, G)$ of q-dimensional cochains of M modulo M_0 over G as the subgroup of $C^q(M, G)$ which consists of the cochains $f \in C^q(M, G)$ such that $c \in C_q(M_0)$ implies $f(c) = 0$. (4.3) implies that

$$\delta(C^q(M, M_0, G)) \subset C^{q+1}(M, M_0, G). \tag{5.4}$$

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[3]) Cohomology over a topological coefficient group can be defined in an obvious way; however throughout the present work we shall not consider topology in the coefficient groups.

A cochain $f \in C^q(M, G)$ is called a *cocycle* if $\delta f = 0$. The set of all q-dimensional cocycles is denoted by $Z^q(M, G)$. Being the kernel of the coboundary homomorphism (5.1), $Z^q(M, G)$ is a subgroup of $C^q(M, G)$. A cochain $f \in C^q(M, G)$ is called a *coboundary* if $f = \delta\varphi$ for some $\varphi \in C^{q-1}(M, G)$. The set of all q-coboundaries is the subgroup $B^q(M, G) = \delta C^{q-1}(M, G)$. Further, (5.3) implies the following inclusion

$$B^q(M, G) \subset Z^q(M, G). \tag{5.5}$$

The quotient group

$$H^q(M, G) = Z^q(M, G)/B^q(M, G)$$

is defined to be the *q-dimensional cohomology group* of M over G.

A cochain $f \in C^q(M, G)$ is said to be *equivariant* if it satisfies

$$f(wc) = wf(c) \tag{5.6}$$

for all $w \in W$ and all $c \in C_q(M)$. They form a subgroup $C^q_e(M, G)$ of $C^q(M, G)$. It follows easily that

$$C^q_e(M, G) \subset C^{q+1}_e(M, G). \tag{5.7}$$

Setting $C^q_e(M, M_0, G) = C^q(M, M_0, G) \cap C^q_e(M, G)$, we shall define:

$$Z^q_e(M, M_0, G) = Z^q(M, G) \cap C^q_e(M, M_0, G),$$
$$B^q_e(M, M_0, G) = \delta C^{q-1}_e(M, M_0, G).$$

It follows from (5.3), (5.4) and (5.7) that

$$B^q_e(M, M_0, G) \subset Z^q_e(M, M_0, G).$$

The quotient group

$$H^q_e(M, M_0, G) = Z^q_e(M, M_0, G)/B^q_e(M, M_0, G)$$

is called the *q-dimensional equivariant cohomology group of M modulo M_0 over G*. When $M_0 = 0$, it is called the *q-dimensional equivariant cohomology group of M over G*, denoted by $H^q_e(M, G)$.

6. Chain transformations

We shall return to the concept of a chain transformation as defined in § 2. Let W be a group of operators for two Mayer chain complexes M and N. Let M_0, N_0 be segregated subcomplexes respectively of M, N. By a chain transformation

$$\tau : (M, M_0) \to (N, N_0) \tag{6.1}$$

of the pair (M, M_0) into the pair, (N, N_0), we understand a chain transformation $\tau : M \to N$ such that $\tau C_q(M_0) \subset C_q(N_0)$ for each dimension q. The chain transformation (6.1) is said to be *equivariant* if $\tau w = w\tau$ for every $w \in W$.

Now let (6.1) be an equivariant chain transformation and let G be an abelian additive group with W as a group of left operators. For each cochain $f \in C^q(N, G)$, we define a cochain $\tau^* f \in C^q(M, G)$ by taking for each $c \in C_q(M)$

$$(6.2) \qquad (\tau^* f)(c) = f(\tau c).$$

Obviously the correspondence $f \to \tau^* f$ defines a homomorphism

$$(6.3) \qquad \tau^* : C^q(N, G) \to C^q(M, G)$$

for every dimension q. The following relations are easily verified:

$$\tau^* C^q(N, N_0, G) \subset C^q(M, M_0, G),$$
$$\tau^* C^q_e(N, G) \subset C^q_e(M, G), \quad \tau^*\delta = \delta\tau^*.$$

These imply that τ^* induces homomorphisms of the groups of equivariant cocycles and coboundaries of N modulo N_0 over G into the corresponding ones of M modulo M_0 over G. Consequently we obtain homomorphisms:

$$\tau^q_e : H^q_e(N, N_0, G) \to H^q_e(M, M_0, G).$$

7. Chain homotopies

Two chain transformations $\tau : M \to N$ and $\varrho : M \to N$ are said to be *chain homotopic* if, for each dimension q, a homomorphism

$$(7.1) \qquad \mathrm{D} : C_q(M) \to C_{q+1}(N)$$

is given in such a manner that for each $c \in C_q(M)$

$$(7.2) \qquad \partial \mathrm{D} c = \tau c - \varrho c - \mathrm{D}\partial c.$$

The collection D of homomorphisms (7.1) is called a *chain homotopy* between τ and ϱ, denoted by $\mathrm{D} : \tau \simeq \varrho$.

If the chain transformations τ and ϱ map $C_q(M_0)$ into $C_q(N_0)$ for each dimension q, i.e.

$$(7.3) \qquad \tau : (M, M_0) \to (N, N_0), \quad \varrho : (M, M_0) \to (N, N_0),$$

then τ and ϱ are said to be *chain homotopic relative to* $\{M_0, N_0\}$ provided that there exists a chain homotopy $\mathrm{D} : \tau \simeq \varrho$ satisfying for each dimension q

$$(7.4) \qquad \mathrm{D} C_q(M_0) \subset C_{q+1}(N_0).$$

Such a chain homotopy $\mathrm{D} : \tau \simeq \varrho$ will be called a *chain homotopy* relative to $\{M_0, N_0\}$, denoted by $\mathrm{D} : \tau \simeq \varrho$ rel. $\{M_0, N_0\}$.

A chain homotopy $D : \tau \simeq \varrho$ rel. $\{M_0, N_0\}$ is said to be *equivariant* if for every $w \in W$ and each $c \in C_q(M)$

$$(7.5) \qquad Dwc = wDc.$$

Now let $D : \tau \simeq \varrho$ rel. $\{M_0, N_0\}$ be an equivariant chain homotopy between two equivariant chain transformations τ and ϱ in (7.3). For each dimension q, we define a homomorphism

$$(7.6) \qquad D^* : C^q(N, G) \to C^{q-1}(M, G)$$

by taking for each $f \in C^q(N, G)$ and each $c \in C_{q-1}(M)$

$$(7.7) \qquad D^*f(c) = f(Dc).$$

Condition (7.2) implies that

$$(7.8) \qquad \delta D^*f = \tau^*f - \varrho^*f - D^*\delta f.$$

If W operates on the left of G, then D^*f is equivariant if f is so. Formula (7.8) implies that if f is an equivariant cocycle then $\tau^*f - \varrho^*f$ is an equivariant coboundary δD^*f. Hence the chain transformations τ and ϱ have the same effect on the equivariant cohomology groups; i.e. $\tau_e^q = \varrho_e^q$ for every dimension q.

8. Obstructions

Throughout the sections 8—15, let M and N be two Mayer chain complexes with W as a group of operators for both M and N, and let M_0 be a segregated subcomplex of M. According to § 4, the subcomplex

$$\overline{M}^n = M_0 \cup M^n$$

is segregated for every integer n.

All the chain transformations and the chain homotopies considered in the remainder of the paper are equivariant with respect to the group W of operators; therefore, we shall occasionally omit the descriptive word „equivariant" when there is no danger of ambiguity.

For each dimension q, we shall denote by

$$(8.1) \qquad \eta : Z_q(N) \to H_q(N)$$

the natural homomorphism (with kernel $B_q(N)$) of the integral cycles onto the homology classes. W operates on the left of $H_q(N)$ in an obvious way and

$$(8.2) \qquad w\eta = \eta w.$$

Now let $\tau : \overline{M}^n \to N$ be an arbitrarily given equivariant chain transformation. For each chain $c \in C_{n+1}(M)$, we have

$$\partial c \in C_n(M) = C_n(\overline{M}^n).$$

Then the chain $\tau\partial c \in C_n(N)$ is well-defined. Since $\partial\tau\partial c = \tau\partial\partial c = 0$, $\tau\partial c \in Z_n(N)$. Hence we obtain a homomorphism

$$l_\tau^{n+1} = \eta\tau\partial : C_{n+1}(M) \to H_n(N). \tag{8.3}$$

LEMMA 8.1 *Each equivariant chain transformation $\tau : \overline{M}^n \to N$ determines as in* (8.3) *an equivariant cocycle $l_\tau^{n+1} \in Z_e^{n+1}(M, M_0, H_n(N))$, called the obstruction of τ.*

Proof. According to (8.3), $l_\tau^{n+1} \in C^{n+1}(M, H_n(N))$. For each $w \in W$ and every $c \in C_{n+1}(M)$, we have

$$l_\tau^{n+1}wc = \eta\tau\partial wc = \eta\tau w\partial c = \eta w\tau\partial c = w\eta\tau\partial c = wl_\tau^{n+1}c.$$

This implies that $l_\tau^{n+1} \in C_e^{n+1}(M, H_n(N))$. Next, for each $c \in C_{n+1}(M_0)$ the chain $\tau c \in C_{n+1}(N)$ is defined and $\tau\partial c = \partial\tau c \in B_n(N)$. Hence we have $l_\tau^{n+1}c = \eta\tau\partial c = 0$, i.e. $l_\tau^{n+1} \in C^{n+1}(M, M_0, H_n(N))$. Last, for each $c \in C_{n+2}(M)$, we have

$$(\delta l_\tau^{n+1})c = l_\tau^{n+1}(\partial c) = \eta\tau\partial\partial c = 0.$$

Hence $l_\tau^{n+1} \in Z^{n+1}(M, H_n(N))$. This completes the proof that

$$l_\tau^{n+1} \in Z_e^{n+1}(M, M_0, H_n(N)). \qquad \text{Q.E.D.}$$

Let $\tau : \overline{M}^t \to N$ and $\varrho : \overline{M}^r \to N$ be two given equivariant chain transformations with $t \geqq n$ and $r \geqq n$. Let $\tau_* = \tau \mid \overline{M}^{n-1}$ and $\varrho_* = \varrho \mid \overline{M}^{n-1}$. Let $\mathrm{D} : \tau_* \simeq \varrho_*$ be an equivariant chain homotopy defined on $\overline{M}^{n-1}$ into N, (see § 7).

For each $c \in C_n(M)$, $\partial c \in C_{n-1}(M) = C_{n-1}(\overline{M}^{n-1})$. Then the chain $\tau c - \varrho c - \mathrm{D}\partial c$ of $C^n(N)$ is well-defined. Further, it follows from (7.5) that

$$\begin{aligned}\partial[\tau c - \varrho c - \mathrm{D}\partial c] &= \tau\partial c - \varrho\partial c - \partial\mathrm{D}\partial c = \tau^*\partial c - \varrho_*\partial c - \partial\mathrm{D}\partial c\\ &= \varrho_*\partial c + \mathrm{D}\partial\partial c + \partial\mathrm{D}\partial c - \varrho_*\partial c - \partial\mathrm{D}\partial c = 0.\end{aligned}$$

Hence $\tau c - \varrho c - \mathrm{D}\partial c \in Z_n(N)$ and we obtain an operator homomorphism

$$\mathrm{E} = \tau - \varrho - \mathrm{D}\partial : C_n(M) \to Z_n(N), \tag{8.4}$$

whence we deduce a homomorphism

$$\xi_\mathrm{D}^n = \eta\mathrm{E} : C_n(M) \to H_n(N). \tag{8.5}$$

LEMMA 8.2. *Each equivariant chain homotopy* D: $\tau_* \simeq \varrho_*$ *of two equivariant chain transformations τ and ϱ determines as in* (8.5) *an equivariant cochain* $\xi^n_{\mathrm{D}} \in C^n_e(M, M_0, H_n(N))$, *called the obstruction of* D.

Proof. According to (8.5), $\xi^n_{\mathrm{D}} \in C^n(M, H_n(N))$. For each $w \in W$ and every $c \in C_n(M)$, we have

$$\begin{aligned}\xi^n_{\mathrm{D}} wc &= \eta \mathrm{E} wc = \eta\tau wc - \eta\varrho wc - \eta \mathrm{D}\partial wc \\ &= w\eta\tau c - w\eta\varrho c - w\eta\mathrm{D}\partial c = w\eta\mathrm{E}c = w\xi^n_{\mathrm{D}} c.\end{aligned}$$

This implies that $\xi^n_{\mathrm{D}} \in C^n_e(M, H_n(N))$. Next, for each $c \in C_n(M_0)$, the chain $\mathrm{D}c \in C_{n+1}(N)$ is defined and it follows from (7.5) that

$$\mathrm{E}c = \tau c - \varrho c - \mathrm{D}\partial c = \tau_* c - \varrho_* c - \mathrm{D}\partial c = \partial \mathrm{D} c \in B_n(N).$$

This completes the proof that $\xi^n_{\mathrm{D}} \in C^n_e(M, M_0, H_n(N))$. Q.E.D.

LEMMA 8.3 *If the equivariant chain transformation* $\tau : \overline{M}^n \to N$ *has an equivariant extension* $\tau' : \overline{M}^{n+1} \to N$, *then* $l^{n+1}_\tau = 0$.

Proof. Suppose that τ has an equivariant extension $\tau' : \overline{M}^{n+1} \to N$. For each $c \in C_{n+1}(M)$, we have

$$\tau\partial c = \tau'\partial c = \partial\tau' c \in B_n(N).$$

Therefore $l^{n+1}_\tau c = \eta\tau\partial c = 0$. This proves that $l^{n+1}_\tau = 0$.

LEMMA 8.4 *If* $\tau, \varrho : \overline{M}^n \to N$ *be equivariant chain transformations and* $\mathrm{D} : \tau \mid M^{n-1} \simeq \varrho \mid M^{n-1}$ *be an equivariant chain homotopy, then*

$$\partial\xi^n_{\mathrm{D}} = l^{n+1}_\tau - l^{n+1}_\varrho. \tag{8.6}$$

Hence if τ *and* ϱ *have equivariant extensions* $\tau', \varrho' : \overline{M}^{n+1} \to N$, *then* $\xi^n_{\mathrm{D}} \in Z^n_e(M, M_0, H_n(N))$.

Proof. For each $c \in C_{n+1}(M)$, we have

$$(\delta\xi^n_{\mathrm{D}})c = \xi^n_{\mathrm{D}}\partial c = \eta\mathrm{E}\partial c = \eta\tau\partial c - \eta\varrho\partial c - \eta\mathrm{D}\partial\partial c = l^{n+1}_\tau c - l^{n+1}_\varrho c.$$

This proves that $\delta\xi^n_{\mathrm{D}} = l^{n+1}_\tau - l^{n+1}_\varrho$. The last assertion of the lemma follows from (8.6) and Lemma 8.3 Q.E.D.

9. The presentable and the regular subgroups

Throughout the present chapter, let us denote by Ω the set of all equivariant chain transformations $\tau : M \to N$. Obviously Ω form an abelian group with functional addition as the group operation. Similarly, for each integer n, let Ω^n denote the group of equivariant chain transformations $\tau : \overline{M}^n \to N$.

LEMMA 9.1. *The correspondence* $\tau \to l^{n+1}_\tau$ *induces a homomorphism*

$$\omega_n : \Omega^n \to H^{n+1}_e(M, M_0, H_n(N)). \tag{9.1}$$

Proof. According to Lemma 8.1 every chain transformation $\tau \in \Omega^n$ determines an obstruction $l_\tau^{n+1} \in Z_e^{n+1}(M, M_0, H_n(N))$. l_τ^{n+1} represents an element $\omega_n(\tau)$ of the equivariant cohomology group $H_e^{n+1}(M, M_0, H_n(N))$. To prove that ω_n is a homomorphism, let $\tau, \varrho \in \Omega^n$ and $c \in C_{n+1}(M)$. Then

$$l_{\tau+\varrho}^{n+1} c = \eta(\tau + \varrho)\partial c = \eta(\tau\partial c + \varrho\partial c) = \eta\tau\partial c + \eta\varrho\partial c = l_\tau^{n+1}c + l_\varrho^{n+1}c.$$

This implies that $\omega_n(\tau + \varrho) = \omega_n(\tau) + \omega_n(\varrho)$. Q.E.D.

The image $\omega_n(\Omega^n)$ will be called the *presentable subgroup* of $H_e^{n+1}(M, M_0, H_n(N))$, denoted by

$$P_e^{n+1} = P_e^{n+1}(M, M_0, H_n(N)).$$

The elements of P_e^{n+1} are called the *presentable elements* of $H_e^{n+1}(M, M_0, H_n(N))$. For each $\tau \in \Omega^n$, the element $\omega_n(\tau) \in p_e^{n+1}$ is said to be *presented* by τ; and τ is called *presenter* of $\omega_n(\tau)$.

A chain transformation $\tau \in \Omega^n$ is said to be *regular* if $c \in C_q(M_0)$ implies $\tau c = 0$ for each dimension q. The regular chain transformations $\tau \in \Omega^n$ form a subgroup Ω_r^n of Ω^n.

The image $\omega_n(\Omega_r^n)$ will be called the *regular subgroup* of $H_e^{n+1}(M, M_0, H_n(N))$, denoted by

$$R_e^{n+1} = R_e^{n+1}(M, M_0, H_n(N)).$$

The elements of R_e^{n+1} are called the *regular elements* of $H_e^{n+1}(M, M_0, H_n(N))$. Each $\tau \in \Omega_r^n$ is called a *regular presenter* of $\omega_n(\tau) \in R_e^{n+1}$. Clearly we have $R_e^{n+1} \subset P_e^{n+1}$. We shall use the following notation:

$$J_e^{n+1} = J_e^{n+1}(M, M_0, H_n(N)) = P_e^{n+1}/R_e^{n+1}.$$

Now let Δ^{n-1} denote the set of all triples $(\tau, \varrho, \mathrm{D})$, where $\tau, \varrho \in \Omega$ and $\mathrm{D} : \tau \mid \overline{M}^{n-1} \simeq \varrho \mid \overline{M}^{n-1}$ is an equivariant chain homotopy defined on $\overline{M}^{n-1}$. For any two triples $(\tau, \varrho, \mathrm{D})$ and $(\tau', \varrho', \mathrm{D}')$ of Δ^{n-1}, we define the homomorphisms $\tau + \tau'$, $\varrho + \varrho'$, $\mathrm{D} + \mathrm{D}'$ by adding the functional values. Obviously $\tau + \tau'$ and $\varrho + \varrho'$ are equivariant chain transformations of M into N and

$$\mathrm{D} + \mathrm{D}' : \tau + \tau' \mid \overline{M}^{n-1} \simeq \varrho + \varrho' \mid \overline{M}^{n-1}$$

is an equivariant chain homotopy defined on $\overline{M}^{n-1}$. We define

$$(\tau, \varrho, \mathrm{D}) + (\tau', \varrho', \mathrm{D}') = (\tau + \tau', \varrho + \varrho', \mathrm{D} + \mathrm{D}').$$

Under this addition, the set Δ^{n-1} of triples $(\tau, \varrho, \mathrm{D})$ form an abelian group.

LEMMA 9.2. *The correspondence* $(\tau, \varrho, \mathrm{D}) \to \xi^n_{\mathrm{D}}$ *induces a homomorphism*

$$\mu_n : \Delta^{n-1} \to H^n_e(M, M_0, H_n(N)). \tag{9.2}$$

Proof. According to Lemma 8.2, every triple $(\tau, \varrho, \mathrm{D}) \in \Delta^{n-1}$ determines an obstruction cochain $\xi^u_{\mathrm{D}} \in C^n_e(M, M_0, H_n(N))$. By Lemma 8.4, ξ^n_{D} is a cocycle and hence represents an element $\mu_n(\tau, \varrho, \mathrm{D})$ of $H^n_e(M, M_0, H_n(N))$. To prove that μ_n is a homomorphism, let $(\tau, \varrho, \mathrm{D})$, $(\tau', \varrho', \mathrm{D}')$ be any two triples of Δ^{n-1} and $c \in C_n(M)$. Then

$$\begin{aligned}\xi^n_{\mathrm{D}+\mathrm{D}'}c &= \eta(\tau + \tau' - \varrho - \varrho' - \mathrm{D}\partial - \mathrm{D}'\partial)c \\ &= \eta(\tau - \varrho - \mathrm{D}\partial)c + \eta(\tau' - \varrho' - \mathrm{D}'\partial)c = \xi^n_{\mathrm{D}}c + \xi^n_{\mathrm{D}'}c.\end{aligned}$$

This implies that μ_n is a homomorphism.

The image $\mu_n(\Delta^{n-1})$ will be called the *presentable subgroup* of $H^n_e(M, M_0, H_n(N)$, denoted by

$$\mathfrak{P}^n_e = \mathfrak{P}^n_e(M, M_0, H_n(N)).$$

The elements of $\mathfrak{P}^n_e$ are called the *presentable elements* of $H^n_e(M, M_0, H_n(N))$. For each triple $(\tau, \varrho, \mathrm{D}) \in \Delta^{n-1}$, the element $\mu_n(\tau, \varrho, \mathrm{D}) \in \mathfrak{P}^n_e$ is said to be *presented* by $(\tau, \varrho, \mathrm{D})$ is called a *presenter* of $\mu_n(\tau, \varrho, \mathrm{D})$.

A triple $(\tau, \varrho, \mathrm{D}) \in \Delta^{n-1}$ is said to be *regular*, if $\tau = 0 = \varrho$ and $c \in C_q(M_0)$ implies $\mathrm{D}c = 0$ for each dimension q. The regular triples of Δ^{n-1} form a subgroup Δ^{n-1}_r of Δ^{n-1}.

The image $\mu_n(\Delta^{n-1}_r)$ will be called the *regular subgroup* of $H^n_e(M, M_0, H_n(N))$, denoted by

$$\mathfrak{R}^n_e = \mathfrak{R}^n_e(M, M_0, H_n(N)).$$

The elements of $\mathfrak{R}^n_e$ are called the *regular elements* of $H^n_e(M, M_0, H_n(N))$. Each triple $(\tau, \varrho, \mathrm{D}) \in \Delta^{n-1}_r$ is called a *regular presenter* of $\mu_n(\tau, \varrho, \mathrm{D}) \in \mathfrak{R}^n_e$. Clearly $\mathfrak{R}^n_e \subset \mathfrak{P}^n_e$. We shall use the following notation:

$$\mathfrak{J}^n_e = \mathfrak{J}^n_e(M, M_0, H_n(N)) = \mathfrak{P}^n_e/\mathfrak{R}^n_e.$$

10. Free operators

Following Eilenberg-MacLane [2, p. 55], we shall say that W *operates freely* on $C_q(M)$ if (i) $C_q(M)$ is a free abelian group and there is a collection $\{c_\alpha\}$ of chains $c_\alpha \in C_q(M)$ such that the totality of the chains wc_α, for all $w \in W$ and all $c_\alpha \in \{c_\alpha\}$ constitute a set of free generators of $C_q(M)$; the collection $\{c_\alpha\}$ itself is called a *W-base* of $C_q(M)$.

W is said to *operate freely* on the pair $(C_q(M), C_q(M_0))$, if it operates freely on $C_q(M)$ with a W-base $\{c_\alpha\}$ such that $\{c_\alpha\} \cap C_q(M_0)$ constitute a W-base of $C_q(M_0)$. We denote

$$\{c^0_\beta\} = \{c_\alpha\} \cap C_q(M_0), \quad \{c^*_\gamma\} = \{c_\alpha\} / C_q(M_0).$$

LEMMA 10.1 If *W operates freely on the pair $(C_q(M), C_q(M_0))$, then the totality of the chains wc^*_γ, for all $w \in W$ and all $c^*_\gamma \in \{c^*_\gamma\}$, constitute a set of free generators of a subgroup $C^*_q(M)$ of $C_q(M)$, and $C_q(M)$ is the direct sum $C_q(M_0) \oplus C^*_q(M)$,* of *$C_q(M_0)$ and $C^*_q(M)$.*

Proof. The free generators wc^*_γ, $(w \in W,\ c^*_\gamma \in \{c^*_\gamma\})$, generate a subgroup $C^*_q(M)$. It remains to prove that

$$C_q(M) = C_q(M_0) \oplus C^*_q(M).$$

Since $\{c_\alpha\} = \{c^0_\beta\} \cup \{c^*_\gamma\}$, we have $C_q(M) = C_q(M_0) + C^*_q(M)$. Now let $c \in C_q(M_0) \cap C^*_q(M)$. It follows from our definition that

$$c = \sum_{i=1}^{r} a_i w_i c^0_{\beta_i} = \sum_{i=1}^{r'} a'_i w'_i c^*_{\gamma_i},$$

where the coefficients a_i and a'_i are integers. Since the chains $w_i c^0_{\beta_i}$ and $w_i c^*_{\gamma_i}$ are free generators of $C_q(M)$ and [4]) $\{c^0_\beta\} \cup \{c^*_\gamma\} = \square$, this implies that $c = 0$. Hence $C_q(M_0) \cap C^*_q(M) = 0$ and the sum is direct. Q.E.D.

COROLLARY 10.2. *If W operates freely on the pair $(C_q(M), C_q(M_0))$, then W operates freely on both $C_q(M_0)$ and $C^*_q(M)$.*

11. Extension of chain transformations

LEMMA 11.1. *Assume that W operates freely on $(C_{n+1}(M), C_{n+1}(M_0))$. An equivariant chain transformation $\tau : \overline{M}^n \to N$ has an equivariant extension $\tau' : \overline{M}^{n+1} \to N$ if $l^{n+1}_\tau = 0$.*

Proof. Assume that $l^{n+1}_\tau = 0$. Since W operates freely on $(C_{n+1}(M), C_{n+1}(M_0))$, there is a direct decomposition of the chain group $C_{n+1}(M)$ as follows:

$$C_{n+1}(M) = C_{n+1}(M_0) \oplus C^*_{n+1}(M).$$

W operates freely on both $C_{n+1}(M_0)$ and $C^*_{n+1}(M)$.

Since $l^{n+1}_\tau = 0$, the operator homomorphism $\tau\partial : C_{n+1}(M) \to C_n(N)$ maps $C_{n+1}(M)$ and hence $C^*_{n+1}(M)$ into $B_n(N)$. Since W operates freely on $C^*_{n+1}(M)$, the homomorphism $\tau\partial \mid C^*_{n+1}(M)$ can be lifted to an operator homomorphism $\tau^* : C^*_{n+1}(M) \to C_{n+1}(N)$ such that

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[4]) Following A. D. WALLACE, we denote by the square $\square$ the vacuous set.

$\partial\tau^*c = \tau\partial c$ for each $c \in C^*_{n+1}(M)$, [2, p. 55]. Define an operator homomorphism $\tau' : C_{n+1}(M) \to C_{n+1}(N)$ as follows. For each chain $c \in C_{n+1}(M)$, there exists a unique pair of chains $c_0 \in C_{n+1}(M_0)$ and $c_* \in C^*_{n+1}(M)$ such that $c = c_0 + c_*$. We define

$$\tau'(c) = \tau(c_0) + \tau^*(c_*).$$

Immediately we have $\partial\tau'c = \tau\partial c$ for each $c \in C_{n+1}(M)$. Hence we obtain an equivariant extension $\tau' : \overline{M}^{n+1} \to N$ by defining $\tau' = \tau$ for the dimensions $q \leqq n$.

LEMMA 11.2. *Assume that W operates freely on* $(C_n(M), C_n(M_0))$ *Given an equivariant chain transformation* $\tau : \overline{M}^n \to N$ *and an equivarent cocycle* $l^{n+1} = l^{n+1}_\tau + \delta\xi^n$ *with* $\xi^n \in C^n_e(M, M_0, H_n(N))$, *there exists an equivariant chain transformation* $\varrho : \overline{M}^n \to N$ *with* $l^{n+1}_\varrho = l^{n+1}$ *and* $\varrho \mid \overline{M}^{n-1} = \tau \mid \overline{M}^{n-1}$.

Proof. Since W operates freely on $(C_n(M), C_n(M_0))$, there is a direct decomposition of $C_n(M)$ as follows:

$$C_n(M) = C_n(M_0) \oplus C^*_n(M).$$

W operates freely on both $C_n(M_0)$ and $C^*_n(M)$.

$\xi^n \in C^n_e(M, M_0, H_n(N))$ means that ξ^n is an operator homomorphism

$$\xi_n : C_n(M) \to H_n(N)$$

which maps $C_n(M_0)$ into zero. Since W operates freely on $C^*_n(M)$, it follows from a lemma of Eilenberg-MacLane, [2, p. 55] ,that the partial homomorphism $\xi^n \mid C^*_n(M)$ can be lifted to an operator homomorphism

$$\mathrm{h}^n : C^*_n(M) \to Z_n(N)$$

such that $\eta\mathrm{h}^n = \xi^n \mid C^*_n(M)$. Define an operator homomorphism

$$\mathrm{E} : C_n(M) \to Z_n(N)$$

as follows: For each chain $c \in C_n(M)$, there exists a unique pair of chains $c_0 \in C_n(M_0)$ and $c_* \in C^*_n(M)$ such that $c = c_0 + c_*$. We define $\mathrm{E}(c) = \mathrm{h}^n(c_*)$. Obviously $\eta\mathrm{E} = \xi^n$ and E maps $C_n(M_0)$ into zero. Define an equivariant chain transformation $\varrho : \overline{M}^n \to N$ by taking $\varrho \mid \overline{M}^{n-1} = \tau \mid \overline{M}^{n-1}$ and

$$\varrho(c) = \tau(c) + \mathrm{E}(c)$$

for every chain $c \in C_n(M)$. To justify this definition, we note firstly that $\mathrm{E}(C_n(M_0)) = 0$ implies the two definitions of ϱ on $C_n(M_0)$ agree, and secondly, for each $c \in C_n(M)$,

$$\partial\varrho(c) = \partial\tau(c) + \partial\mathrm{E}(c) = \partial\tau(c) = \tau(\partial c) = \varrho(\partial c)$$

because $\mathrm{E}(c)$ is a cycle, τ is a chain transformation, and $\varrho \mid \overline{M}^{n-1} = \tau \mid \overline{M}^{n-1}$. The obstruction of ϱ is given as follows: For each $c \in C_{n+1}(M)$

$$l_\varrho^{n+1}(c) = \eta\varrho\partial c = \eta\tau\partial c + \eta \mathrm{E}\partial c = \eta\tau\partial c + \xi^n \partial c = l_\tau^{n+1}(c) + \delta\xi^n(c).$$

Hence $l_\varrho^{n+1} = l_\tau^{n+1} + \delta\xi^n = l^{n+1}$. This completes the proof.

COROLLARY 11.3. *Assume that W operates freely on both $(C_n(M), C_n(M_0))$ and $(C_{n+1}(M), C_{n+1}(M_0))$. The obstruction l_τ^{n+1} of an equivariant chain transformation $\tau : \overline{M}^n \to N$ is an equivariant coboundary modulo M_0 if and only if there exists an equivariant chain transformation $\tau' : \overline{M}^{n+1} \to N$ such that $\tau' \mid \overline{M}^{n-1} = \tau \mid \overline{M}^{n-1}$.*

An equivariant chain transformation $\tau : M_0 \to N$ is said to be *extensible over M* if τ admits an equivariant extension $\tau' : M \to N$. τ is said to be *n-extensible over M* if it is extensible over $\overline{M}^n$.

As in § 9, the equivariant chain transformations $\tau : M_0 \to N$ form an additive abelian group Ω_0 with functional addition as the group operation. The following lemma is obvious.

LEMMA 11.4. *The extensible (n-extensible) elements of the group Ω_0 form a subgroup $\mathfrak{E}$ (a subgroup $\mathfrak{E}^n$).*

If there exist two integers a and b, $a < b$, such that $C_q(M) = C_q(M_0)$ unless $a < q \leqq b$, then we have obviously

$$\Omega_0 = E^a \supset \ldots \supset E^n \supset \ldots \supset E^b = E.$$

Now let $\tau : M_0 \to N$ be an equivariant chain transformation n-extensible over M, and let $\tau' : \overline{M}^n \to N$ be an arbitrary equivariant extension of τ. According to Lemma 8.1 and 9.1, τ' presents a presentable cohomology class

$$\gamma_{\tau'}^{n+1} = \omega_n(\tau') \in P_e^{n+1}(M, M_0, H_n(N)).$$

THEOREM 11.5. Fundamental Extension Theorem. *Assume that W operates freely on $(C_n(M), C_n(M_0))$ and on $(C_{n+1}(M), C_{n+1}(M_0))$. τ is $(n+1)$-extensible over M if and only if $\gamma_{\tau'}^{n+1}$ is regular.*

Proof. *Necessity.* Suppose that τ admits an equivariant extension $\tau^* : \overline{M}^{n+1} \to N$. Let $\tau'' = \tau^* \mid \overline{M}^n$. According to Lemma 8.3, $\gamma_{\tau''}^{n+1} = \omega_n(\tau'') = 0$. Let $\varrho = \tau' - \tau''$; then

$$\gamma_{\tau'}^{n+1} = \omega_n(\tau') - \omega_n(\tau'') = \omega_n(\varrho).$$

Since $\tau' \mid M_0 = \tau = \tau'' \mid M_0$, ϱ is regular. This proves that $\gamma_{\tau'}^{n+1} \in R_e^{n+1}(M, M_0, H_n(N))$.

Sufficiency. Suppose that $\gamma_{\tau'}^{n+1}$ be regular. By the definition of regularity, there exists a regular presenter $\varrho : \overline{M}^n \to N$ of $\gamma_{\tau'}^{n+1}$. Let $\tau'' = \tau' - \varrho$. Since ϱ is regular, $\tau'' \mid M_0 = \tau' \mid M_0 = \tau$. Since

$$\gamma_{\tau''}^{n+1} = \omega_n(\tau' - \varrho) = \omega_n(\tau') - \omega_n(\varrho) = \gamma_{\tau'}^{n+1} - \gamma_{\tau'}^{n+1} = 0,$$

the obstruction $l_{\tau''}^{n+1}$ is an equivariant coboundary modulo M_0. It follows by Corollary 11.3 that there exists an equivariant chain transformation $\tau^* : \overline{M}^{n+1} \to N$ such that $\tau^* \mid \overline{M}^{n-1} = \tau'' \mid \overline{M}^{n-1}$. τ^* is an equivariant extension of τ. Hence τ is $(n + 1)$-extensible over M. This completes the proof.

Lemma 11.6. *$\gamma_{\tau'}^{n+1}$ determines a unique element $\beta_n(\tau)$ of the group $J_e^{n+1}(M, M_0, H_n(N))$ which depends only on $\tau : M_0 \to N$.*

Proof. $\gamma_{\tau'}^{n+1}$ does determine an element $\beta_n(\tau')$ of J_e^{n+1} by means of the projection of P_e^{n+1} onto the quotient group J_e^{n+1}. It remains to prove that $\beta_n(\tau')$ depends only on τ. Let $\tau'' : \overline{M}^n \to N$ be another equivariant extension of τ, then $\varrho = \tau' - \tau''$ is regular. Hence we obtain

$$\gamma_{\tau'}^{n+1} - \gamma_{\tau''}^{n+1} = \omega_n(\varrho) \in R_e^{n+1}(M, M_0, H_n(N)).$$

This implies that $\beta_n(\tau') = \beta_n(\tau'')$.

Theorem 11.7. *Assume that W operates freely on $(C_n(M), C_n(M_0))$ and $(C_{n+1}(M), C_{n+1}(M_0))$. The correspondence $\tau \to \beta_n(\tau)$ is a homomorphism of $\mathfrak{E}^n$ onto $J_e^{n+1}(M, M_0, H_n(N))$, denoted by*

$$\beta_n : \mathfrak{E}^n \to J_e^{n+1}(M, M_0, H_n(N)),$$

whose kernel is the subgroup $\mathfrak{E}^{n+1}$ of Ω_0. Hence the quotient group $\mathfrak{E}^n/\mathfrak{E}^{n+1}$ is isomorphic with $J_e^{n+1}(M, M_0, H_n(M))$.

Proof. Let $\tau, \varrho : M_0 \to N$ be any two chain transformations belonging to the subgroup $\mathfrak{E}^n$ of Ω_0. Let $\tau', \varrho' : \overline{M}^n \to N$ be equivariant extensions of τ and ϱ respectively. Then the equivariant chain transformation $\tau' + \varrho' : M^n \to N$ is an extension of $\tau + \varrho : M_0 \to N$. Obviously, we have

$$l_{\tau'+\varrho'}^{n+1} = l_{\tau'}^{n+1} + l_{\varrho'}^{n+1}.$$

By projecting these cocycles first into the presentable cohomology classes and then into the elements in the quotient group $J_e^{n+1} = P_e^{n+1}/R_e^{n+1}$, we obtain

$$\beta_n(\tau + \varrho) = \beta_n(\tau) + \beta_n(\varrho)$$

by the aid of Lemma 11.6. This proves that β_n is a homomorphism.

Let $p \in P_e^{n+1}$ be an arbitrary presentable cohomology class. Then there exists an equivariant presenter $\tau' : \overline{M}^n \to N$ of p, i.e. $\gamma_{\tau'}^{n+1} = p$. Let $\tau = \tau' \mid M_0$. Then $\tau \in \mathfrak{E}^n$ and $\beta_n(\tau) \in J_e^{n+1}$ is the element which contains p. Hence β_n is onto.

So far we have not used the assumption that W operates freely on $(C_n(M), C_n(M_0))$ and $(C_{n+1}(M), C_{n+1}(M_0))$. We shall use this assumption implicitly in what follows. According to the funsamental extension theorem, any chain transformation $\tau \in \mathfrak{E}^n$ is $(n+1)$-extensible over M if and only if $\gamma_{\tau'}^{n+1} \in R_e^{n+1}$ for any equivariant extension $\tau' : \overline{M}^n \to N$ of τ. In other words, τ is in $\mathfrak{E}^{n+1}$ if and only if $\beta_n(\tau) = 0$. This completes the proof.

For the remainder of this section, we assume that there exist two integers a and b, $a < b$, such that $C_q(M) = C_q(M_0)$ unless $a < q \leqq b$. As an immediate consequence of Theorem 11.7, we have the following

THEOREM 11.8. *Assume that W operates freely on $(C_q(M), C_q(M_0))$ for each q with $a < q \leqq b$. A necessary and sufficient condition that every equivariant chain transformation $\tau : M_0 \to N$ can be equivariantly extended over M is that*

$$J_e^{n+1}(M, M_0, H_n(N)) = 0 \qquad (a \leqq n < b).$$

12. Extension of chain homotopies

Throughout the present section, let τ, $\varrho : M \to N$ be two equivariant chain transformations.

LEMMA 12.1. *Assume that W operates freely on $(C_n(M), C_n(M_0))$ An equivariant chain homotopy $\mathrm{D} : \tau \mid \overline{M}^{n-1} \simeq \varrho \mid \overline{M}^{n-1}$ has an equivariant extension $\mathrm{D}' : \tau \mid \overline{M}^n \simeq \varrho \mid \overline{M}^n$ if and only if $\xi_{\mathrm{D}}^n = 0$.*

Proof. *Necessity.* Suppose the existence of an equivariant extension D' of D. For each $c \in C_n(M)$, we have

$$\mathrm{E}(c) = \tau(c) - \varrho(c) - \mathrm{D}\partial c = \partial \mathrm{D}'c \in B_n(N).$$

Hence $\xi_{\mathrm{D}}^n = \eta \mathrm{E} = 0$. Note that the assumption that W operates freely on $(C_n(M), C_n(M_0))$ is not used in the proof of necessity.

Sufficiency. Assume $\xi_{\mathrm{D}}^n = 0$. Since W operates freely on $(C_n(M), C_n(M_0))$, there is a direct decomposition of the chain group $C_n(M)$ as follows:

$$C_n(M) = C_n(M_0) \oplus C_n^*(M).$$

W operates freely on both $C_n(M_0)$ and $C_n^*(M)$. Since $\xi_{\mathrm{D}}^n = \eta \mathrm{E} = 0$, the operator homomorphism $\mathrm{E} : C_n(M) \to C_n(N)$ maps $C_n(M)$ and hence $C_n^*(M)$ into $B_n(N)$. Since W operates freely on $C_n^*(M)$, the homomorphism $\mathrm{E} \mid C_n^*(M)$ can be lifted to an operator homomorphism $\mathrm{D}^* : C_n^*(M) \to C_{n+1}(N)$ such that $\partial \mathrm{D}^* c = \mathrm{E} c$ for each chain $c \in C_n^*(M)$, [2, p. 55]. For each chain $c \in C_n(M)$, there exists a unique pair of chains $c_0 \in C_n(M_0)$ and $c_* \in C_n^*(M)$ such that $c = c_0 + c_*$. We define an operator homomorphisn $\mathrm{D}' : C_n(M) \to C_{n+1}(N)$ by taking

$$\mathrm{D}'(c) = \mathrm{D}(c_0) + \mathrm{D}^*(c_*).$$

Immediately we have

$$\partial \mathrm{D}^*(c) = \mathrm{E}(c) = \tau(c) - \varrho(c) - \mathrm{D}(\partial c)$$

for each chain $c \in C_n(M)$. Hence we obtain an equivariant extension $\mathrm{D}' : \tau \mid \overline{M}^n \simeq \varrho \mid \overline{M}^n$ of the chain homotopy D by defining $\mathrm{D}' = \mathrm{D}$ for the dimension $q < n$. This completes the proof.

LEMMA 12.2. *Assume that* W *operates freely on* $(C_{n-1}(M), C_{n-1}(M_0))$. *Given an equivariant chain homotopy* $\mathrm{D} : \tau \mid \overline{M}^{n-1} \simeq \varrho \mid \overline{M}^{n-1}$ *and an equivariant cocycle* $\xi^n = \xi^n_{\mathrm{D}} + \partial h^{n-1}$ *with* $h^{n-1} \in C_e^{n-1}(M, M_0, H_n(N))$, *there exists an equivariant chain homotopy* $\mathrm{D}^* : \tau \mid \overline{M}^{n-1} \simeq \varrho \mid \overline{M}^{n-1}$ *with* $\xi^n_{\mathrm{D}^*} = \xi^n$ and $\mathrm{D}^* \mid M^{n-2} = \mathrm{D} \mid \overline{M}^{n-2}$.

Proof. Since W operates freely on $(C_{n-1}(M), C_{n-1}(M_0))$, there is a direct decomposition of the chain group as follows:

$$C_{n-1}(M) = C_{n-1}(M_0) \oplus C_{n-1}^*(M).$$

W operates freely on both $C_{n-1}(M_0)$ and $C_{n-1}^*(M)$. $h^{n-1} \in C_e^{n-1}(M, M_0, H_n(N))$ means that h^{n-1} is an operator homomorphism

$$h^{n-1} : C_{n-1}(M) \to H_n(N)$$

which maps $C_{n-1}(M_0)$ into zero. Since W operates freely on $C_{n-1}^*(M)$, the partial homomorphism $h^{n-1} \mid C_{n-1}^*(M)$ can be lifted to an operator homomorphism

$$k^{n-1} : C_{n-1}^*(M) \to Z_n(N)$$

such that $\eta k^{n-1} = h^{n-1} \mid C_{n-1}^*(M)$. Define an operator homomorphism

$$F : C_{n-1}(M) \to Z_n(N)$$

as follows: For each chain $c \in C_{n-1}(M)$, there exists a unique pair of chains $c_0 \in C_{n-1}(M_0)$ and $c_* \in C_{n-1}^*(M)$ such that $c = c_0 + c_*$. We define $F(c) = k^{n-1}(c_*)$. Clearly $\eta F = h^{n-1}$ and F maps $C_{n-1}(M_0)$ into zero. Define an equivariant chain homotopy $\mathrm{D}^* : \tau \mid \overline{M}^{n-1} \simeq \varrho \mid \overline{M}^{n-1}$ by taking $\mathrm{D}^* \mid \overline{M}^{n-2} = \mathrm{D} \mid \overline{M}^{n-2}$ and

$$\mathrm{D}^*(c) = \mathrm{D}(c) - F(c)$$

for every chain $c \in C_{n-1}(M)$. That D^* is an equivariant chain homotopy is easily verified. The obstruction of D^* is given as follows: For each $c \in C_n(M)$

$$\begin{aligned}\xi^n_{\mathrm{D}*}(c) &= \eta E^*(c) = \eta(\tau c - \varrho c - \mathrm{D}^*\partial c) = \eta(\tau c - \varrho c - \mathrm{D}\partial c + F\partial c)\\ &= \eta E(c) + \eta F(\partial c) = \eta E(c) + h^{n-1}(\partial c) = \xi^n_{\mathrm{D}}(c) + \delta h^{n-1}(c).\end{aligned}$$

Hence $\xi^n_{\mathrm{D}*} = \xi^n_{\mathrm{D}} + \delta h^{n-1}$. This completes the proof.

Corollary 12.3. *Assume that W operates freely on both $(C_{n-1}(M), C_{n-1}(M_0))$ and $(C_n(M), C_n(M_0))$. The obstruction ξ^n_{D} of an equivariant chain homotopy $\mathrm{D} : \tau \mid \overline{M}^{n-1} \simeq \varrho \mid \overline{M}^{n-1}$ is an equivariant coboundary module M_0 if and only if there exists an equivariant chain homotopy $\mathrm{D}' : \tau \mid \overline{M}^n \simeq \varrho \mid \overline{M}^n$ such that $\mathrm{D}' \mid \overline{M}^{n-2} = \mathrm{D} \mid \overline{M}^{n-2}$.*

Theorem 12.4. *Assume that $\mathrm{D} : \tau \mid \overline{M}^{n-1} \simeq \varrho \mid \overline{M}^{n-1}$ be an equivariant chain homotopy and that W operates freely on both $(C_{n-1}(M), C_{n-1}(M_0))$ and $(C_n(M), C_n(M_0))$. There exists an equivariant chain homotopy $\mathrm{D}' : \tau \mid \overline{M}^n \simeq \varrho \mid \overline{M}^n$ with $\mathrm{D}' \mid M_0 = D|M_0$, if and only if the obstruction ξ^n_{D} represents a regular cohomology class,* i.e. $\mu_n(\tau, \varrho, \mathrm{D}) \in \mathfrak{R}^n_e(M, M_0, H_n(N))$.

Proof. *Necessity.* Suppose the existence of the equivariant chain homotopy $\mathrm{D}' : \tau \mid \overline{M}^n \simeq \varrho \mid \overline{M}^n$ with $\mathrm{D}' \mid M_0 = \mathrm{D} \mid M_0$. Let $\mathrm{D}^* = \mathrm{D}' \mid \overline{M}^{n-1}$. Obviously

$$(\tau, \varrho, \mathrm{D}) - (\tau, \varrho, \mathrm{D}^*) \in \Delta^{n-1}_r.$$

Since μ_n is a homomorphism according to Lemma 9.2, we have

$$\mu_n(\tau, \varrho, \mathrm{D}) - \mu_n(\tau, \varrho, \mathrm{D}^*) \in \mathfrak{R}^n_e(M, M_0, H_n(N)).$$

Since D^* has an extension D' over $\overline{M}^n$, it follows from Lemma 12.1 that $\mu_n(\tau, \varrho, \mathrm{D}^*) = 0$. Hence $\mu_n(\tau, \varrho, \mathrm{D})$ is regular. Note that the assumption on free operators is not used in this part of the proof.

Sufficiency. Suppose that $\mu_n(\tau, \varrho, \mathrm{D})$ be regular. By the definition of regularity, there exists a regular triple $(0, 0, \mathrm{D}_r)$ such that $\mu_n(0, 0, \mathrm{D}_r) = \mu_n(\tau, \varrho, \mathrm{D})$. Call $\mathrm{D}^* = \mathrm{D} - \mathrm{D}_r$. Then we have $(\tau, \varrho, \mathrm{D}^*) = (\tau, \varrho, \mathrm{D}) - (0, 0, \mathrm{D}_r)$ and therefore

$$\mu_n(\tau, \varrho, \mathrm{D}^*) = \mu_n(\tau, \varrho, \mathrm{D}) - \mu_n(0, 0, \mathrm{D}_r) = 0.$$

This implies that the obstruction $\xi^n_{\mathrm{D}*}$ is an equivariant coboundary. By Corollary 12.3, there exists an equivariant chain homotopy $\mathrm{D}' : \tau \mid \overline{M}^n \simeq \varrho \mid \overline{M}^n$ such that $\mathrm{D}' \mid \overline{M}^{n-2} = \mathrm{D}^* \mid \overline{M}^{n-2}$. Since D_r is regular, we have

$$\mathrm{D}' \mid M_0 = \mathrm{D}^* \mid M_0 = \mathrm{D} \mid M_0 - \mathrm{D}_r \mid M_0 = \mathrm{D} \mid M_0.$$

This completes the proof.

Now let $D : \tau \mid M_0 \simeq \varrho \mid M_0$ be an equivariant chain homotopy, where $\tau, \varrho : M \to N$ are two given equivariant chain transformations. D is said to be *extensible* over M if there exists an equivariant extension $D' : \tau \simeq \varrho$. D is said to be *q-extensible* over M if it is extensible over $\overline{M}^q$. Assume that D be $(n-1)$-extensible over M and $D^* : \tau \mid \overline{M}^{n-1} \simeq \varrho \mid \overline{M}^{n-1}$ be an arbitrary equivariant extension of D over $\overline{M}^{n-1}$. We are going to prove the following

Lemma 12.5. *The cohomology class $\mu_n(\tau, \varrho, D^*)$ determines a unique element $\nu_n(\tau, \varrho, D)$ of the group $\mathfrak{J}^n_e(M, M_0, H_n(N))$ which depends only on (τ, ϱ, D).*

Proof. $\mu_n(\tau, \varrho, D^*)$ does determine an element $\nu_n(\tau, \varrho, D^*)$ of $\mathfrak{J}^n_e$ by means of the projection of $\mathfrak{P}^n_e$ into the quotient group $\mathfrak{J}^n_e = \mathfrak{P}^n_e/\mathfrak{R}^n_e$. It remains to prove that $\nu_n(\tau, \varrho, D^*)$ depends only on (τ, ϱ, D). Let $D' : \tau \mid \overline{M}^{n-1} \simeq \varrho \mid \overline{M}^{n-1}$ be another equivariant extension of D, then $(\tau, \varrho, D^*) - (\tau, \varrho, D') \in \Delta^{n-1}_r$. Hence we have

$$\mu_n(\tau, \varrho, D^*) - \mu_n(\tau, \varrho, D') \in \mathfrak{R}^n_e.$$

This implies $\nu_n(\tau, \varrho, D^*) = \nu_n(\tau, \varrho, D')$. Q. E. D.

Assume that there exist two integers a and b, $a < b$, such that $C_q(M) = C_q(M_0)$ unless $a < q \leq b$, and assume that W operates freely on $(C_q(M), C_q(M_0))$ for each q with $a < q \leq b$. The following theorem is an immediate consequence of Theorem 12.4.

Theorem 12.6. *Every equivariant chain homotopy $D : \tau \mid M_0 \simeq \varrho \mid M_0$ of any two equivariant chain transformations $\tau, \varrho : M \to N$ is extensible over M, if*

$$\mathfrak{J}^n_e(M, M_0, H_n(N)) = 0 \qquad (a < n \leq b).$$

The condition is also necessary as will be seen later.

13. Homotopy of chain transformations

As mentioned at the beginning of § 9, the set of all equivariant chain transformations of M into N form an abelian group Ω with functional addition as the group operation. The zero element 0 of the group Ω is the null chain transformation $0 : M \to N$ defined by $0(c) = 0$ for each $c \in C_q(M)$ with every dimension q.

For a given $\tau_0 \in \Omega$, let us denote by $\Omega(\tau_0)$ the subset of Ω which consists of the chain transformations $\tau \in \Omega$ such that $\tau \mid M_0 = \tau_0 \mid M_0$. Obviously $\Omega(0)$ is a subgroup of Ω and $\Omega(\tau_0)$ is the coset $\tau_0 + \Omega(0)$.

Two chain transformations $\tau, \varrho \in \Omega(\tau_0)$ are said to be *equi-*

variantly chain homotopic relative to M_0 if there exists an equivariant chain homotopy $\mathrm{D} : \tau \simeq \varrho$ such that $\mathrm{D} \mid M_0 = 0$, where $0 : \tau \mid M_0 \simeq \varrho \mid M_0$ denotes the trivial chain homotopy defined by $0(c) = 0$ for each $c \in C_q(M_0)$ with every dimension q. τ and ϱ are said to be *equivariantly chain q-homotopic relative to* M_0 if there is an equivariant chain homotopy $\mathrm{D} : \tau \mid \overline{M}^q \simeq \varrho \mid \overline{M}^q$ such that $\mathrm{D} \mid M_0 = 0$.

The following theorem is an immediate consequence of Lemma 12.5 and Theorem 12.4.

THEOREM 13.1. *Any two equivariant chain transformations* τ, ϱ *of* $\Omega(\tau_0)$ *which are equivariantly chain* $(n-1)$*-homotopic relative to* M_0 *determine a unique element* $v_n(\tau, \varrho) = v_n(\tau, \varrho, 0)$ *of* $\mathfrak{J}_e^n(M, M_0, H_n(N))$. $v_n(\tau, \varrho) = 0$ *if* τ *and* ϱ *are equivariantly chain n-homotopic relative to* M_0.

Conversely, we have the next theorem which follows from Theorem 12.4 and the first part of Theorem 13.1.

THEOREM 13.2. *If W operates freely on both* $(C_{n-1}(M), C_{n-1}(M_0))$ *and* $(C_n(M), C_n(M_0))$, *then* $v_n(\tau, \varrho) = 0$ *implies that* τ *and* ϱ *are equivariantly chain n-homotopic relative to* M_0.

Assume that there are two integers a and b, $a < b$, such that $C_q(M) = C_q(M_0)$ unless $a < q \leqq b$, and assume that W operates freely on $(C_q(M), C_q(M_0))$ for each q with $a < q \leqq b$. Then the following theorem is a particular case of Theorem 12.6.

THEOREM 13.3. *Any two equivariant chain transformations* τ, $\varrho \in \Omega(\tau_0)$ *are equivariantly chain homotopic relative to* M_0 *if*

$$\mathfrak{J}_e^n(M, M_0, H_n(N)) = 0 \qquad (a < n \leqq b).$$

The condition is also necessary as will be seen later.

14. Classification of chain transformation

The relation that two chain transformations $\tau, \varrho \in \Omega(\tau_0)$ are equivariantly chain homotopic relative to M_0 is obviously reflexive, symmetric and transitive; therefore, it is an equivalence relation. The chain transformations of $\Omega(\tau_0)$ are divided by this relation into disjoint equivalence classes, called the *equivariant homotopy classes relative to* M_0. The classification problem is to enumerate these classes by means of some convenient invariants.

LEMMA 14.1. *The correspondence* $\tau \to \tau + \tau_0$ *defines a one-to-one transformation of* $\Omega(0)$ *onto* $\Omega(\tau_0)$; *it maps equivariantly chain homotopic (q-homotopic) chain transformations relative to* M_0 *into such and conversely. Hence the same correspondence defines a one-*

to-one transformation of the equivariant homotopy classes of $\Omega(0)$ relative to M_0 onto those of $\Omega(\tau_0)$.

Proof. Since $\Omega(0)$ is a subgroup of Ω and $\Omega(\tau_0)$ is the coset $\tau_0 + \Omega(0)$, it follows immediately that the correspondence $\tau \to \tau + \tau_0$ is one-to-one and maps $\Omega(0)$ onto $\Omega(\tau_0)$. Now for any two chain transformations $\tau, \varrho \in \Omega(0)$, $D : \tau \simeq \varrho$ is equivalent with $D : \tau + \tau_0 \simeq \varrho + \tau_0$ ($D : \tau \mid \overline{M}^q \simeq \varrho \mid \overline{M}^q$ is equivalent with $D : \tau + \tau_0 \mid \overline{M}^q \simeq \varrho + \tau_0 \mid \overline{M}^q$). This completes the proof.

Lemma 14.2. *Those chain transformations of $\Omega(0)$ which are equivariantly chain homotopic with 0 relative to M_0 form a subgroup $\Omega_0(0)$ of $\Omega(0)$, and the equivariant chain homotopy classes of $\Omega(0)$ relative to M_0 constitute the quotient group $\mathfrak{G} = \Omega(0)/\Omega_0(0)$.*

Proof. For any $\tau, \varrho \in \Omega(0)$, $D' : \tau \simeq 0$ and $D'' : \varrho \simeq 0$ imply that $D' - D'' : \tau - \varrho \simeq 0$. This proves the first part of the lemma. For any $\tau, \varrho \in \Omega(0)$, $D : \tau \simeq \varrho$ is equivalent with $D : \tau - \varrho \simeq 0$. Hence the equivariant chain homotopy classes of $\Omega(0)$ relative to M_0 are the cosets of $\Omega_0(0)$ in $\Omega(0)$. Q.E.D.

Similarly, those chain transformations of $\Omega(0)$ which are equivariantly chain q-homotopic with 0 relative to M_0 form a subgroup $\Omega_0^q(0)$ of $\Omega(0)$. Obviously

$$\Omega_0(0) \subset \Omega_0^{q+1}(0) \subset \Omega_0^q(0) \subset \Omega(0).$$

Call $\mathfrak{G}^q = \Omega_0^q(0)/\Omega_0(0)$. We obtain a sequence of groups

$$\mathfrak{G} \supset \ldots \supset \mathfrak{G}^q \supset \mathfrak{G}^{q+1} \supset \ldots, \tag{14.1}$$

and obviously we have

$$\mathfrak{G}^q/\mathfrak{G}^{q+1} \approx \Omega_0^q(0)/\Omega_0^{q+1}(0). \tag{14.2}$$

Theorem 14.3. *Every chain transformation $\tau \in \Omega_0^{n-1}(0)$ determines a unique element $\nu_n(\tau) = \nu_n(\tau, 0)$ of the group $\mathfrak{J}_e^n(M, M_0, H_n(N))$. $\nu_n(\tau) = 0$ if $\tau \in \Omega_0^n(0)$. The correspondence $\tau \to \nu_n(\tau)$ is a homomorphism*

$$\nu_n\colon \quad \Omega_0^{n-1}(0) \to \mathfrak{J}_e^n(M, M_0, H_n(N)). \tag{14.3}$$

Proof. The first two assertions are merely specializations of those in Theorem 13.1. It remains to prove that ν_n is a homomorphism. Let $\tau, \varrho \in \Omega_0^{n-1}(0)$. Then there are equivariant chain homotopies

$$D' : \tau \mid \overline{M}^{n-1} \simeq 0 \mid \overline{M}^{n-1}, \quad D'' : \varrho \mid \overline{M}^{n-1} \simeq 0 \mid \overline{M}^{n-1}$$

such that $D' \mid M_0 = 0 = D'' \mid M_0$. Obviously we have

$$\mu_n(\tau, 0, D') + \mu_n(\varrho, 0, D'') = \mu_n(\tau + \varrho, 0, D' + D'').$$

Projecting from $\mathfrak{P}_e^n$ to the quotient group $\mathfrak{J}_e^n$, we obtain $\nu_n(\tau) + \nu_n(\varrho) = \nu_n(\tau + \varrho)$. This completes the proof.

Throughout the remainder of this section, we assume that W operates freely on $(C_q(M), C_q(M_0))$ whenever $C_q(M) \neq C_q(M_0)$.

LEMMA 14.4. Homotopy Extension Lemma. *For any two equivariant chain transformations* $\tau, \varrho : M \to N$ *and any equivariant chain homotopy* $\mathrm{D} : \tau \mid M_0 \simeq \varrho \mid M_0$, *there exist an equivariant extension* $\tau^* : M \to N$ *of the partial chain transformation* $\tau \mid M_0$ *and an equivariant extension* $\mathrm{D}^* : \tau^* \simeq \varrho$ *of* D.

Proof. Since W operates freely on $(C_q(M), C_q(M_0))$, there is a direct decomposition of $C_q(M)$ as follows:

$$C_q(M) = C_q(M_0) \oplus C_q^*(M), \qquad (q > d).$$

W operates freely on both $C_q(M_0)$ and $C_q^*(M)$. For every $c \in C_q(M)$ with each dimension q, there is a unique pair of chain $c_0 \in C_q(M_0)$ and $c_* \in C_q^*(M)$ such that $c = c_0 + c_*$. We define a collection D^* of operator homomorphisms

$$\mathrm{D}^* : C_q(M) \to C_{q+1}(N)$$

by taking $\mathrm{D}^*(c) = \mathrm{D}(c_0)$ and a collection τ^* of operator homomorphisms

$$\tau^* : C_q(M) \to C_q(N)$$

by taking for each $c \in C_q(M)$

$$\tau^*(c) = \varrho(c) + \mathrm{D}^*(\partial c) + \partial \mathrm{D}^*(c). \tag{14.4}$$

To verify that τ^* is a chain transformation, it is sufficient to note the following relation

$$\tau^* \partial c = \varrho \partial c + \partial \mathrm{D}^* \partial c = \partial \varrho c + \partial \mathrm{D}^* \partial c = \partial \tau^* c$$

for each $c \in C_q(M)$. When $c \in C_q(M_0)$, $\mathrm{D}^*(c) = \mathrm{D}(c)$ by definition. Then (14.4) becomes

$$\tau^*(c) = \varrho(c) + \mathrm{D}\partial(c) + \partial \mathrm{D}(c) = \tau(c)$$

since $\mathrm{D} : \tau \mid M_0 \simeq \varrho \mid M_0$. It follows from (14.4) that $\mathrm{D}^* : \tau^* \simeq \varrho$. This completes the proof.

LEMMA 14.5. Existence Lemma. *For an arbitrary element* $\zeta \in \mathfrak{J}_e^n(M, M_0, H_n(N))$, *there exists a chain transformation* $\tau \in \Omega_0^{n-1}(0)$ *such that* $\tau \mid \overline{M}^{n-1} = 0$ *and* $\nu_n(\tau) = \zeta$.

Proof. Let $p : \mathfrak{P}_e^n \to \mathfrak{J}_e^n$ denote the projection of $\mathfrak{P}_e^n$ onto the factor group $\mathfrak{J}_e^n$. Let $\zeta \in \mathfrak{J}_e^n$ be an arbitrary element. Choose an element $\xi_0 \in \mathfrak{P}_e^n$ such that $p(\xi_0) = \zeta$. Since $\mathfrak{P}_e^n = \mu_n(\Delta^{n-1})$, there exists a triple $(\tau_0, \varrho, \mathrm{D}) \in \Delta^{n-1}$ such that $\mu_n(\tau_0, \varrho, \mathrm{D}) = \xi_0$.

Applying the homotopy extension lemma to the subcomplex $\overline{M}^{n-1}$, there exist an equivariant extension $\tau_1 : M \to N$ of the partial chain transformation $\tau_0 \mid \overline{M}^{n-1}$ and an equivariant extension $D^* : \tau_1 \simeq \varrho$ of D. Call $\xi_1 = \mu_n(\tau_1, \varrho, D)$. Since D has an extension $D^* : \tau_1 \simeq \varrho$, $\mu_n(\tau_1, \varrho, D) \in \mathfrak{R}_e^n$ by Theorem 12.4. Let $\tau = \tau_0 - \tau_1$, then $\tau \mid \overline{M}^{n-1} = 0$ and

$$\nu_n(\tau) = p\mu_n(\tau_0, \varrho, D) - p\mu_n(\tau_1, \varrho, D) = p(\xi_0) - p(\xi_1) = \zeta.$$

This completes the proof.

THEOREM 14.6. *The homomorphism ν_n in* (14.3) *maps $\Omega_0^{n-1}(0)$ onto $\mathfrak{J}_e^n(M, M_0, H_n(N))$ and its kernel is $\Omega_0^n(0)$. Hence we have the following isomorphisms*:

$$\mathfrak{G}^{n-1}/\mathfrak{G}^n \approx \Omega_0^{n-1}(0)/\Omega_0^n(0) \approx \mathfrak{J}_e^n(M, M_0, H_n(N)).$$

Proof. That ν_n is onto follows immediately from the existence lemma. Since $\nu_n(\tau) = \nu_n(\tau, 0)$, it follows from Theorem 13.2 that the kernel of ν_n is $\Omega_0^n(0)$. Q. E. D.

For the following main classification theorem, we shall assume one more condition, namely that there exist two integers a and b, $a < b$, such that $C_q(M) = C_q(M_0)$ unless $a < q \leqq b$. Then $\Omega_0^a(0) = \Omega(0)$ and $\Omega_0^b(0) = \Omega_0(0)$. By Lemma 14.1, Theorem 14.6, and a purely group-theoretic argument, we obtain the

THEOREM 14.7. Classification Theorem. *The equivariant homotopy classes of the equivariant chain transformations $\Omega(\tau_0)$ relative to M_0 are in a one-to-one correspondence with the elements of the group* (*direct sum*):

$$\mathfrak{J}_e^{a+1}(M, M_0, H_{a+1}(N)) \oplus \mathfrak{J}_e^{a+2}(M, M_0, H_{a+2}(N)) \oplus \dots \oplus \mathfrak{J}_e^b(M, M_0, H_b(N)).$$

As an immediate consequence, we deduce that the conditions in Theorems 12.6 and 13.3 are also necessary.

If $M_0 = 0$, then Theorem 14.7 becomes the following

COROLLARY 14.8. *The equivariant homotopy classes of the equivariant chain transformations Ω are in a one-to-one correspondence with the elements of the group*:

$$\mathfrak{J}_e^{a+1}(M, H_{a+1}(N)) \oplus \mathfrak{J}_e^{a+2}(M, H_{a+2}(N)) \oplus \dots \oplus \mathfrak{J}_e^b(M, H_b(N)).$$

15. A few special cases

Throughout the present section, we assume that there exist two integers a and b, $a < b$, such that $C_q(M) = C_q(M_0)$ unless

$a < q \leqq b$. We also assume that W operates freely on $(C_q(M), C_q(M_0))$ for each dimension q with $a < q \leqq b$.

LEMMA 15.1. *If* $\mathfrak{J}_e^q(\overline{M}^n, M_0, H_q(N)) = 0$ *for each dimension* q *with* $a < q < n$, *then* $R_e^{n+1}(M, M_0, H_n(N)) = 0$ *and hence* $J_e^{n+1}(M, M_0, H_n(N))$ *is isomorphic with* $P_e^{n+1}(M, M_0, H_n(N))$.

Proof. Let $\xi \in R_e^{n+1}$ be arbitrarily given; then there is a regular presenter $\tau : \overline{M}^n \to N$, $\tau \mid M_0 = 0 \mid M_0$ of ξ, i.e. $\omega_n(\tau) = \xi$. By the successive application of Theorem 14.6, we deduce from our hypothesis of the lemma that there exists an equivariant chain homotopy

$$\mathrm{D} : \tau \mid \overline{M}^{n-1} \simeq 0 \mid \overline{M}^{n-1}.$$

Let $\varrho = 0 \mid \overline{M}^n$. By Lemma 8.3, $l_\varrho^{q+1} = 0$. By Lemma 8.4, $\delta\xi_{\mathrm{D}}^n = l_\tau^{n+1} - l_\varrho^{n+1} = l_\tau^{n+1}$. Hence $\xi = \omega_n(\tau) = 0$. This completes the proof.

Define a Mayer chain complex N_* with operators W by taking $C_q(N_*) = C_{q+1}(N)$ for each dimension q. Then we have $H_q(N_*) = H_{q+1}(N)$ for every q. We shall use the following notations:

$$P_e^q(M, M_0, H_{q+1}(N)) = \mathfrak{P}_e^q(M, M_0, H_q(N_*)),$$
$$R_e^q(M, M_0, H_{q+1}(N)) = \mathfrak{R}_e^q(M, M_0, H_q(N_*)),$$
$$J_e^q(M, M_0, H_{q+1}(N)) = P_e^q(M, M_0, H_{q+1}(N))/R_e^q(M, M_0, H_{q+1}(N)).$$

LEMMA 15.2. *If* $J_e^q(\overline{M}^{n-1}, M_0, H_{q+1}(N)) = 0$ *for each dimension* q *with* $a < q < n-1$, *then* $\mathfrak{R}_e^n(M, M_0, H_n(N)) = 0$ *and hence* $\mathfrak{J}_e^n(M, M_0, H_n(N))$ *is isomorphic with* $\mathfrak{P}_e^n(M, M_0, H_n(N))$.

Proof. Let $\xi \in \mathfrak{R}_e^n$ be arbitrarily given; then there is a regular triple $(0, 0, \mathrm{D}) \in \Delta_\tau^{n-1}$ such that $\mu_n(0, 0, \mathrm{D}) = \xi$. Since $\mathrm{D} : 0 \mid \overline{M}^{n-1} \simeq 0 \mid \overline{M}^{n-1}$, we deduce $\partial \mathrm{D} c = -\mathrm{D}\partial c$ for every $c \in C_q(\overline{M}^{n-1})$ with every dimension q. Define an equivariant chain transformation $\tau : \overline{M}^{n-1} \to N_*$ by taking $\tau c = (-1)^q \mathrm{D} c$ for each $c \in C_q(\overline{M}^{n-1})$ with every dimension q. The definition of τ is justified by the following relation:

$$\partial\tau c = (-1)^q \partial \mathrm{D} c = (-1)^{q-1}\mathrm{D}\partial c = \tau\partial c, \quad (c \in C_q(\overline{M}^{n-1})).$$

It follows from (8.3)—(8.5) and the relations given above that

$$\xi_{\mathrm{D}}^n = \eta \mathrm{E} = -\eta \mathrm{D}\partial = (-1)^n \eta\tau\partial = (-1)^n l_\tau^n.$$

Since $\mathrm{D} \mid M_0 = 0 = \tau \mid M_0$, l_τ^n represents a regular element of $H^n(M, M_0, H_{n-1}(N_*))$. According to Lemma 15.1 and the hypothesis of the present lemma, l_τ^n is an equivariant coboundary of M modulo M_0 and hence so is ξ_{D}^n. This implies $\mu_n(0, 0, \mathrm{D}) = 0$. Q.E.D.

Lemma 15.3. *If $R^{q+1}(M, M_0, H_q(N)) = 0$ for each dimension q with $n < q < b$, then*

$$\mathfrak{P}^n_e(M, M_0, H_n(N)) = H^n_e(M, M_0, H_n(N)).$$

Proof. Let $f \in Z^n_e(M, M_0, H_n(N))$ be arbitrarily given. f is an operator homomorphism:

$$f : C_n(M) \to H_n(N); \quad f(c) = 0 \quad (c \in C_n(M_0) + B_n(M)).$$

Since W operates freely on $(C_n(M), C_n(M_0))$, $C_n(M)$ has a direct decomposition

$$C_n(M) = C_n(M_0) \oplus C^*_n(M).$$

W operates freely on both $C_n(M_0)$ and $C^*_n(M)$. The partial homomorphism $f \mid C^*_n(M)$ can be lifted to an operator homomorphism

$$\varphi^* : C^*_n(M) \to Z_n(N)$$

such $\eta\varphi^* = f \mid C^*_n(N)$. Define an operator homomorphism

$$\varphi : C_n(M) \to Z_n(N)$$

as follows: For each chain $c \in C_n(M)$, there exists a unique pair of chains $c_0 \in C_n(M_0)$ and $c_* \in C^*_n(M)$ such that $c = c_0 + c_*$. We define $\varphi(c) = \varphi^*(c_*)$. Obviously $\eta\varphi = f$ and φ maps $C_n(M_0)$ into zero. Define an equivariant chain transformation $\tau : \overline{M}^n \to N$ by taking $\tau \mid \overline{M}^{n-1} = 0$ and $\tau = \varphi$ on $C_n(M)$. This definition is justified by $\varphi(c) = 0$ when $c \in C_n(M_0)$ and $\partial\varphi c = 0$ for every $c \in C_n(M)$. The obstruction of τ is given by

$$l^{n+1}_\tau = \eta\tau\partial = \eta\varphi\partial = f\partial = 0;$$

hence, by Lemma 11.1, τ has an equivariant extension $\tau_1 : \overline{M}^{n+1} \to N$. Since $\tau_1 \mid M_0 = \tau \mid M_0 = 0$, τ_1 is a regular presenter of the element $\omega_{n+1}(\tau_1) \in R^{n+2}_e = 0$. This implies that the obstruction $l^{n+2}_{\tau_1}$ is an equivariant coboundary of M modulo M_0. According to corollary 11.3, there exists an equivariant chain transformation $\tau_2 : \overline{M}^{n+2} \to N$ such that $\tau_2 \mid \overline{M}^n = \tau$. By the successive application of the same argument, one can prove the existence of an equivariant chain transformation $\tau^* : M \to N$ with $\tau^* \mid \overline{M}^n = \tau$. Since $\tau^* \mid \overline{M}^{n-1} = 0 \mid \overline{M}^{n-1}$, $(\tau^*, 0, 0)$ is a triple of Δ^{n-1}. The element $\mu_n(\tau^*, 0, 0)$ of $\mathfrak{P}^n_e$ is represented by the cocycle $\eta\tau^* = \eta\tau = f$. This completes the proof.

Theorem 15.4. *If $H_q(N) = 0$ for each dimension q with $a \leqq q < n$, then every equivariant chain transformation $\tau : M_0 \to N$ is n-extensible over M and determines a unique cohomology class*

$\beta_n(\tau)$ of $H_e^{n+1}(M, M_0, H_n(N))$ called the characteristic class of τ. $\beta_n(\tau) = 0$ if and only if τ is $(n+1)$-extensible over M.

Proof. The hypothesis of the theorem implies the following weaker conditions:

$$(15.1) \qquad \mathfrak{J}_e^q(\overline{M}^n, M_0, H_q(N)) = 0, \qquad (a < q < n);$$

$$(15.2) \qquad J_e^{q+1}(M, M_0, H_q(N)) = 0, \qquad (a \leqq q < n).$$

We shall prove the theorem by means of (15.1) and (15.2) only.

By means of the successive application of Theorem 11.7, (15.2) implies that every equivariant chain transformation $\tau : M_0 \to N$ is n-extensible over M. Let $\tau' : \overline{M}^n \to N$ be an arbitrary equivariant extension of τ. According to Lemma 11.6, the obstruction $l_{\tau'}^{n+1}$ determines an element $\beta_n(\tau)$ of $J_e^{n+1}(M, M_0, H_n(N))$. In accordance with Lemma 15.1, the condition (15.1) implies that $R_e^{n+1}(M, M_0, H_n(N)) = 0$. Hence $\beta_n(\tau)$ is a single presentable cohomology class of $H_e^{n+1}(M, M_0, H_n(N))$. This proves the first assertion of the theorem. The last assertion of the theorem is an immediate consequence of Theorem 11.7. Q.E.D.

Taking $b = n + 1$, we obtain the following corollary which is an analogue of the Hopf extension theorem for continuous maps.

COROLLARY 15.5. *If $b = n + 1$ and $H_q(N) = 0$ for each dimension q with $a \leqq q < n$, then every equivariant chain transformation $\tau : M_0 \to N$ is n-extensible over M and determines a characteristic cohomology class $\beta_n(\tau)$. $\beta_n(\tau) = 0$ if and only if τ is extensible over M.*

The following theorem is an analogue of the Steenrod classification theorem for continuous maps, [6].

THEOREM 15.6. *If $b \leqq n + 1$ and $H_q(N) = 0$ for each dimension q with $a < q < n$, then the equivariant homotopy classes of the equivariant chain transformations $\Omega(\tau_0)$ relative to M_0 are in a one-to-one correspondence with the elements of the group (direct sum):*

$$H_e^n(M, M_0, H_n(N)) \oplus \mathfrak{J}_e^{n+1}(M, M_0, H_{n+1}(N)),$$

where

$$\mathfrak{J}_e^{n+1}(M, M_0, H_{n+1}(N)) = H_e^{n+1}(M, M_0, H_{n+1}(N)) / \mathfrak{R}_e^{n+1}(M, M_0, H_{n+1}(N)).$$

Proof. The hypothesis of the theorem implies that the following weaker conditions hold:

$$(15.3) \quad J_e^q(\overline{M}^{n-1}, M_0, H_{q+1}(N)) = 0, \qquad (a < q < n-1);$$

$$(15.4) \quad \mathfrak{J}_e^q(M, M_0, H_q(N)) = 0, \qquad (a < q < n);$$

$$(15.5) \quad \mathfrak{J}_e^q(M, M_0, H_q(N)) = 0, \qquad (q > n + 1);$$

$$(15.6) \quad R^{q+1}(M, M_0, H_q(N)) = 0, \qquad (n < q < b).$$

We shall prove the theorem by means of these weaker conditions (15.3)—(15.6).

In accordance with Lemma 15.2, the condition (15.3) implies that $\mathfrak{R}_e^n = 0$. According to Lemma 15.3, the condition (15.6) implies that $\mathfrak{P}_e^n = H_e^n$ and $\mathfrak{P}_e^{n+1} = H_e^{n+1}$. These and the conditions (15.4)—(15.5) give our theorem as a special case of Theorem 14.7. Q.E.D.

Taking $b = n$ and $M_0 = 0$, we obtain the following corollary which is an analogue of the Hopf classification theorem of continuous maps, [3; 7].

COROLLARY 15.7. *If $b = n$ and $H_q(N) = 0$ for each dimension q with $a < q < n$, then every equivariant chain transformation $\tau : M \to N$ is equivariantly chain $(n-1)$-homotopic with 0 and determines a unique equivariant cohomology class $v_n(\tau) \in H_e^n(M, H_n(N))$, called the characteristic cohomology class of τ, which depends only on the equivariant homotopy class $[\tau]$ of τ. The association $[\tau] \to v_n(\tau)$ gives a one-to-one correspondence between the equivariant homotopy classes of the chain transformations Ω and the elements of the equivariant cohomology group $H_e^n(M, H_n(N))$.*

For the last theorem of this section, we shall take $N = M$ and $M_0 = 0$.

THEOREM 15.8. *For every integer $n > a$, the following conditions are equivalent:*

(i)	$H_q(M) = 0,$	$(a < q < n);$
(ii)	$H_e^q(M, H_q(M)) = 0,$	$(a < q < n);$
(iii)	$\mathfrak{J}_e^q(M, H_q(M)) = 0,$	$(a < q < n).$

Proof. That (i) → (ii) and (ii) → (iii) is obvious. It follows from (iii) and Theorem 14.6 that the identity chain transformation $\iota : M \to M$ is equivariantly chain $(n-1)$-homotopic with the null chain transformation 0. By a standard argument, we deduce that

$$H_q(M) = H_q(0) = 0, \qquad (a < q < n).$$

This completes the proof.

Note that the integer a in Theorem 15.8 has the meaning that $C_q(M) = 0$ whenever $q \leqq a$. If W operates *simply* on M, i.e. $wc = c$ for all $w \in W$ and all $c \in C_q(M)$ with every dimension q, then the condition that W operates freely on $(C_q(M), C_q(M_0))$ requires only that $C_q(M)$ be a free group with $C_q(M_0)$ as a direct summand. In this case, $H_e(M, H_q(M))$ reduces to the ordinary cohomology group $H_e^q(M, H_q(M))$ and we shall denote $\mathfrak{J}_e^q(M,$

$H_q(M))$ simply by $\mathfrak{I}^q(M, H_q(M))$. We have the following corollary of Theorem 15.8.

COROLLARY 15.9. *For every integer $n > a$, the following conditions are equivalent*:

(i) $H_q(M) = 0, \qquad (a < q < n)$;

(ii) $H^q(M, H_q(M)) = 0, \qquad (a < q < n)$;

(iii) $\mathfrak{I}^q(M, H_q(M)) = 0, \qquad (a < q < n)$.

16. Induced homomorphisms

Throughout the present section, let L, M, N be three Mayer chain complexes with the same group W of operators. Let $L_0 \subset L$ and $M_0 \subset M$ be segregated subcomplexes. Let

$$\tau : (L, L_0) \to (M, M_0) \tag{16.1}$$

be a given equivariant chain transformation (for definition, see § 6).

In accordance with § 6, τ induces for each q a homomorphism

$$\tau^* : C^q(M, G) \to C^q(L, G) \tag{16.2}$$

defined by $(\tau^* f)c = f(\tau c)$ for each $f \in C^q(M, G)$ and each $c \in C_q(L)$, where G is an arbitrary abelian additive group with W as a group of left operators. τ^* maps $C^q(M, M_0, G)$ into $C^q(L, L_0, G)$, and $C^q_e(M, G)$ into $C^q_e(L, G)$. Further, $\tau^*\delta = \delta\tau^*$. Hence τ^* induces for each q a homomorphism

$$\tau^q_e : H^q_e(M, M_0, G) \to H^q_e(L, L_0, G). \tag{16.3}$$

Now let $\theta : \overline{M}^n \to N$ be any equivariant chain transformation. Call $\theta\tau = \theta\tau \mid L^n$. θ determines an obstructions $l^{n+1}_e \in Z^{n+1}_e(M, M_0, H_n(N))$ by Lemma 8.1, and $\theta\tau$ determines an obstruction $l^{n+1}_{\theta\tau} \in Z^{n+1}_e(L, L_0, H_n(N))$.

LEMMA 16.1. *The homomorphism τ^* maps l^{n+1}_θ into $l^{n+1}_{\theta\tau}$*, i.e. $l^{n+1}_{\theta\tau} = \tau^* l^{n+1}_\theta$.

Proof. By the definition of $l^{n+1}_{\theta\tau}$, we have

$$l^{n+1}_{\theta\tau}(c) = \eta\theta\tau\partial c = \eta\theta\partial(\tau c) = l^{n+1}_\theta(\tau c) = \tau^* l^{n+1}_\theta(c)$$

for every $c \in C_{n+1}(L)$. This proves the lemma.

THEOREM 16.2. *For each integer n, the homomorphism*

$$\tau^{n+1}_e : H^{n+1}_e(M, M_0, H_n(N)) \to H^{n+1}_e(L, L_0, H_n(N))$$

maps the presentable subgroup and the regular subgroup into such

Hence τ_e^{n+1} *induces a homomorphism*

$$\tau^{n+1} : J_e^{n+1}(M, M_0, H_n(N)) \to J_e^{n+1}(L, L_0, H_n(N)).$$

Proof. That τ_e^{n+1} maps $P_e^{n+1}(M, M_0, H_n(N))$ into $P_e^{n+1}(L, L_0, H_n(N))$ is a direct consequence of Lemma 16.1. If $\theta : \overline{M}^n \to N$ is regular, i.e. $\theta \mid M_0 = 0$, so is $\theta\tau$. Hence τ_e^{n+1} maps $R_e^{n+1}(M, M_0, H_n(N))$ into $R_e^{n+1}(L, L_0, H_n(N))$. This completes the proof.

Now let $\theta, \varrho : M \to N$ be equivariant chain transformations and $\mathrm{D} : \theta \mid \overline{M}^{n-1} \simeq \varrho \mid \overline{M}^{n-1}$ be an equivariant chain homotopy defined on $\overline{M}^{n-1}$. The triple $(\theta, \varrho, \mathrm{D})$ determines an obstruction $\xi_{\mathrm{D}}^n \in Z_e^n(M, M_0, H_n(N))$. Call $\mathrm{D}\tau = \mathrm{D}\tau \mid \overline{L}^{n-1}$. Then $\mathrm{D}\tau : \theta\tau \mid \overline{L}^{n-1} \simeq \varrho\tau \mid \overline{L}^{n-1}$ and the triple $(\theta\tau, \varrho\tau, \mathrm{D}\tau)$ determines an obstruction $\xi_{\mathrm{D}\tau}^n \in Z_e^n(L, L_0, H_n(N))$.

Lemma 16.3. *The homomorphism* τ^* *maps* ξ_{D}^n *into* $\xi_{\mathrm{D}\tau}^n$, i.e. $\xi_{\mathrm{D}\tau}^n = \tau^*\xi_{\mathrm{D}}^n$.

Proof. By definition of $\xi_{\mathrm{D}\tau}^n$, we have

$$\begin{aligned} \xi_{\mathrm{D}\tau}^n(c) &= \eta\theta\tau(c) - \eta\varrho\tau(c) - \eta\mathrm{D}\tau\partial(c) \\ &= \eta\theta(\tau c) - \eta\varrho(\tau c) - \eta\mathrm{D}\partial(\tau c) \\ &= \xi_{\mathrm{D}}^n(\tau c) = \tau^*\xi_{\mathrm{D}}^n(c) \end{aligned}$$

for every $c \in C_n(L)$. This proves the Lemma.

Analogous to Theorem 16.2, one can prove the following

Theorem 16.4. *For each integer* n, *the homomorphism*

$$\tau_e^n : H_e^n(M, M_0, H_n(N)) \to H_e^n(L, L_0, H_n(N))$$

maps the presentable subgroup and the regular subgroup into such. Hence τ_e^n *induces a homomorphism*

$$\tau_*^n : \mathfrak{J}_e^n(M, M_0, H_n(N)) \to \mathfrak{J}_e^n(L, L_0, H_n(N)).$$

17. Augmentable abstract complexes

Following Eilenberg [1, p. 379], let K be a collection of abstract elements σ_q called *cells*. With each cell there is associated an integer $q \geqq 0$ called the dimension of σ_q. To any two cells σ_{q+1}, $\sigma_q \in K$ there corresponds an integer $[\sigma_{q+1} : \sigma_q]$ called the *incidence number*. K will be called a (closure finite) *abtsract complex* provided the incidence numbers satisfy the following conditions:

(17.1) Given σ_{q+1}, $[\sigma_{q+1} : \sigma_q] \neq 0$ only for a finite number of q-cells σ_q;

(17.2) Given σ_{q+1} and σ_{q-1}, $\sum_{\sigma_q} [\sigma_{q+1} : \sigma_q][\sigma_q : \sigma_{q-1}] = 0$.

The q-cells σ_q are taken as free generators of an abelian group $C_q(K)$, the group of (integral) q-chains of K. The *boundary operator* ∂ is a homomorphism.

$$\partial : C_q(K) \to C_{q-1}(K), \qquad (q > 0), \tag{17.3}$$

defined for each generator σ_q as

$$\partial\sigma_q = \sum_{\sigma_{q-1}} [\sigma_q : \sigma_{q-1}]\sigma_{q-1}. \tag{17.4}$$

Condition (17.1) justifies the definition (17.4) and condition (17.2) implies that $\partial\partial = 0$.

For each integral 0-chain $c = \sum a_i\sigma_e^i \in C_0(K)$, define $I(c) = \sum a_i$. The abstract complex is said to be *augmentable* if $I(\partial c) = 0$ for every $c \in C_1(K)$. All abstract complexes considered hereafter in this section are assumed to be augmentable.

For a given augmentable abstract complex K, let us define a Mayer chain complex $M = M_K$ as follows: Take $C_q(M) = C_q(K)$ for each $q \geqq 0$; define $C_{-1}(M)$ to be the additive group of integers; and let $C_q(M) = 0$ for every $q < -1$. For the boundary homomorphisms

$$\partial : C_q(M) \to C_{q-1}(M),$$

we extend those of K by defining $\partial c = I(c) \in C_{-1}(M)$ for each $c \in C_0(M)$ and $\partial c = 0$ for every $c \in C_q(M)$ with dimension $q < 0$.

We shall say that W *operates* on the augmentable abstract complex K (on the left), provided that W operates on the associated Mayer chain complex M_K in such a way that $wc = c$ for every $c \in C_{-1}(M_K)$ and every $w \in W$.

Assume that K be an augmentable abstract complex with operators W and $M = M_K$. We shall use the following notation:

$$H_q(K) = H_q(M) \tag{17.5}$$

for each dimension q, where $H_q(M)$ is defined in § 3. It is easy to see that the group $H_0(K)$, defined by (17.5), is the usual *reduced* 0-th homology group of K.

Let M_0 denote the (-1)-dimensional skeleton M^{-1} of $M = M_K$. For an arbitrary abelian group G with left operators W, we denote

$$H_e^q(K, G) = H_e^q(M, M_0, G)$$

for each dimension q, where $H_e^q(M, M_0, G)$ is defined in § 5. It is easy to see that, for each $q \geqq 0$, the group $H_e^q(K, G)$, defined by (17.6), is exactly the same as defined by Eilenberg, [1, p. 383].

Now let K' be another augmentable abstract complex with operators W and $M' = M_{K'}$. We shall use the following notations:

$$J_e^{n+1}(K, H_n(K')) = J_e^{n+1}(M, M_0, H_n(M')),$$
$$\mathfrak{J}_e^n(K, H_n(K')) = \mathfrak{J}_e^n(M, M_0, H_n(M')).$$

and similar notations for the presentable and the regular subgroups of § 9.

Since both $C_{-1}(M)$ and $C_{-1}(M')$ are the group of integers, we can identify M_0 with $M'_0 \subset M'$. This gives an identity chain transformation

$$\iota : M_0 \to M'$$

which is obviously equivariant. K is said to have *index* $\geqq n$ with respect to K' if ι has an equivariant extension $\iota^* : \overline{M}^n \to M'$.

In the remainder of the present section, we shall assume that W operates freely on K, i.e. W operates freely on $C_q(K)$ for each $q \geqq 0$.

THEOREM 17.1. *The following statements are equivalent:—*

(i) *K has index $\geqq n$ with respect to K'.*

(ii) *Every chain transformation $\tau : M_0 \to M'$ has an equivariant extension $\tau^* : \overline{M}^n \to M'$.*

(iii) $J_e^{q+1}(K, H_q(K')) = 0$ *for every $q < n$.*

Proof. (i) → (ii). Since both $C_{-1}(M)$ and $C_{-1}(M')$ are the group of integers, $\tau : M_0 \to M'$ is determined by the integer $a = \tau(1)$, namely, $\tau = a\iota$. Hence τ has an equivariant extension $\tau^* = a\iota^* : \overline{M}^n \to M'$. That (ii) and (iii) are equivalent follows from successive application of Theorem 11.7. Q.E.D.

The following theorem is an immediate application of § 11.

THEOREM 17.2. *If K has index $\geqq n$ with respect to K', then the obstruction $l_{\iota^*}^{n+1}$ determines an element* (not depending on the choice of ι^*) *$\chi^{n+1}(K, K')$ of $J_e^{n+1}(K, H_n(K'))$. K has index $\geqq n+1$ with respect to K' if and only if $\chi^{n+1}(K, K') = 0$.*

THEOREM 17.3. *If K has index $\geqq n$ with respect to K', then the group $J_e^{n+1}(K, H_n(K'))$ is cyclic and is generated by $\chi^{n+1}(K, K')$.*

Proof. By Theorem 17.1, every chain transformation $\tau : M_0 \to M'$ is n-extensible over M. Denoting by Ω_0 the group of all chain transformations $\tau : M_0 \to M'$. Then Ω_0 is a free cyclic group with $i : M_0 \to M'$ as a generator. According to Theorem 11.7

$$J_e^{n+1}(K, H_n(K')) = J_e^{n+1}(M, M_0, H_n(M')) = \beta_n(\Omega_0),$$

and $\chi^{n+1}(K, K') = \beta_n(\iota)$ by definition. This proves the theorem.

Now let K_W denote the standard associated complex with operators W, [1, p. 394]. Then K_W is augmentable; W operates freely on K_W; and $H_q(K_W) = 0$ for every dimension q in the sense of (17.5), [2, p. 60].

An augmentable abstract complex K is said to have *rank* $\geqq n$ if K_W has index $\geqq n$ with respect to K. We shall use the notation

$$J^{n+1}(W, H_n(K)) = J_e^{n+1}(K_W, H_n(K)).$$

The following corollary is an immediate result of the Theorem 17.1—17.3.

COROLLARY 17.4. *K has rank $\geqq n$ if and only if $J^{q+1}(W, H_q(K)) = 0$ for each $q < n$. Then K determines an element $\chi^{n+1}(K) = \chi^{n+1}(K_W, K)$ of $J^{n+1}(W, H_n(K))$ called the $(n+1)$-dimensional characteristic element K, and the group $J^{n+1}(W, H_n(K))$ is cyclic with $\chi^{n+1}(K)$ as generator.*

As before, let $M = M_K$ and denote by L the corresponding Mayer chain complex of the abstract complex K_W. Let $L_0 = L^{-1}$ and $M_0 = M^{-1}$. Since both $C_{-1}(L)$ and $C_{-1}(M)$ are the group of integers, we may identify L_0 with M_0. Let

$$i : L_0 \to M, \qquad j : M_0 \to L$$

denote the identity chain transformations, which are clearly equivariant. Let us use the notation:

$$\mathfrak{J}_e^q(K^n, H_q(K)) = \mathfrak{J}_e^q(M^n, M_0, H_q(M)), \qquad (q < n).$$

THEOREM 17.5. *If $J^{q+1}(W, H_q(K)) = 0 = \mathfrak{J}_e^q(K^n, H_q(K))$ for every $q < n$, then both i and j are equivariantly n-extensible over L and M respectively. For arbitrary equivariant extensions*

$$i^* : L^n \to M, \qquad j^* : M^n \to L$$

of i and j, the composed chain transformations $j^*i^* \mid L^{n-1}$ and $i^*j^* \mid M^{n-1}$ are both equivariantly chain homotopic with the identities relative to $L_0 = M_0$.

The above theorem is an easy consequence of the Theorems 11.7 and 13.1 under the hypothesis of the theorem together with the fact that $H_q(L) = 0$ for every dimension q.

Following Eilenberg [1, p. 395], we adopt the notation

$$H^q(W, G) = H_e^q(K_W, G)$$

for each dimension q and every coefficient group G with W as left operators. According to § 6 and § 7, Theorem 17.5 gives immediately the following

COROLLARY 17.6. *If $J^{q+1}(W, H_q(K)) = 0 = \mathfrak{J}_e^q(K^n, H_q(K))$ for every $q < n$, any equivariant extension $i^* : L^n \to M$ induces* *isomorphisms of* $H^q_e(K, G)$ *onto* $H^q_e(K_W, G)$ *and of* $H_q(K)$ *onto* $H_q(K)$ *for every* $q < n$; *hence*

$$H^q_e(K, G) \approx H^q(W, G), \qquad (0 \leqq q < n), \tag{17.6}$$

$$H_q(K) = 0, \qquad (0 \leqq q < n). \tag{17.7}$$

THEOREM 17.7. *The following statements are equivalent*:

(i) $J^{q+1}(W, H_q(K)) = 0 = \mathfrak{J}^q_e(K^n, H_q(K))$, $(0 \leqq q < n)$.

(ii) $H_q(K) = 0$, $(0 \leqq q < n)$.

(iii) $H^{q+1}(W, H_q(K)) = 0 = H^q_e(K, H_q(K))$, $(0 \leqq q < n)$.

(iv) $H^q(K, H_q(K)) = 0$, $(0 \leqq q < n)$.

Proof. That (i) → (ii) follows from Corollary 17.6. That (ii) → (iii) and (iii) → (i) is obvious. If we take W to be the group consisting of a single element 1, then (iii) becomes (iv). Hence (ii) and (iv) are also equivalent. Q.E.D.

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