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by

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This paper was inspired by a theorem of Lázár 1) which we shall state below.

Let \( M \) be a set of positive measure. To every \( x \in M \) we adjoin a set of elements \( y \neq x \) of \( M \) which may have the cardinal number of the continuum. This means that we define a function \( y = \varphi(x) \) of \( x \in M \), where \( y \in M \), which may assume continuously many values for every \( x \), whereas the equation \( x = \varphi(x) \) cannot occur. If neither of the two equations \( y = \varphi(x) \) and \( x = \varphi(y) \) holds, the two elements \( x \) and \( y \) are called independent.

Let \( \{\varphi(x)\} \) denote the set of values \( \varphi(x) \) for a given \( x \); assume moreover that \( x \) is not a point of accumulation of \( \{\varphi(x)\} \) and that \( \{\varphi(x)\} \) is of measure zero.

**THEOREM.** We can find a set of positive exterior measure of elements of \( M \) so that any pair of its elements are independent.

**Proof.** As the set \( \{\varphi(x)\} \) does not contain \( x \) and as \( x \) is no point of accumulation, its complement \( C(\varphi(x)) \) with respect to \( M \) contains \( x \) and an interval \( I_x \) surrounding \( x \). Now for every \( x \) let us choose in \( I_x \) a closed segment \( S(x) \) with rational end-points. Hence all values of \( \varphi(x) \) are situated outside \( S(x) \).

Now the segments \( S(x) \) form an enumerable system \( S_1, S_2, \ldots \). To every \( S_n \) there belongs at least one \( x \in M \), such that \( S(x) = S_n \). Let \( N_n \) denote the set of all \( x \in M \) with \( S(x) = S_n \). We assert that there is at least one segment \( S_n \) for which \( N_n \) has positive exterior measure. For suppose that to every \( S_n \) the set \( N_n \) would be of measure zero. Then, as \( M \subseteq N_1 + N_2 + \ldots \), by the well-known theorem that the sum of enumerably many sets of measure zero is a set of measure zero, we must conclude that \( M \) was a set of measure zero.

The elements \( x \in N_n \) all belong to \( S_n \); they are evidently independent as the adjoined values are all outside the segment \( S_n \).

Lázár's theorem 2) only states that under the same assumptions we can find a set with the power of the continuum.

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2) Evidently Lázár uses the word "condensationpoint" where only a point of accumulation is meant.

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