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On the ultimate boundedness of the solutions of certain differential equations

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In this paper alternative proofs are given, under somewhat less restrictive conditions on the functions g and p (see para. 4 and 5), of some theorems recently proved by Cartwright and Littlewood [1]), on differential equations of the type

$$(1) \qquad \ddot{x} + k\dot{x}\, f(x) + g(x) = kp(t), \qquad (k > 0)$$

(dots denoting differentiation for t) [2]). The general method is to compare trajectories of (1) with those of

$$(2) \qquad \ddot{x} + h\dot{x} + g(x) = 0 \qquad (h > 0)$$

and to confirm the physically plausible conjecture that if $kf(x) \geqq 2h$ for large x, a trajectory T_1 of (1) that starts within a trajectory T_2 of (2) will stay there, except possibly near the origin.

1. The functions f and g are to be continuous for every x, and $p(t)$ is to be such that (1) has a solution for any assigned initial values of $x(t)$ and $\dot{x}(t)$ [3]). If an arc of a trajectory of (1), i.e. of a solution of

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[1]) M. L. Cartwright and J. E. Littlewood, Annals of Math. 48 (1947) 472—494, here called "C and L".

[2]) In C and L, g may depend on k. The slight modifications required in the proofs in this case are referred to in para. 8.

[3]) The form of existence theorem required is: given a block $|x - x_0| \leqq \alpha$, $|y - y_0| \leqq \beta$, $|t - t_0| \leqq y$, in (x, y, t)-space, an arc of a solution $x = \xi(t)$, $y = \dot{\xi}(t)$ exists, passing through x_0, y_0, t_0 and having its end points on the boundary of the block. A sufficient condition on p is that it has only a finite number of discontinuities; the functions ξ and $\dot{\xi}$ are everywhere continuous and (1) is satisfied except at the discontinuities of p. A more general condition is that p be summable in every finite interval, (see e.g. Caratheodory, Reelle Funktionen, (1918), p. 682). In this case $\xi(t)$ is a *solution* in an interval if it and $\dot{\xi}$ are absolutely continuous in the interval and satisfy (1) almost everywhere. The absolute continuity of ξ and $\dot{\xi}$ justifies the ordinary processes of analysis used, and no further reference will be made

$$(3) \qquad \frac{dx}{dt} = y,\ \frac{dy}{dt} + kyf(x) + g(x) = kp(t),$$

regarded as a curve in the (x, y) plane with parameter t, ("time"), lies entirely in one of the half-planes $y \geqq 0$ or $y \leqq 0$, $x(t)$ is monotonic on it, and t and y are single valued functions, $y(x)$, $t(x)$. The function $y(x)$ is a solution of

$$(4) \quad y' + kf(x) + g(x)/y = kp(t)/y \quad (y' = dy/dx,\ t = t(x)).$$

Lemma 1. *Let Γ_1 and Γ_2 be arcs in the same half-plane* ($y \geqq 0$ *or* $y \leqq 0$) *of trajectories of two equations of type* (1), *satisfying*

$$(5a) \quad y_1' + kf_1(x) + g(x)/y_1 = kp_1(t_1)/y_1$$

$$(5b) \quad y_2' + kf_2(x) + g(x)/y_2 = kp_2(t_2)/y_2.$$

Suppose that $|\, p_i(t) \,| \leqq K_i$, $(i = 1, 2)$ *on the arcs. Then if* $y_0 \neq 0$, Γ_1 *cannot meet* Γ_2 *from within at a point* (x_0, y_0) *where*

$$(6) \quad f_1(x_0) > f_2(x_0) + (K_1 + K_2)/|\, y_0 \,|.$$

"Γ_1 meets Γ_2 from within at (x_0, y_0)" means that (i) $y_1(x_0) = y_2(x_0) = y_0$, and (ii) $y_2(x) - y_1(x)$ has the sign of y_0 in an open interval immediately preceding x_0 in time, (i.e. to the left of x_0 if $y_0 > 0$, to the right if $y_0 < 0$).

We have [4])

$$(7) \quad y_2'(x) - y_1'(x) \geqq k(f_1(x) - f_2(x)) + \frac{g(x)}{y_1 y_2}(y_2 - y_1) - k\frac{K_1}{|\, y_1 \,|} - k\frac{K_2}{|\, y_2 \,|}$$

$$k \to \left[(f_1(x_0) - f_2(x_0)) - \frac{K_1 + K_2}{|\, y_0 \,|} \right] \text{ as } x \to x_0.$$

If, then, (6) is satisfied, $y_2'(x) - y_1'(x) > 0$ in some open interval I containing x_0. Integrating from x to x_0, $y_2(x) - y_1(x) \leqq 0$ or $\geqq 0$, as x (of I) $\leqq x_0$ or $\geqq x_0$. This is inconsistent with Γ_1 meeting Γ_2 at x_0 from within.

Lemma 2. *If, in Lemma* 1, $p_1(t) = p_2(t) = 0$ *for all t and if* $x_0 \neq 0$, Γ_1 *cannot meet* Γ_2 *from within, rel.* 0, *at* $(x_0, 0)$ *if* $f_1(x_0) \geqq f_2(x_0)$ *and* $g(x_0)/x_0 > 0$.

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[4]) It is agreed once for all that inequalities and equalities involving derivatives, deduced from differential equations, are asserted only for values of x and t for which the derivatives exist and satisfy the differential equations (cf. footnote [3]).

"Γ_1 meets Γ_2 from within, rel. 0, at $(x_0, 0)$" means that $y_2(x_0) = y_1(x_0) = 0$ and $y_2(x) - y_1(x)$ has the sign of x in an open interval immediately preceding x_0 (in time).

Suppose that $g(x_0)/x_0 > 0$ and that $f_1(x_0) \geqq f_2(x_0)$. Equation (1) shows that, since $p = 0$, $-\ddot{x}$ has the same sign as $g(x)$, i.e. as x, at a point where a trajectory meets the x-axis. Thus $|x|$ has a maximum on both curves at $x = x_0$, and, therefore, $y_1/x > 0$, $y_2/x > 0$ in an open interval I immediately preceding $(x_0, 0)$ in time. If then $(y_2(x) - y_1(x))/x > 0$ in I,

$$g(x)\left(\frac{1}{y_1} - \frac{1}{y_2}\right) > 0$$

and hence, by (7), with $p_1 = p_2 = 0$,

$$y_2'(x) - y_1'(x) > k(f_1(x_0) - f_2(x_0)) \geqq 0,$$

at all points of I. This is inconsistent with Γ_1 meeting Γ_2 from within at $(x_0, 0)$.

2. *Trajectories of the equation* (2). We now assume that $g(x)/x > 0$ when $|x| > a_0$, a certain non-negative constant [5]), and that $G(x) = \int_0^x g(x)dx \to \infty$ as $|x| \to \infty$.

The integrals

$$\tfrac{1}{2}y^2 + G(x) = \tfrac{1}{2}y_0^2 + G(x_0)$$

of the equation $\ddot{x} + g(x) = 0$ have at every point the topological character of a simple arc, and owing to their symmetry about $y = 0$ their components are either simple closed curves or open arcs not meeting $y = 0$.

Lemma 3. *Any trajectory T of* (2) *is bounded for $t \geqq$* constant. If T passes through (x_0, y_0), choose $|Y_0| > |y_0|$ such that

$$\tfrac{1}{2}Y_0^2 + G(x_0) > \overline{bd}\, G(x) \text{ in } \langle -a_0, a_0 \rangle.$$

Since $G(x) \to \infty$, the integral curve of $\ddot{x} + g(x) = 0$ through (x_0, Y_0) meets the x-axis on both sides of the origin and so is closed. It contains (x_0, y_0) within it, and cuts the x-axis in points where $g(x)/x > 0$. Hence Lemma 3 follows from Lemmas 1 and 2.

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[5]) Constants denoted by italic letters other than x, y, Y, t are fixed throughout the paper, save that C is used in the usual way as an „ambiguous" constant, independent of k. The meanings of Greek letters, and of x_0, y_0 etc., may vary.

Lemma 4. *Given $\varepsilon > 0$, every trajectory T of (2) meets the set $|x| \leqq a_0 + \varepsilon$, $|y| \leqq \varepsilon$ for arbitrarily large positive values of t.*

The integral

$$\text{(8)} \qquad \tfrac{1}{2}y^2 + h\int_0^t y^2 dt = \tfrac{1}{2}y_0^2 + G(x_0) - G(x)$$

of (2) has a bounded right-hand side as $t \to \infty$ (by Lemma 3). Hence $\int y^2 dt$ is convergent as $t \to +\infty$. Since x and y $(= \dot{x})$ are bounded, it follows from (2) that $\ddot{x}$ $(= \dot{y})$ is bounded. The convergence of $\int y^2 dt$ therefore implies that $y \to 0$ as $t \to \infty$. Hence, by (2), $\dot{y} + g(x) \to 0$ as $t \to \infty$. Since T is bounded for $t \to +\infty$, and $g(x)/x > 0$ in $|x| > a_0$, the function $|g|$ has a positive lower bound δ on the part of T in $|x| \geqq a_0 + \varepsilon$. If then T remained in (say) $x > a_0 + \varepsilon$ for $t > t_1$, $|\dot{y}|$ would ultimately remain $> \frac{1}{2}\delta$; and this is not consistent with $y \to 0$.

3. Let $Q(u)$ denote, for each positive u, the least Q such that $|g(x)| \geqq u$ if $|x| \geqq Q$. (If $|g| < u$ for all x, $Q(u) = \infty$).

Lemma $\left\langle\begin{matrix}5a.\\5b.\end{matrix}\right\rangle$ *If $\eta > 0$, an arc in quadrant $\left\langle\begin{matrix}2\\4\end{matrix}\right\rangle$ of a trajectory T of (2) crosses the line $\begin{matrix}y = -\eta\\ y = \eta\end{matrix}$ at most once in $|x| \geqq Q = Q(h\eta)$.*

(The quadrants are numbered $\frac{4\,|\,1}{3\,|\,2}$, in accordance with the positive sense of description of a trajectory.)

Only $5a$ need be proved. Between two intersections of T with $y = -\eta$ is a point where $y'(x) = 0$, i.e. a point on the curve C: $hy + g(x) = 0$. The part of $y = -\eta$ outside $\langle 0, Q \rangle$ lies above C and below $y = 0$. But in this region (since $y < 0$), $-h - \dfrac{g(x)}{y} > 0$, i.e. $y' > 0$. Hence y decreases with decreasing x, that is with increasing time. Thus T cannot get from C to $y = -\eta$ outside $|x| \leqq Q$.

Lemma 6. *If $\eta > 0$, any trajectory of (2) through a point (x_0, y_0) of quadrants 1 or 3, where $|x_0| \geqq \max(a_0, Q(h\eta))$, $|y_0| \geqq \eta$, will cross first the x-axis, and then the line $x = x_0$ at a point (x_0, y_1) where $|y_1| > \frac{1}{2}\eta$.*

It is sufficient to consider the case $x_0 > 0$, and therefore $g(x_0) > 0$, $y_0 > 0$. On the upper arc, since for $x > x_0$

$$y'(x) = -h - \frac{g(x)}{y} < -h,$$

we have $y(x) < y_0$; and the arc meets $y = 0$ in $(\xi_0, 0)$, where $x_0 < \xi_0 < x_0 + y_0/h$. On the upper arc, for $x_0 < x \leqq \xi_0$,

$$\tfrac{1}{2}\frac{d}{dx}(y^2) = -hy - g(x) > -hy_0 - g(x),$$

giving by integration

$$-\tfrac{1}{2}y_0^2 > -hy_0(\xi_0 - x_0) - G(\xi_0) + G(x_0),$$

$$\tfrac{1}{2}y_0^2 < (\xi_0 - x_0)\,(hy_0 + g(\xi)), \tag{9}$$

where $x_0 < \xi < \xi_0$.

If the lower arc crosses $y = -\eta$, then by Lemma *5a* it remains below until it crosses $x = x_0 \geqq Q(h\eta)$; and so $|y_1| \geqq \eta$. We may therefore assume $y > -\eta$ at all points on the lower arc, whence

$$y'(x) > -h + \frac{g(x)}{\eta} \geqq 0$$

since $x \geqq x_0$. Hence $|y_1|$ is the maximum value of $|y|$ on the arc; and since y is now negative,

$$\tfrac{1}{2}\frac{d}{dx}(y^2) = -hy - g(x) = h|y| - g(x) \leqq h|y_1| - g(x);$$

giving on integration from x_0 to ξ_0,

$$-\tfrac{1}{2}y_1^2 \leqq (\xi_0 - x_0)(h|y_1| - g(\xi)), \text{ (same } \xi!).$$

Since $g(\xi) - h|y_1| > 0$, we deduce from this and (9)

$$y_1^2(g(\xi) + hy_0) > y_0^2(g(\xi) - h|y_1|),$$

$$|y_1| > \frac{g(\xi)y_0}{g(\xi) + hy_0} \geqq \frac{h\eta \cdot y_0}{h\eta + hy_0} \geqq \frac{h\eta \cdot \eta}{h\eta + h\eta} = \tfrac{1}{2}\eta.$$

[Lemma 6 may be extended similarly to trajectories of the equation

$$\ddot{x} + \dot{x}f(x) + g(x) = 0,$$

given that $|g(x)| > \eta f(x) > 0$ for $|x| \geqq |x_0|$. The inequality (9) is replaced by $\tfrac{1}{2}y_0^2 \leqq (\xi_0 - x_0)(y_0 f(\xi) + g(\xi))$ and we finally obtain $|y_1| \geqq y_0 g(\xi)/(y_0 f(\xi) + g(\xi))$.]

4. The following conditions, besides those of continuity stated in para. 1, are assumed in Lemmas 7, 8, 9, and Theorem 1.

(*i*) $g(x)/x > 0$ when $|x| > a_0$;

(*ii*) $f(x) \geqq 2h > 0$ when $|x| \geqq a_0$;

(*iii*) $|p(t)|$ and $\int_t^{t'} p\,d\tau$ are bounded in $\langle 0, \infty)$ — say both remain $\leqq K$.

We assume also that $k \geqq 1$.

The pattern equation

$$\frac{d^2x}{d\tau^2} + h\frac{dx}{d\tau} + g(x) = 0, \tag{10}$$

with which (1) will be compared, has parameter τ and the equations of its trajectories are written in the form

$$\frac{dx}{d\tau} = Y, \; \frac{dY}{d\tau} + hY + g(x) = 0.$$

The letter t always denotes the parameter of (1); $x(t)$ is a solution of (1) and $y(t) = \dot{x}(t)$. T_y denotes a „half-trajectory" of (1), i.e. the part $t \geqq$ constant; and Γ_y is an arc $t_0 \leqq t \leqq t_1$. T_Y, Γ_Y have similar meanings in relation to (10). On an arc Γ_Y lying in a half-plane $y \geqq 0$ or $y \leqq 0$, Y is a single-valued function $Y(x)$ satisfying

$$Y'(x) = -h - g(x)/Y. \tag{11}$$

From this and the analogous equation (4) for an arc of T_y lying in one half-plane, we have, putting $u(x) = Y(x) - y(x)$,

$$u'(x) = kf(x) - h + \frac{ug(x)}{yY} - k\frac{p(t)}{y}. \tag{12}$$

Lemma 7. *An arc Γ_y cannot meet an arc Γ_Y from within* [6]) *at (x_0, y_0), where $x_0 \geqq a_0$ and $|y_0| > d_0 = K/h$.* Follows immediately from Lemma 1, since $f(x_0) - h \geqq h$.

Lemma 8*a* [7]). *Let an arc Γ_y start at (x_0, y_0, t_0), where $x_0 \geqq a_0$. Let an arc Γ_Y start on $x = a_0$ above the x-axis, cut $x = x_0$ above (x_0, y_0), and end at (α, β), where $\beta \geqq d_0 = K/h$, and $\alpha \geqq x_0 + (K + 2\beta)/h$. Then if Γ_y does not meet $x = a_0$ when $t > t_0$, it meets neither Γ_Y nor $x = \alpha$.*

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[6]) See Lemma 1.

[7]) The name "8*a*" implies, that as in the case of Lemma 5, a corresponding "8*b*" is also asserted, with interchange of quadrants 1 and 3, 2 and 4. Only trivial modifications are needed in the proof. This applies to Lemmas 9*a*, 10*a*, etc. below.

The ordinate of Γ_Y is a single-valued function $Y(x)$ of x; that of Γ_y is not, in general. Suppose that Γ_y remains in $< a_0, \alpha)$ at least for $t_0 \leqq t < t_1$. Then $U(t) = Y(x(t)) - y(t)$ is a well-defined function of t in $< t_0, t_1)$. By hypothesis $U(t_0) > 0$, and by Lemma 7, applied to (4) and (10), $U(t)$ must remain positive in $\langle t_0, t_1)$, for at its first zero, Γ_y would meet Γ_Y from within.

By (12), since $y(t) = dx/dt$, we have (cf. footnote [4])),

$$\frac{dU}{dt} = y(t)(kf(x) - h) + g(x)\frac{U(t)}{Y(x)} - kp(t), \quad (x = x(t))$$
$$> \frac{dx}{dt}(kf(x) - h) - kp(t),$$

giving, on integration from t_0 to t_1,

$$U(t_1) > U(t_1) - U(t_0) > k\int_{x(t_0)}^{x(t_1)} (f(x) - h)dx - k\int_{t_0}^{t_1} p(t)dt,$$
$$\geqq kh(x(t_1) - x_0) - kK.$$

If then $x(t_1) = \alpha$,

$$U(\alpha) - y(t_1) = U(t_1) > kh(\alpha - x_0) - kK \qquad (13)$$
$$> 2k\beta \geqq 2\beta.$$

But $Y(\alpha) = \beta$ and therefore $y(t_1) \leqq -\beta < 0$. This is impossible: the first intersection of Γ_y with $x = \text{constant} > 0$ must be above $y = 0$. Thus Γ_y does not meet $x = \alpha$, and the relation $U(t) > 0$, proved in $< t_0, t_1)$, holds throughout Γ_y.

Lemma 9*a*. *Under the conditions of* 8*a*, $Y(x(t)) \geqq 2\beta$, *whenever* $y(t) = 0$ *and* $x(t) \geqq x_0 + (K + 2\beta)/h$.

From the inequality for $x(t)$ it follows, as in (13), that $Y(x(t)) - y(t) \geqq 2\beta$.

Theorem 1. *If* f, g, p *satisfy* (*i*), (*ii*), (*iii*) *and if* $\varepsilon > 0$, *no* T_y *can remain ultimately in the set* $|x| \geqq a_0 + \varepsilon$.

Suppose e.g. that T_y starts at (x_0, y_0, t_0) where $x_0 \geqq a_0 + \varepsilon$, and remains in $x \geqq a_0 + \varepsilon$. If Y_1 is large enough, a trajectory T_Y starting at (a_0, Y_1) cuts first the line $x = x_0$, above (x_0, y_0), and then $y = K/h = d_0$ at $x = \alpha(x_0, y_0) > x_0 + (d_0 + K)/h$. [8])
Hence, by Lemma 8*a*, $\alpha = \alpha(x_0, y_0)$ is an upper bound of $x(t)$ on T_y. If $|f| \leqq C$, $|g| \leqq C$ in $\langle a_0, \alpha \rangle$, it follows from (4) that on any arc of T_y in $|y| \geqq \varepsilon$,

$$|y'| \leqq kC + \varepsilon^{-1}(C + kK) = J_k \text{ say.}$$

[1]) If $|g| \leqq C$ in $\langle a_0, x_0 + (d_0 + K)/h \rangle$, then $|Y'| \leqq h + C/d_0$ in the same interval and we can put $Y_1 = (h + C/d_0)(x_0 + (d_0 + K/h) + y_0 + d_0)$.

Therefore T_y cannot cross $y = \pm (y_0 + J_k(\alpha - a_0))$, i.e. $y(t)$ is also bounded.

Consider equation (3). Integrating from t_1 to t,

$$y(t) - y(t_1) + k\int_{x(t_1)}^{x(t)} f(x)dx + \int_{t_1}^{t} g(x(t))dt = k\int_{t_1}^{t} p(t)dt.$$

All the terms of this equation have been shown to be bounded except the g-integral. Therefore this also is bounded. But this is impossible, for since $g(x) > 0$ in $\langle a_0 + \varepsilon, \alpha \rangle$ it has a positive lower bound there.

Corollary. *Given $\varepsilon > 0$, every T_y meets either $x = 0$ or the rectangle $|x| \leqq a_0 + \varepsilon$, $|y| \leqq d_0$, after any assigned time t_0.*

Suppose T_y starts at (x_0, y_0, t_0), where, say, $x_0 > 0$, and does not meet the rectangle. If $|x_0| \leqq a_0$, then $|y| = |\dot{x}| \geqq d_0$ so long as $|x| \leqq a_0$, and therefore T_y meets $x = \pm a_0$ at a finite time. If it meets $x = -a_0$ there is nothing more to prove. Suppose it meets $x = a_0$. Since $\dot{x}(t)$ remains $\geqq \frac{1}{2}d_0$ in a further positive t-interval, T_y crosses $x = a_0 + \delta$, $\delta > 0$, when $t = t_1 > t_0$. By Theorem 1 it later crosses $x = a_0 + \min(\varepsilon, \delta)$. The first such crossing after t_1 is from the right, and therefore is in $y < 0$. Thus $y = \dot{x}$ remains $< -d_0$ as long as x is in $\langle -a_0 - \varepsilon, a_0 + \varepsilon \rangle$, showing that T_y meets $x = 0$.

5. The further condition

$$(iv)_0 \quad Q(2K) \text{ is finite}$$

is now imposed on g. (For Q see para. 3).

Lemma 10*a*. *Suppose (i) to (iii) and $(iv)_0$ satisfied, and that $\alpha_0 \geqq a_0$. Let Γ_Y start at (α_0, Y_0), where $Y_0 > d_0 = K/h$, cross $y = d_0$ to the right of $x = \alpha_1 = \alpha_0 + (2d_0 + K)/h$, and end on $x = a_0$, $y < 0$. Let Γ_y start at (α_0, y_0) where $0 < y_0 < Y_0$ and remain in $|x| \geqq a_0$.*

Then if Γ_y does not meet the rectangle $|x| \leqq Q(2K)$, $|y| \leqq d_0$, it does not meet Γ_Y.

Suppose that Γ_y does not meet the rectangle. By Lemma 8*a* no sub-arc of Γ_y in $y \geqq 0$ can meet Γ_Y. It follows that Γ_Y meets $y = d_0$ first in $x > Q(2K)$. Let Γ_y^1 be a sub-arc of Γ_y in $y \leqq 0$, starting at $(x_1, 0)$, and let $Y(x)$ and $Y^*(x)$ be the ordinates of the upper and lower arcs of Γ_Y. Since $x > \alpha_1$, Lemma 9*a* gives $Y(x_1) \geqq 2d_0$. Since also $x_1 > Q(2K) = Q(2hd_0)$, Lemma 6 is applicable, with $\eta = 2d_0$, and gives $Y^*(x_1) < -d_0$. By Lemma 5*a*, $Y^*(x) < -d_0$ at least until Γ_Y meets $x = Q(2K)$ and in $|x| \leqq Q(2K)$ we have $y < -d_0$ on Γ_y^1. Thus any intersection of Γ_Y and Γ_y^1 must be in $y < -d_0$, which is impossible (Lemma 7).

6. The special case $a_0 = 0$ of Theorem 2 now follows.

Theorem 2*a* (case $a_0 = 0$). *Suppose (i) to (iii) and $(iv)_0$ satisfied, with $a_0 = 0$, A T_y starting at $(0, y_0)$ remains enclosed by a T_Y starting at $(0, Y_0)$, where $Y_0 > y_0 > 0$, until (possibly) T_y meets the rectangle R_0,* $= [|x| \leqq a_1^0, |y| \leqq d_0]$, *where* $a_1^0 = \max(Q(2K), 2(d_0 + K)/h)$.

The meaning of "T_y remains enclosed by T_Y" is as follows. Let T_y and T_Y cut the y-axis successively (alternately above and below the x-axis) at $y = y_0, Y_0; y_1, Y_1; y_2, Y_2; \ldots$ respectively until T_y enters R_0 (or ad infinitum if this does not occur). Then the arc T_y^n of T_y from $(0, y_n)$ to $(0, y_{n+1})$ lies in the domain D_n bounded by the straight segment and the arc of T_Y^n with the common end points $(0, Y_n)$ and $(0, Y_{n+1})$. The point Y_{n+1} is outside R_0 if T_y^n does not meet R_0.

The theorem follows from repeated applications of Lemmas 10*a* and 10*b* with $d_0 = 0$. By Lemma 7, T_y^n and T_Y^n cannot meet in $|y| > d_0$. Therefore if T_y^n does not meet R_0, T_Y^n cuts $y = d_0$ outside $|x| \leqq (2d_0 + K)/h$, as required in Lemma 10.

Corollary. *Given (i) to (iii) with $a_0 = 0$, and $(iv)_0$, every T_y ultimately meets R_0.* By Theorem 1, Corollary, we may suppose T_y to start on $x = 0$, say with $y_0 > 0$. If then T_Y starts at $(0, Y_0)$, where $Y_0 > y_0$, and if T_y never enters R_0, T_Y also never enters R_0, contrary to Lemma 4.

7. In the general case, $a_0 > 0$, a stronger condition than $(iv)_0$ is needed. Let A_0 be a bound for both $|f|$ and $|g|$ in $\langle -a_0, a_0 \rangle$ and let

$$a_1 = \frac{a_0}{h}(12A_0 + 4K + 5h), \qquad B_1 = bd\,|g| \text{ in } \langle -a_1, a_1 \rangle,$$

$$d_1 = \max(1, 2K/h, \sqrt{a_1 B_1}), \qquad d_2 = d_1 + (h + B_1/d_1)a_1$$

$$c_0 = \max\big(Q(2ha_1),\ a_1 + h^{-1}(K + 2d_1)\big).$$

Let R_1 denote the rectangle $|x| \leqq c_0$, $|y| \leqq d_1$.

The new condition on g is

(iv) $Q(2hd_1)$ is finite.

This may evidently be replaced by the simpler but stronger

(iv') $|g(x)| \to \infty$ as $|x| \to \infty$.

Lemma 11*a*. *An arc Γ_Y starting at $(0, Y_0)$, where $Y_0 > d_2$, and lying in $y \geqq 0$, $|x| \leqq a_1$, cannot meet $y = d_1$.*

So long as $Y \geqq d_1$ we have, by (11), $Y'(x) \geqq -h - B_1/d_1$. If ξ were the first point of Γ_Y on $y = d_1$, with $0 \leqq \xi \leqq a_1$, we should have

$$d_1 - Y_0 = Y(\xi) - Y(0) \geqq -(h + B_1/d_1)\xi \geqq -(h+B_1/d_1)a_1,$$

i.e. $Y_0 \leqq d_2$, contrary to hypothesis.

Lemma 12*a*. *If the arcs Γ_y, Γ_Y lie in $|y| > d_1$ and run from $x = 0$ to $x = a_0$, then* $|u(a_0) - u(0)| \leqq \frac{1}{4}kh(a_1 - a_0)$. (For $u(x)$ see before equation (12)). By (4) we have in $\langle 0, a_0 \rangle$:

$$|y'| \leqq kA_0 + A_0/d_1 + kK/d_1 \leqq k(2A_0 + K),$$

giving

$$|y(a_0) - y(0)| \leqq ka_0(2A_0 + K).$$

Similarly, by (11),

$$|Y(a_0) - Y(0)| \leqq ka_0(h + A_0).$$

Therefore

$$|u(a_0) - u(0)| \leqq ka_0(3A_0 + h + K) = \tfrac{1}{4}kh(a_1 - a_0).$$

Lemma 13. *If Γ_y, Γ_Y lie either both in $y > d_1$ or both in $y < -d_1$, and run from $x = x_0$ to $x = x_1$, where $a_0 \leqq x_0 < x_1 \leqq a_1$, and if $u(x) \leqq 0$ in $\langle x_0, x_1 \rangle$, then $u(x_0) < -\frac{1}{4}kh(x_1 - x_0)$.*

By (12), since $u(x) \leqq 0$ in $\langle x_0, x_1 \rangle$, we have there

$$\frac{du}{dx} \geqq (2k - 1)h + B_1\frac{u(x)}{d_1^2} - k\frac{K}{d_1}$$

$$\geqq \tfrac{1}{2}kh + ju(x), \text{ putting } j = B_1/d_1^2 \leqq 1/a_1.$$

Thus

$$\frac{d}{dx}(ue^{-jx}) \geqq \tfrac{1}{2}khe^{-jx}$$

in $\langle x_0, x_1 \rangle$, giving

$$u(x_1)e^{-jx_1} - u(x_0)e^{-jx_0} \geqq -\tfrac{1}{2}k\frac{h}{j}(e^{-jx_1} - e^{-jx_0})$$

$$u(x_0) \leqq u(x_1)e^{-j(x_1-x_0)} - \tfrac{1}{2}\frac{kh}{j}(1 - e^{-j(x_1-x_0)}).$$

Now if $0 < s \leqq 1$, $1 - e^{-s} > \frac{1}{2}s$. Since $j \leqq 1/a_1$, $j(x_1 - x_0) \leqq 1$, and $u(x_1) \leqq 0$ by hypothesis. Therefore $u(x_0) < -\frac{1}{4}kh(x_1 - x_0)$.

Theorem 2*a*. (Case $a_0 > 0$). *If (i) to (iii) and (iv) or (iv)′ hold, and if $Y_0 > y_0 > 0$, a T_y starting at $(0, y_0)$ remains enclosed by by T_Y starting at $(0, Y_0)$, until (possibly) T_y enters the rectangle $R_2 = [|x| \leqq c_0, |y| \leqq d_2]$; save that T_y may lie outside T_Y in the ranges $0 < x < a_1$, $y > 0$ and $0 > x > -a_1$, $y < 0$.*

The meaning is that (with the notation of the case $a_0 = 0$) T_y^n lies in D_n, save possibly for part of the initial arc in $|x| < a_1$.

By Lemma 11*a*, Γ_Y does not meet $y = d_1$ before crossing

$x = a_1$. Since $u(0) > 0$, Lemma 12a gives $u(a_0) > -\frac{1}{4}kh(a_1 - a_0)$. Hence, by Lemma 13, $u(x) > 0$ at some point of $\langle a_0, a_1 \rangle$. Let ξ be the first point at which $u(\xi) = 0$, so that $\xi < a_1$.

By (12)

$$(14) \qquad u'(\xi) \geqq (2k - 1)h - kK/d_2 > \tfrac{1}{2}kh > 0.$$

If there were another zero of u in $(\xi, a_1\rangle$, u' would be $\leqq 0$ at the first zero after ξ, contrary to (14). Thus u remains positive in $\langle \xi, a_1 \rangle$, and in particular $u(a_1) > 0$.

By Lemma 7, T_y and T_Y do not meet in $x \geqq a_1$, $y > d_2$, so that $Y(x) \geqq d_2$ in $\langle a_1, c_0 \rangle$. Hence, putting $\alpha_0 = a_1$ in Lemma 10a (and therefore $\alpha_1 \leqq c_0$), T_y up to its first meeting with $x = a_1$ in $y < 0$ is enclosed by T_Y. In particular, $Y^*(a_1) < y^*(a_1)$, the star denoting the arc below the x-axis; i.e. $u^x(a_1) < 0$. Suppose u^* remains negative in an open interval (ξ, a_1). By Lemma 13,

$$(15) \qquad u^*(\xi) < -\tfrac{1}{4}kh(a_1 - \xi) < 0.$$

To suppose ξ a zero of u^* would therefore lead to a contradiction. There is therefore no zero, and (15) holds throughout $\langle a_0, a_1 \rangle$. Hence $u^*(a_0) < -\frac{1}{4}kh(a_1 - a_0)$, and so finally, by Lemma 12a, $u^*(0) < 0$, i.e. $Y^*(0) < y^*(0)$.

Corollary. *Every T_y ultimately meets R_2* (cf. case $a_0 = 0$).

Theorem 3. *If (i) to (iii), and (iv) or (iv)', hold, every T_y remains ultimately in $|x| \leqq C$, $|y| \leqq kC$:*

By Theorem 2, Corollary, T_y can be assumed to start at (x_0, y_0, t_0) on yR_2, — say on $y = d_2$ or $x = c_0$.

If $y_0 = d_2$ then $-c_0 \leqq x_0 \leqq a_0$, since $y' < 0$ when $x > a_0$ and $y > d_2$. By (4), $|y'| < kC$ when $y \geqq d_2$ and $-c_0 \leqq x \leqq a_0$; therefore $y(a_0) < d_2 + kC(a_0 + c_0) = kH_0$ say. If $x \geqq a_0$ and $y > d_1$, then

$$(16) \qquad y' < -2kh + K/d_2 < -kh,$$

by (4). Thus T_y meets $y = d_2$ at a point $x < 2H_0/h$, and having once entered $y \leqq d_2$ it cannot leave it again in $x \geqq a_0$, by (16). Thus, whether T_y starts on $x = c_0$ or on $y = d_2$, it meets $x = 2H_0/h$, if at all, in $0 \leqq y \leqq d_2$.

Let a fixed T_Y^1 be chosen, starting at $(0, Y_0)$ where $Y_0 > 0$, having an initial arc Γ_Y which does not meet R_2, cuts $y = d_2$ first to the right of $x = 2H_0/h + (K + 2d_0)/h$ and ends on $x = 0$. By Lemma 10a, T_y is enclosed by Γ_Y^1 up to its first meeting with $x = a_1$ below the x-axis. Hence, as in Theorem 2a, T_y meets the negative y-axis first within Γ^1_Y. It now follows from Theorem 2b that T_v remains enclosed by T^1_Y, save in $|x| \leqq a_1$, until it reenters R_2. It follows that the minimal strip $|x| \leqq C$ containing T^1_Y also contains T_v. Similarly T_v's starting on $y = -d_2$ or $x = -c_0$ remain in a fixed set $|x| \leqq C$.

Since $|y'| < kC$ when $|x| \leqq C$ and $|y| \geqq d_1$, all T_v's starting on yR_2 remain in $|y| \leqq kC$.

Note. It may be proved, as in C and L, § 22, that if $a_0 = 0$, T_v remains ultimately in $|x| \leqq C$, $|y| \leqq C$.

8. Theorems 1, 2, 3 can be extended and modified in a number of ways.

(A) *If the bounds of x in Theorem 3 are not required to be independent of k, f can be a function $f(x, y, t)$, provided that $Q(kK)$ is finite, and that in addition to satisfying (ii) f is uniformly bounded in every closed x-interval, relative to y and t.*[9])

Direct use is made above of "$f =$ function of x alone", only in putting

$$\int_{t_0}^{t_1} f(x(t)) \frac{dx}{dt} dt = \int_{x(t_0)}^{x(t_1)} f(x) dx$$

in the proofs of Lemmas 8a and 8b, and Theorem 1. If in Lemma 8a it is assumed that $x_0 \geqq Q(kK)$, then at points of T_v on $y = 0$, $-\ddot{x} = g(x) - kp(t)$ has the sign of $g(x)$, i.e. of x. Therefore T_v cannot cross $y = 0$ twice before recrossing $x = x_0$: it lies in $y > 0$ up to its furthest point from 0, and on this arc we may put $y = y(x)$, $t = t(x)$. The calculations of Lemma 8a can then be performed in terms of $y(x)$ and $u(x)$; and similarly at other relevant points of the argument, — the details are easily supplied.

(B) The function g can depend on k[10]), $g = g(x, k)$, in Theorems 1, 2, 3 if (iv) is sharpened to

(iv^*) $Q(2hd_1)$, independent of k in $k \geqq k_0$, exists; and if further

(v) g is uniformly bounded in every finite x-interval.

Only trivial changes are needed in the proofs.

9. The convergence theorem, Theorem $2(iv)$ of C and L, can now be proved as in their text, § 12. Since $a_0 = 0$ is assumed, only the 'basic' conditions (i) to (iii) and $(iv)_0$ of the present paper are needed (but $Q(2K)$ must be independent of k if $g = g(x, k)$). It may be noted that the new conditions on g' and g'' imposed in C and L, need only hold in $|x| \leqq B =$ the

[9]) C and L, Theorem 1. The function f is to be continuous in x and y for each t, summable in t for each x and y.

[10]) As throughout C and L.

constant of our Theorem 3. If g is independent of k it is therefore sufficient to assume that
(*vi*) $g'(x) > 0$ in $|x| \leqq B$, (*vii*) $g''(x)$ exists in $|x| \leqq B$, where B is the "C" of theorem 3. It then follows that $g'(x)$ has a positive lower bound, which is all that is needed in the proof.

The methods of C and L can be used to prove the following theorem on desturbances in the force-function. Let the functions f and g satisfy the conditions (*i*) to (*iii*) with $a_0 = 0$, $(iv)_0$, (*vi*) and (*vii*), and let $p_1(t)$ and $p_2(t)$ be bounded summable functions of bounded integral. Let $E(t) = \int_0^t (p_1 - p_2)dt$.

Theorem 4. *Let $x_i(t)$ be, for $i = 1, 2$, any solution of*

$$\ddot{x} + k\dot{x}f(x) + g(x) = kp_i(t). \tag{17}$$

Then if k exceeds a certain k_0, the quantities

$$X = \left(\int_0^t (x_1 - x_2)^2 dt\right)^{\frac{1}{2}} \text{ and } \Theta = \left(\int_0^t (E(t) - E(\tau))^2 d\tau\right)^{\frac{1}{2}}$$

satisfy the inequality $X^2 \leqq C_1 \Theta X + C_2$ for all positive t, where C_1 and C_2 are positive constants; and

$$\int_0^t (\dot{x}_1 - \dot{x}_2)^2 dt \leqq C_3 t^{\frac{1}{2}} X + C_4.$$

If l and L are (positive) lower and upper bounds of g' in $|x| \leqq B$, C_1 can be taken to be L/hl.

Corollary 1. *If $\int_0^t (E(t) - E(\tau))^2 d\tau$ is bounded, then $x_1 - x_2 \to 0$ and $\dot{x}_1 - \dot{x}_2 \to 0$.* For the integral $\int (x_1 - x_2)^2 dt$ being then convergent, the assertions follow from the boundedness of $x(t)$ and $\dot{x}(t)$ (cf. C and L, § 12).

Corollary 2. *If, for all t, $|E(t)| < \varepsilon$, then*

$$\left(t^{-1}\int_0^t (x_1 - x_2)^2 dt\right)^{\frac{1}{2}} \leqq C\varepsilon + O(t^{-1}), \quad \left(t^{-1}\int_0^t (\dot{x}_1 - \dot{x}_2)^2 dt\right)^{\frac{1}{2}} \leqq C\sqrt{\varepsilon} + O(t^{-1}).$$

Proof of Theorem 4. Let $\xi(t) = x_1(t) - x_2(t)$, $F(x) = \int_0^t f(x)dz$, $\Delta F = F(x_1) - F(x_2)$, $\Delta g = g(x_1) - g(x_2)$. From equations (17) we have

$$\ddot{\xi} + k\frac{d}{dt}(\Delta F) + \Delta g = k(p_1(t) - p_2(t)).$$

Therefore

$$\int_0^t (\Delta g)dt = -[\dot{\xi}]_0^t - k[\Delta F]_0^t + kE(t),$$

and

$$\int_0^{t_0} (p_1 - p_2)\int_0^t \Delta g d\tau = -\int_0^{t_0} (p_1 - p_2)(\dot{\xi} + k\Delta F)dt + \tfrac{1}{2}kE^2(t_0) + CE(t_0),$$

$$= -k^{-1}\int_0^{t_0} (\dot{\xi} + k\Delta F)\left(\frac{d}{dt}(\dot{\xi} + k\Delta F) + \Delta g\right)dt + O(1),$$

by (17) and the hypothesis on E,

$$= -\tfrac{1}{2}k^{-1}[(\dot{\xi} + k\Delta F)^2]_0^{t_0} - k^{-1}\int_0^{t_0} (\dot{\xi} + k\Delta F)\Delta g dt + O(1).$$

The first term is bounded by theorems already proved. Further,

$$\int_0^t \dot{\xi}\Delta g dt = \tfrac{1}{2}\left[\xi^2\frac{\Delta g}{\xi}\right]_0^t - \tfrac{1}{2}\int_0^t \xi^2\frac{d}{dt}\left(\frac{\Delta g}{\xi}\right)dt,$$

giving

$$(18)\quad \int_0^t (\dot{\xi} + k\Delta F)\Delta g dt - = \tfrac{1}{2}[\xi\Delta g]_0^t + \int_0^t \xi^2\left(k\frac{\Delta F}{\xi}\frac{\Delta g}{\xi} - \tfrac{1}{2}\frac{d}{dt}\frac{\Delta g}{\xi}\right)dt.$$

From the conditions imposed on g it follows, as in C and L, § 12, that

$\left|\frac{d}{dt}\left(\frac{\Delta g}{\xi}\right)\right| \leqq C_0$, where C_0 is independent of k, and

$$\frac{\Delta F}{\xi} \geqq 2h,\quad \frac{\Delta g}{\xi} \geqq l = \underline{bd}\ g' \text{ in } |x| \leqq B.$$

Therefore the integral on the right of (18) is not less than

$$(19)\qquad (2hkl - C_0)\int_0^t \xi^2 dt \geqq hkl\int_0^t \xi^2 dt$$

if $k \geqq C_0/hl$.

We have also

$$\left(\int_0^{t_0} (p_1 - p_2)\int_0^t \Delta g d\tau\right)^2 = \left(\int_0^{t_0} (E(t_0) - E(t))\Delta g dt\right)^2$$

$$\leqq \int_0^{t_0} (E(t_0) - E(t))^2 dt \int_0^{t_0}\left(\frac{\Delta g}{\xi}\right)\xi^2 dt$$

$$\leqq L^2\int_0^{t_0}(E(t_0) - E(t))^2 dt \int_0^{t_0} \xi^2 dt.$$

Combining this with our other inequalities we have

$$L\left(\int_0^{t_0} (E(t_0)-E(t))^2dt\right)^{\frac{1}{2}}\left(\int_0^{t_0}\xi^2dt\right)^{\frac{1}{2}} \geqq hl\int_0^{t_0}\xi^2dt - \text{const.},$$

that is, $L\Theta X \geqq hlX^2 - \text{const.}$

Finally

$$\int_0^t \dot\xi^2dt = [\xi\dot\xi]_0^t - \int_0^t \xi\ddot\xi dt \leqq C_4 + C_3\int_0^t |\xi|\,dt,$$

since ξ, $\dot\xi$ and $\ddot\xi$ are bounded,

$$\leqq C_4 + C_3\left(t\int_0^t \xi^2dt\right)^{\frac{1}{2}}.$$

This completes the proof of Theorem 4.

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