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G. Fodor

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On a theorem in the theory of binary relations

by

G. Fodor

Szeged.

This paper was inspired by a theorem of Lázár [1]) which we shall state below.

Let M be a set of positive measure. To every $x \in M$ we adjoin a set of elements $y \neq x$ of M which may have the cardinal number of the continuum. This means that we define a function $y = \varphi(x)$ of $x \in M$, where $y \in M$, which may assume continuously many values for every x, whereas the equation $x = \varphi(x)$ cannot occur. If neither of the two equations $y = \varphi(x)$ and $x = \varphi(y)$ holds, the two elements x and y are called independent.

Let $\{\varphi(x)\}$ denote the set of values $\varphi(x)$ for a given x; assume moreover that x is not a point of accumulation of $\{\varphi(x)\}$ and that $\{\varphi(x)\}$ is of measure zero.

THEOREM. We can find a set of positive exterior measure of elements of M so that any pair of its elements are independent.

Proof. As the set $\{\varphi(x)\}$ does not contain x and as x is no point of accumulation, its complement $C\{\varphi(x)\}$ with respect to M contains x and an interval I_x surrounding x. Now for every x let us choose in I_x a closed segment $S(x)$ with rational endpoints. Hence all values of $\varphi(x)$ are situated outside $S(x)$.

Now the segments $S(x)$ form an enumerable system $S_1, S_2, \dots$ To every S_n there belongs at least one $x \in M$, such that $S(x) = S_n$. Let N_n denote the set of all $x \in M$ with $S(x) = S_n$. We assert that there is at least one segment S_n for which N_n has positive exterior measure. For suppose that to every S_n the set N_n would be of measure zero. Then, as $M \subset N_1 + N_2 + \dots$, by the well-known theorem that the sum of enumerably many sets of measure zero is a set of measure zero, we must conclude that M was a set of measure zero.

The elements $x \in N_n$ all belong to S_n; they are evidently independent as the adjoined values are all outside the segment S_n.

Lázár's theorem [2]) only states that under the same assumptions we can find a set with the power of the continuum.

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[1]) Compositio Mathematica 3 (1936), p. 304.

[2]) Evidently Lázár uses the word "condensationpoint" where only a point of accumulation is meant.

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