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Classification and geometrical applications**

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# On ordinary quantities and $W$ -quantities

Classification and geometrical applications

by

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Delft

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## 1. Interior and exterior orientation.

We consider an  $E_n$ <sup>1)</sup> in which an orientation ( $n$ -dimensional screw-sense) is determined by an ordered sequence of  $n$  independent directions each with a definite sense<sup>2)</sup>. Now suppose an  $E_p$ ,  $0 < p < n$  to be given in  $E_n$ . This  $E_p$  in  $E_n$  determines uniquely an  $E_{n-p}$  (not lying in  $E_n$ ) in the following way: all  $E_p$ 's totally parallel with the given one can be considered as elements of a set, which is an  $(n-p)$ -dimensional plane manifold in which an affine geometry is induced, i.e. an  $E_{n-p}$ . This process of obtaining the  $E_{n-p}$  is called „Zusammenlegung“ by Weyl (analogous to „stetige Zerlegung“ in topology) and is also described by saying that in each  $E_p$  all points are „identified“.

Now we can either define a  $p$ -dimensional orientation in the  $E_p$  or an  $(n-p)$ -dimensional orientation in the  $E_{n-p}$ . In the first case we say that the  $E_p$  as well as its  $p$ -direction has got an *interior orientation*, in the second case we say that it is provided with an *exterior orientation*. The notions of interior and exterior orientation were introduced by Veblen and Whitehead<sup>3)</sup>. For  $p = 0$  and  $p = n$  we define the orientation as follows:

The interior orientation of an  $E_0$  is a  $+$  or a  $-$ -sign, the exterior one is an ordinary orientation of the  $E_n$ . The interior orientation of the  $E_n$  is just this ordinary orientation, the exterior one is a  $+$  or  $-$ -sign.

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1)  $E_n = n$ -dimensional space with ordinary affine geometry.

2) Comp. E. I (Einführung in die neueren Methoden der Differentialgeometrie by J. A. SCHOUTEN and D. J. STRUIK, Vol I [Noordhoff 1935]), p. 16.

3) O. VEULEN & J. H. C. WHITEHEAD, The foundations of differential geometry [Cambr. Tracts in Math. 29 (1932)], 55, 56.

**2. Contra- and covariant  $p$ -vectors. <sup>4)</sup>**

Let  $x^\kappa$  ( $\kappa, \lambda, \mu, \nu, \pi, \varrho, \sigma, \tau=1, \dots, n$ ) be cartesian coordinates in an  $E_n$ . Any other system  $x^{\kappa'}$  ( $\kappa', \lambda', \dots, \tau'=1', \dots, n'$ ) of cartesian coordinates in this  $E_n$  is connected with the first one by equations of the form:

$$(1) \quad x^{\kappa'} = A_{\kappa}^{\kappa'} x^\kappa + A_0^{\kappa'}; \quad \text{Det} (A_{\kappa}^{\kappa'}) \neq 0$$

where the  $A_{\kappa}^{\kappa'}$  and  $A_0^{\kappa'}$  are constants.

A *contravariant*  $p$ -vector  $v^{\kappa_1 \dots \kappa_p}$  is defined by its transformation formula

$$(2) \quad v^{\kappa'_1 \dots \kappa'_p} = A_{\kappa_1}^{\kappa'_1} \dots A_{\kappa_p}^{\kappa'_p} v^{\kappa_1 \dots \kappa_p}$$

and its property of being alternating with respect to all suffixes. If it is simple (i.e. the alternated product of  $p$  vectors) it can be represented by a (e.g. simply connected) part of an  $E_p$  with an *interior* orientation. Two such parts determine the same  $p$ -vector if and only if 1<sup>o</sup> the  $E_p$ 's are totally parallel, 2<sup>o</sup> the two  $p$ -dimensional volumes are equal, 3<sup>o</sup> the orientations are the same. The  $(\kappa_1, \dots, \kappa_p)$ -component is determined by the projection of the oriented part of the  $E_p$  upon the  $E_p$  of the contravariant

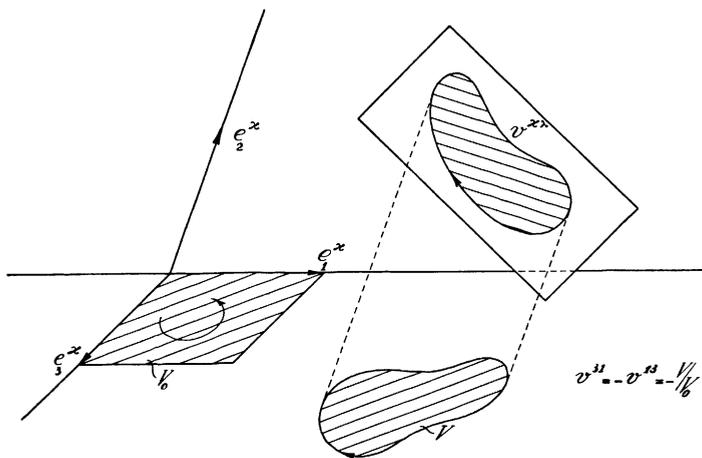


Fig. 1 <sup>5)</sup>.

measuring vectors  $e^{\kappa_1}, \dots, e^{\kappa_p}$ ; the  $(n-p)$ -direction in which the

<sup>4)</sup> The results of § 2 — 5 were first published in J. A. SCHOUTEN, Über die geometrische Deutung von gewöhnlichen  $p$ -Vektoren und  $W$ - $p$ -Vektoren und den korrespondierenden Dichten. [Proc. Amsterdam, 41 (1938), 709—716].

<sup>5)</sup> The orientation of the parallelogram on  $e^{\kappa_1}, e^{\kappa_2}$  is the one belonging to  $e_3^{[\kappa_1 \kappa_2]}$ .

projection is performed is the  $(n-p)$ -direction common to the  $(n-1)$ -directions of the covariant measuring vectors  $e_{\lambda}^{\kappa_1}, \dots, e_{\lambda}^{\kappa_p}$ . The value of this  $(\kappa_1, \dots, \kappa_p)$ -component is the  $p$ -dimensional volume of the projection as measured by the  $p$ -dimensional volume of the parallelotope with edges  $e_{\kappa_1}^{\kappa}, \dots, e_{\kappa_p}^{\kappa}$  and provided with a factor  $+1$  or  $-1$  if its orientation is the same or opposite as the orientation of  $e_{\kappa_1}^{\kappa}, \dots, e_{\kappa_p}^{\kappa}$  in this order.

The projection of the orientation becomes undetermined if and only if the  $E_p$  of  $v^{\kappa_1} \dots v^{\kappa_p}$  has a direction in common with the  $(n-p)$ -direction of the projection, in which case the volume of the projection is zero.

A covariant  $p$ -vector  $w_{\lambda_1 \dots \lambda_p}$  is defined by its transformation formula

$$(3) \quad w_{\lambda'_1 \dots \lambda'_p} = A_{\lambda'_1}^{\lambda_1} \dots A_{\lambda'_p}^{\lambda_p} w_{\lambda_1 \dots \lambda_p}$$

and its property of being alternating. If it is simple it can be represented by a cylinder (the interior of which may be chosen simply connected) consisting of  $\infty^{p-1}$  ( $2$  for  $p=1$ ) totally parallel  $E_{n-p}$ 's (the generators) with an exterior orientation of their  $(n-p)$ -direction. Hence the set of  $\infty^{p-1}$   $E_{n-p}$ 's is oriented. Two such cylinders determine the same  $p$ -vector if and only if  $1^0$  their  $(n-p)$ -directions are the same,  $2^0$  they intersect from

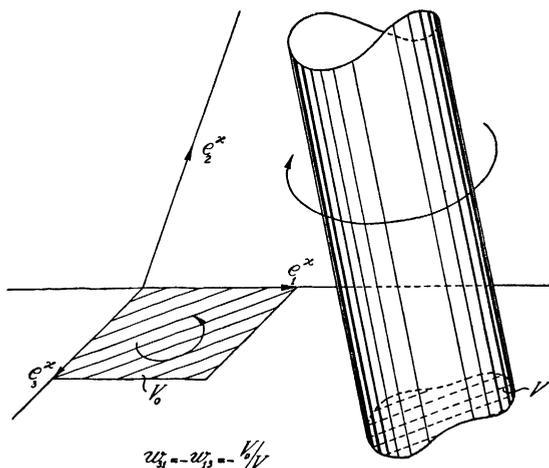


Fig. 2<sup>5</sup>).

one (and then from every)  $E_p$  which has no direction in common with the  $(n-p)$ -direction, parts with equal  $p$ -dimensional volumes,

and  $\mathfrak{B}^0$  the orientations are the same. The  $\lambda_1 \dots \lambda_p$ -component is the reciprocal value of the  $p$ -dimensional volume of the intersection of the cylinder with the  $E_p$  of  $e^{\lambda_1}, \dots, e^{\lambda_p}$ , as measured by the parallelotope of these vectors. It is positive or negative if the orientation of  $w_{\lambda_1 \dots \lambda_p}$  has the same or the opposite sense resp. as the orientation of  $e^{\lambda_1}, \dots, e^{\lambda_p}$  in this order.

**3. Ordinary contra- and covariant  $p$ -vectordensities.**

An ordinary contra- or covariant  $p$ -vectordensity of weight  $w$  is defined by its transformation-formula

$$(4) \quad v^{\lambda'_1 \dots \lambda'_p} = \Delta^{-w} A^{\lambda'_1}_{\lambda_1} \dots A^{\lambda'_p}_{\lambda_p} v^{\lambda_1 \dots \lambda_p}; \quad \Delta = \text{Det} (A^{\lambda'}_{\lambda})$$

$$(5) \quad w_{\lambda'_1 \dots \lambda'_p} = \Delta^{-w} A^{\lambda_1}_{\lambda'_1} \dots A^{\lambda_p}_{\lambda'_p} w_{\lambda_1 \dots \lambda_p}$$

resp. and by its property of being alternating. For  $p = 0$  we get ordinary scaldensities of weight  $w$ .

In an  $E_n$  three quantities are given a priori:

A. The unit affinor  $A^{\lambda}_{\lambda}$  with the components

$$(6) \quad A^{\lambda}_{\lambda} = \begin{cases} 1, & \lambda = \lambda \\ 0, & \lambda \neq \lambda \end{cases};$$

with respect to every system of coordinates.

B. The contravariant unit  $n$ -vectordensity  $\mathfrak{G}^{\lambda_1 \dots \lambda_n}$  of weight  $+1$  defined by

$$(7) \quad \mathfrak{G}^{1 \dots n} = +1; \quad \mathfrak{G}^{\lambda_1 \dots \lambda_n} = \mathfrak{G}^{[\lambda_1 \dots \lambda_n]}$$

with respect to every system of coordinates;

C. The covariant unit  $n$ -vectordensity  $e_{\lambda_1 \dots \lambda_n}$  of weight  $-1$  defined by

$$(8) \quad e_{1 \dots n} = +1; \quad e_{\lambda_1 \dots \lambda_n} = e_{[\lambda_1 \dots \lambda_n]}$$

with respect to every system of coordinates.

Hence a one to one correspondence exists between the set of all contravariant  $p$ -vectors and the set of all covariant  $(n-p)$ -vectordensities of weight  $-1$  and also between the set of all covariant  $p$ -vectors and the set of all contravariant  $(n-p)$ -vectordensities of weight  $+1$ :

$$(9) \quad v_{\lambda_1 \dots \lambda_{n-p}} = \frac{1}{p!} e_{\lambda_1 \dots \lambda_{n-p} \lambda_1 \dots \lambda_p} v^{\lambda_1 \dots \lambda_p}; \quad v^{\lambda_1 \dots \lambda_p} = \frac{1}{(n-p)!} v_{\lambda_1 \dots \lambda_{n-p}} \mathfrak{G}^{\lambda_1 \dots \lambda_{n-p} \lambda_1 \dots \lambda_p}$$

$$(10) \quad w^{\lambda_1 \dots \lambda_{n-p}} = \frac{1}{p!} w_{\lambda_1 \dots \lambda_p} \mathfrak{G}^{\lambda_1 \dots \lambda_{n-p} \lambda_1 \dots \lambda_p}; \quad w_{\lambda_1 \dots \lambda_p} = \frac{1}{(n-p)!} e_{\lambda_1 \dots \lambda_p \lambda_1 \dots \lambda_{n-p}} w^{\lambda_1 \dots \lambda_{n-p}}$$

In particular with a scalar  $p$  correspond both a co- and a contra-variant  $n$ -vectordensity

$$(11) \quad \begin{cases} p'_{\lambda_1 \dots \lambda_n} = p e_{\lambda_1 \dots \lambda_n}; & p = p'_{1 \dots n} = \frac{1}{n!} p'_{\lambda_1 \dots \lambda_n} \mathfrak{E}^{\lambda_1 \dots \lambda_n} \\ p^{\kappa_1 \dots \kappa_n} = p \mathfrak{E}^{\kappa_1 \dots \kappa_n}; & p = p^{\lambda \dots n} = \frac{1}{n!} p^{\kappa_1 \dots \kappa_n} e_{\kappa_1 \dots \kappa_n}. \end{cases}$$

Of course the geometrical representations of corresponding quantities are the same. For densities of other weights no such simple geometrical representations exist.

Table 1 shows the quantities considered here for  $n = 3$ .

Table 1.

Figure	Notation 1	Notation 2	Number of independent components	Orientation
none	$p$ ; scalar	$\begin{cases} p'_{\mu\lambda\kappa}; \text{ cov. triv. dens. } w = -1 \\ p^{\kappa\lambda\mu}; \text{ contr. triv. dens. } w = +1 \end{cases}$	1	none
	$v^\kappa$ ; contr. vect.	$v_{\lambda\kappa}$ ; cov. biv. dens. $w = -1$	3 (proj.)	interior
	$w_\lambda$ ; cov. vect.	$w^{\kappa\lambda}$ ; contr. biv. dens. $w = +1$	$3 \left( \frac{1}{\text{inters.}} \right)$	exterior
	$f^{\kappa\lambda}$ ; contr. biv.	$f_\lambda$ ; cov. vect. dens. $w = -1$	3 (proj.)	interior
	$h_{\lambda\kappa}$ ; cov. biv.	$h^\kappa$ ; contr. vect. dens. $w = +1$	$3 \left( \frac{1}{\text{inters.}} \right)$	exterior
	$\begin{cases} p^{\kappa\lambda\mu}; \text{ contr. triv.} \\ q_{\mu\lambda\kappa}; \text{ cov. triv.} \end{cases}$	$\begin{cases} p; \text{ scalardens. } w = -1 \\ q; \text{ scalardens. } w = +1 \end{cases}$	$\begin{cases} 1 \text{ (vol.)} \\ 1 \left( \frac{1}{\text{vol.}} \right) \end{cases}$	screw

#### 4. $W$ -quantities.

Aside ordinary densities we can also consider densities, in the transformation-formulae of which the *absolute value*  $|\Delta|$  is taken instead of  $\Delta$  itself. As they were introduced by H. Weyl<sup>6)</sup> we call them  $W$ -densities and distinguish them from ordinary densities by a  $\bar{\phantom{x}}$  above the central letter. If the weight is  $w$ , their transformation formulae are

<sup>6)</sup> RZM, § 13, 4th Aufl., 98.

$$(12) \quad \bar{v}^{\kappa'_1 \dots \kappa'_p} = |\Delta|^{-w} A^{\kappa'_1}_{\kappa_1} \dots A^{\kappa'_p}_{\kappa_p} \bar{v}^{\kappa_1 \dots \kappa_p}$$

$$(13) \quad \bar{v}_{\lambda'_1 \dots \lambda'_p} = |\Delta|^{-w} A^{\lambda'_1}_{\lambda_1} \dots A^{\lambda'_p}_{\lambda_p} \bar{v}_{\lambda_1 \dots \lambda_p}.$$

For  $p = 0$  we get  $W$ -scalars densities. As long as we consider only (real) transformations with  $\Delta > 0$ , there is no difference between  $W$ -densities and ordinary densities. But this restriction is equivalent with giving an orientation in  $X_n$ , viz. the orientation of  $e^{\kappa_1}, \dots, e^{\kappa_n}$  in this order<sup>7)</sup>. Hence the geometric interpretation of a  $W$ -density can only differ from the interpretation of a corresponding ordinary density by the orientation. Hence it is to be expected that we may, at least in the cases  $w = +1$  resp.  $w = -1$  where we have simple geometrical representations, obtain  $W$ -densities from ordinary densities by interchanging *interior* and *exterior* orientations.

Take for instance a part of an  $E_p$  with an *exterior* orientation and fix the rules about equivalence and building of components in the same way as for an ordinary  $p$ -vector, the  $(\kappa_1 \dots \kappa_p)$ -component being positive or negative if the projection of the *exterior* orientation has the same or the opposite sense resp. as the orientation of  $e^{\kappa_{p+1}}, \dots, e^{\kappa_n}$  in this order, where  $\kappa_1, \dots, \kappa_n$  is an even permutation of  $1, \dots, n$ . Hence, if the quantity thus defined, which we denote by  $\bar{v}^{\kappa_1 \dots \kappa_p}$ , has the same components as an ordinary contravariant  $p$ -vector with respect to one system of coordinates, this will be true also for all systems of coordinates that can be deduced from the first one by transformations with  $\Delta > 0$ , but the sign changes if we take a transformation with  $\Delta < 0$ . From this follows the transformation-formula of  $\bar{v}^{\kappa_1 \dots \kappa_p}$

$$(14) \quad \bar{v}^{\kappa'_1 \dots \kappa'_p} = \Delta |\Delta|^{-1} A^{\kappa'_1}_{\kappa_1} \dots A^{\kappa'_p}_{\kappa_p} \bar{v}^{\kappa_1 \dots \kappa_p}$$

and, if we define  $\bar{v}_{\lambda_1 \dots \lambda_{n-p}}$  by

$$(15) \quad \bar{v}_{\lambda_1 \dots \lambda_{n-p}} = \frac{1}{p!} e_{\lambda_1 \dots \lambda_{n-p} \kappa_1 \dots \kappa_p} \bar{v}^{\kappa_1 \dots \kappa_p},$$

also:

$$(16) \quad v_{\lambda'_1 \dots \lambda'_{n-p}} = |\Delta| A^{\lambda_1}_{\lambda'_1} \dots A^{\lambda_{n-p}}_{\lambda'_{n-p}} v_{\lambda_1 \dots \lambda_{n-p}}.$$

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<sup>7)</sup> Cf. VEBLEN & WHITEHEAD l.c.

$\bar{v}_{\lambda_1 \dots \lambda_{n-p}}$  is a covariant  $W$ -( $n-p$ )-vectordensity of weight  $-1$ . Accordingly the quantity  $\bar{v}^{\alpha_1 \dots \alpha_p}$  will be called a contravariant  $W$ - $p$ -vector.

In the same way we define a covariant  $W$ - $p$ -vector  $\bar{w}_{\lambda_1 \dots \lambda_p}$  by its transformation-formula

$$(17) \quad \bar{w}_{\lambda'_1 \dots \lambda'_p} = \Delta \left| \Delta \right|^{-1} A_{\lambda'_1}^{\lambda_1} \dots A_{\lambda'_p}^{\lambda_p} \bar{w}_{\lambda_1 \dots \lambda_p}$$

and its alternating property. This quantity is geometrically equivalent with the contravariant  $W$ -( $n-p$ )-vectordensity

$$(18) \quad \bar{w}^{\alpha_1 \dots \alpha_{n-p}} = \frac{1}{p!} \bar{w}_{\lambda_1 \dots \lambda_p} \mathfrak{G}^{\lambda_1 \dots \lambda_p \alpha_1 \dots \alpha_{n-p}}$$

of weight  $+1$  and can, if it is simple, be represented by a cylinder (the interior of which may be chosen simply connected) consisting of  $\infty^{p-1}$  totally parallel  $E_{n-p}$ 's with an *interior* orientation of their ( $n-p$ )-direction. The rules for equivalence and the building of components are the same as in the case of the ordinary covariant  $p$ -vector; the component is positive or negative if the orientation of  $\bar{w}_{\lambda_1 \dots \lambda_p}$  has the same or the opposite sense resp. as the orientation of  $e^{\alpha_1}, \dots, e^{\alpha_n}$  in this order, where  $\lambda_1, \dots, \lambda_n$  is an even permutation of  $1, \dots, n$ .

For  $p = 0$  we get  $W$ -scalars with the transformation-formula

$$(19) \quad \bar{p}^{(\alpha)} = \Delta \left| \Delta \right|^{-1} \bar{p}^{(\alpha)}$$

With  $\bar{p}$  correspond both a co- and a contravariant  $W$ - $n$ -vectordensity of weight  $-1$  and  $+1$  resp.:

$$(20) \quad \begin{aligned} \bar{p}'_{\lambda_1 \dots \lambda_n} &= \bar{p} e_{\lambda_1 \dots \lambda_n}; \quad \bar{p} = \frac{1}{n!} \bar{p}'_{\lambda_1 \dots \lambda_n} \mathfrak{G}^{\lambda_1 \dots \lambda_n} \\ \bar{p}^{\alpha_1 \dots \alpha_n} &= \bar{p} \mathfrak{G}^{\alpha_1 \dots \alpha_n}; \quad \bar{p} = \frac{1}{n!} \bar{p}^{\alpha_1 \dots \alpha_n} e_{\alpha_1 \dots \alpha_n}. \end{aligned}$$

Contrary to an ordinary scalar a  $W$ -scalar has an ( $n$ -dimensional) orientation, viz. the orientation of the coordinate system with respect to which  $p$  is positive. But a  $W$ -scalardensity of weight  $+1$  or  $-1$  has no orientation because it is equivalent with a covariant or contravariant  $W$ - $n$ -vector resp. and is represented by an  $n$ -dimensional volume without orientation.

Table 2 shows the  $W$ -quantities considered here for  $n = 3$ .

Table 2.

Figure	Notation 1	Notation 2	Components	Orientation
none	$\bar{p}$ ; $W$ -scalar	$\bar{p}'_{\mu\lambda\kappa}$ ; cov. $W$ -triv.d.; $w = -1$ $\bar{p}^{\kappa\lambda\mu}$ ; contr. $W$ -triv.d.; $w = +1$	1	screw
	$\bar{v}^\kappa$ ; contr. $W$ -vect.	$\bar{v}_{\lambda\kappa}$ ; cov. $W$ -biv.d.; $w = -1$	3 (proj.)	outside
	$\bar{w}_\lambda$ ; cov. $W$ -vect.	$\bar{w}^{\lambda\kappa}$ ; contr. $W$ -biv.d.; $w = +1$	3 ( $\frac{1}{\text{inters.}}$ )	inside
	$\bar{f}^{\lambda\kappa}$ ; contr. $W$ -biv.	$\bar{f}_\lambda$ ; cov. $W$ -vect.d.; $w = -1$	3 (proj.)	outside
	$\bar{h}_{\lambda\kappa}$ ; cov. $W$ -biv.	$\bar{h}^\kappa$ ; contr. $W$ -vect.d.; $w = +1$	3 ( $\frac{1}{\text{inters.}}$ )	inside
	$\bar{p}^{\kappa\lambda\mu}$ ; contr. $W$ -triv. $\bar{q}_{\mu\lambda\kappa}$ ; cov. $W$ -triv.	$\bar{p}$ ; $W$ -dens.; $w = -1$ $\bar{q}$ ; $W$ -dens.; $w = +1$	1 (vol.) 1 ( $\frac{1}{\text{vol.}}$ )	} none

An example of a  $W$ -scalar is:  $p = +1$  for all right-handed systems and  $p = -1$  for all left-handed systems. Quantities of this kind are called sometimes „pseudoscalars” in physics.  $W$ -vectors are occasionally used in physics but only after the introduction of a metric <sup>8)</sup>. Mr. St. GoLab <sup>9)</sup> has proved by solving a functional equation, that *all* geometric objects with only one component, whose transformation depends on  $\Delta$  only, can be deduced from the four objects: scalars,  $W$ -scalars, ordinary scalardensities and  $W$ -scalardensities. From this theorem follows that the classification we have used here is really exhaustive.

### 5. Identification of quantities.

After introducing a unit of volume, a metric or an  $n$ -dimensional orientation identifications arise between the different quantities derived. We take the case  $n = 3$  as an illustration, the generalisation being obvious. (Cf. table 3)

The four directed quantities occuring after introduction of

<sup>8)</sup> For applications without a metric comp. D. v. DANTZIG, On the phenomenological thermodynamics, [Physica 6 (1939), 673—704]; On relativistic thermodynamics, [Proc. Amsterdam, 42 (1939) 601—607]; On relativistic gas theory, [l.c., 608—625].

<sup>9)</sup> ST. GOLAB, Über die Klassifikation der geometrischen Objekte [Math. Zeitschr. 44 (1938) 104—114].

a unit of volume, are known in literature (from left to right) as polar vector, polar bivector, axial bivector, axial vector. After introduction of a metric the difference between polar and

Table 3.

	$v^x$ $v_{\lambda x}$	$w_\lambda$ $w^{\lambda x}$	$f^{xx}$ $f_\lambda$	$h_{\lambda x}$ $h^x$	$\left\{ \begin{matrix} p^{xx\mu} \\ q \\ \mu_{\lambda x} \\ f, q \end{matrix} \right.$	$\bar{v}^x$ $\bar{v}_{\lambda x}$	$\bar{w}_\lambda$ $\bar{w}^{\lambda x}$	$\bar{f}^{xx}$ $\bar{f}_\lambda$	$\bar{h}_{\lambda x}$ $\bar{h}^x$	$\left\{ \begin{matrix} \bar{p}^{xx\mu} \\ \bar{q} \\ \bar{\mu}_{\lambda x} \\ \bar{f}, \bar{q} \end{matrix} \right.$
After introduction of.										
I. Unit of volume					$w_\lambda$ scalar					Scalar
II. Screwsense										
III. Screwsense and unit of volume					Scalar					Scalar
IV. Metric					$w_\lambda$ scalar					Scalar
V. Screwsense and metric					Scalar					Scalar

axial and between scalars and  $W$ -scalars remains. This is the point of view often found in publications on physics. After introduction of a screw-sense and a metric (this includes I, II, III and IV) all differences between directed quantities and the difference between scalars and  $W$ -scalars vanish.

### 6. Quantities in $X_n$ .

We consider an  $n$ -dimensional differentiable manifold  $X_n$ . Let  $\xi^x(x, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau = 1, \dots, n)$  be a set of coordinates in a sufficiently small part of this  $X_n$  and  $\xi^{x'}(x', \lambda', \dots, \tau' = 1', \dots, n')$  another set of coordinates with

$$(21) \quad \Delta = \text{Det} (A_\lambda^{x'}) \neq 0; A_\lambda^{x'} = \partial_\lambda \xi^{x'}; \partial_\lambda = \frac{\partial}{\partial \xi^\lambda}.$$

It is well known that we define all quantities in the local  $E_n$  of a point of  $X_n$  using in the formulae of transformation  $A_\lambda^{x'} = \partial_\lambda \xi^{x'}$  in stead of the constants  $A_\lambda^{x'}$  in (1). In this way we get in each local  $E_n$  ordinary quantities and  $W$ -quantities.

7. Imbedding of an  $X_m$  in an  $X_n$ .<sup>10)</sup>

If an  $X_m$  is imbedded in an  $X_n$  the following eight cases are important. We write  $k$  for  $n - m$  and suppose  $\eta^a$  ( $a, b, \dots, g = 1, \dots, m$ ) to be coordinates in the  $X_m$ . Further we denote by  $B_b^a$  the unity-affinor in  $X_m$ , by  $\mathfrak{G}'^{a_1 \dots a_m}$  and  $e'_{b_1 \dots b_m}$  the unit- $m$ -vectordensities in  $X_m$  and by  $D$  the transformation-modulus  $D = \det (B_b^{a'})$  in  $X_m$ .

CASE 1. *Pure imbedding without any auxiliary assumptions.*

To every point  $\eta^a$  of the  $X_m$  belongs one and only one point of the  $X_n$ , given by equations of the form

$$(22) \quad \xi^x = \xi^x(\eta^1, \dots, \eta^m).$$

In every point of the  $X_m$  two quantities exist:

1<sup>o</sup> the affinor

$$(23) \quad B_b^x = \frac{\partial \xi^x}{\partial \eta^b},$$

a connecting quantity behaving like a system of  $m$  contravariant vectors with respect to transformations in  $X_n$  and like a system of  $n$  covariant vectors with respect to transformations in  $X_m$ ;

2<sup>o</sup> the simple  $k$ -vectordensity

$$(24) \quad t_{\lambda_1 \dots \lambda_k} = \frac{1}{m!} e_{\lambda_1 \dots \lambda_k} x_1 \dots x_m B_{b_1}^{x_1} \dots B_{b_m}^{x_m} \mathfrak{G}'^{b_1 \dots b_m},$$

a connecting quantity with the weights  $+1$  and  $-1$  with respect to transformations in  $X_m$  and  $X_n$  respectively:

$$(25) \quad t_{\lambda'_1 \dots \lambda'_k} = \Delta D^{-1} A_{\lambda'_1}^{\lambda_1} \dots A_{\lambda'_k}^{\lambda_k} t_{\lambda_1 \dots \lambda_k}.$$

$B_b^x$  as well as  $t_{\lambda_1 \dots \lambda_k}$  are geometrically represented by the tangent  $E_m$  in the local  $E_n$ . They are related by the identity

$$(26) \quad B_b^{\lambda_1} t_{\lambda_1 \dots \lambda_k} = 0.$$

CASE 1'. *Imbedding with exterior orientation.*

The orientation is an exterior orientation of the tangent  $E_m$  in every point of the  $X_m$  and can be given by any quantity  $\omega$  with the absolute value  $+1$  and the transformation-formula

<sup>10)</sup> The eight different cases of imbedding were first treated in J. A. SCHOUTEN, Über die Beziehungen zwischen den geometrischen Größen in einer  $X_n$  und in einer in der  $X_n$  eingebetteten  $X_m$ , [Proc. Amsterdam, 41 (1938) 568—575].

$$(27) \quad (\omega'^{a'}) = \Delta^{-1} | \Delta | D | D |^{-1} (\omega^a).$$

In fact, because of (25), (27) the quantity  $\omega t_{\lambda_1 \dots \lambda_k}$  has the transformation-formula

$$(28) \quad (\omega'^{a'}) t_{\lambda'_1 \dots \lambda'_k} = | \Delta | | D |^{-1} A_{\lambda'_1}^{\lambda_1} \dots A_{\lambda'_k}^{\lambda_k} \omega t_{\lambda_1 \dots \lambda_k},$$

and this is the same as that of a covariant  $k$ -vector except for an always *positive* factor. Hence  $\omega t_{\lambda_1 \dots \lambda_k}$  determines a  $k$ -dimensional orientation in every  $E_k$  having no direction in common with the tangent  $E_m$ . If in any point of the  $X_m$  the orientation of  $e^a, \dots, e^a$  in this order followed by the exterior orientation gives the orientation of  $e^x, \dots, e^x$  in this order, then we choose in that point always  $\omega^{(x,a)} = +1$ .

In each  $E_k$  having no direction in common with the tangent  $E_m$  in case 1' a contravariant alternating quantity  $\pi^{x_1 \dots x_k}$  with the transformation-formula

$$(29) \quad \pi^{x'_1 \dots x'_k} = \Delta^{-1} | \Delta | D | D |^{-1} A_{x'_1}^{x_1} \dots A_{x'_k}^{x_k} \pi^{x_1 \dots x_k},$$

satisfying the invariant condition

$$(30) \quad \pi^{x_1 \dots x_k} t_{x_1 \dots x_k} > 0,$$

is determined except for a *positive* factor. Except for a factor  $\pm 1$  this quantity transforms in the same way as a contravariant  $k$ -vector. Hence it determines such a  $k$ -vector except for the orientation. Now suppose a  $k$ -vector  $p^{x_1 \dots x_k}$  is given except for a factor  $\pm 1$ . Then we can determine a quantity  $\pi^{x_1 \dots x_k}$  by the equation

$$(31) \quad \begin{aligned} \pi^{x_1 \dots x_k} &= \omega p^{x_1 \dots x_k} \operatorname{sgn} (\omega p^{x_1 \dots x_k} t_{x_1 \dots x_k}) \\ &= p^{x_1 \dots x_k} \operatorname{sgn} (p^{x_1 \dots x_k} t_{x_1 \dots x_k}), \end{aligned}$$

(where  $\operatorname{sgn} (z) = \frac{z}{|z|}$  for  $z \neq 0$  and  $\operatorname{sgn} (0) = 0$ ). From this follows, that the geometric representation of a quantity with the transformation (29) satisfying the condition (30) is a part of an  $E_k$  having neither an interior nor an exterior orientation.

CASE 2. *Rigged imbedding.*

An  $X_m$  in an  $X_n$  is called „rigged” („ingespannt”) if in every point of  $X_m$  a  $k$ -direction is given, having no direction in common

with the tangent  $E_m$ . This  $k$ -direction can be given by an affiner  $B_\lambda^\kappa$  with the following properties: 1<sup>o</sup>. its  $\kappa$ -region consists of all contravariant vectors of the local  $E_m$  and its  $\lambda$ -region consists of all covariant vectors whose  $(n-1)$ -direction contains the  $k$ -direction of the rigging, 2<sup>o</sup>:

$$(32) \quad B_\rho^\kappa B_\lambda^\rho = B_\lambda^\kappa.$$

From  $B_b^\kappa$  and  $B_\lambda^\kappa$  an affiner  $B_\lambda^a$  can be uniquely determined by means of the equations

$$(33a) \quad B_\lambda^a B_b^\lambda = B_b^a = \text{unitaffiner of the } X_m.$$

$$(33b) \quad B_\lambda^b B_b^\kappa = B_\lambda^\kappa.$$

The  $\lambda$ -region of  $B_\lambda^a$  is the same as the  $\lambda$ -region of  $B_\lambda^\kappa$ .  $B_b^\kappa$  and  $B_\lambda^a$  together determine  $B_\lambda^\kappa$  uniquely by (33b). The rigging determines also uniquely and is determined by the simple  $k$ -vectordensity

$$(34a) \quad e^{\kappa_1 \dots \kappa_k} = \frac{1}{m!} \mathfrak{G}^{\lambda_1 \dots \lambda_m \kappa_1 \dots \kappa_k} B_{\lambda_1}^{a_1} \dots B_{\lambda_m}^{a_m} e'_{a_1 \dots a_m},$$

a connecting quantity with the weights  $-1$  and  $+1$  with respect to transformations in  $X_m$  and  $X_n$  resp. and satisfying the relations

$$(34b) \quad e^{\kappa_1 \dots \kappa_k} B_{\kappa_1}^a = 0; \quad \frac{1}{k!} e^{\kappa_1 \dots \kappa_k} t_{\kappa_1 \dots \kappa_k} = 1.$$

In the cases 1 and 1'  $B_\lambda^a$  can not be uniquely determined, as then (34) fails, and the solutions of (33) alone contain arbitrary parameters. But they determine  $B_\lambda^a$  partly and well enough in order that the expression  $B_{\kappa_1}^{a_1} \dots B_{\kappa_p}^{a_p} v^{\kappa_1 \dots \kappa_p}$  be uniquely determined if  $v^{\kappa_1 \dots \kappa_p}$  is a simple contravariant  $p$ -vector,  $p \leq m$ , parallel to the tangent  $E_m$ .

We often make use of a special system of coordinates for which in every point of the  $X_m$   $\xi^l = \eta^1, \dots, \xi^m = \eta^m$ <sup>11</sup>). If the  $X_m$  is rigged we choose the parameterlines of  $\xi^{m+1}, \dots, \xi^n$  in such a way that they have in every point of the  $X_m$  a direction lying

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<sup>11)</sup> Coordinate systems of this kind are transformed into each other by transformations of a group studied by A. KAWAGUCHI, The foundation of the theory of displacements II [Proc. Imp. Academy 10 (1934), 45—48]. From this remark follow the relations between our paper and the investigations of KAWAGUCHI, S. HOKARI, Über die Übertragungen, die der erweiterten Transformationsgruppe angehören [Journ. Hokkaido Imp. Univ. 3 (1935), 15—26; 4 (1935) 14—50], S. GOLAB, Über eine Art der Geometrie von Kawaguchi-Hokari [Ann. Soc. Polon. Math. 16 (1937) 25—30], and the investigations of T. HOSOKAWA, A. WUNDHEILER, V. HLATATY, E. CARTAN, T. Y. THOMAS, J. A. SCHOUTEN and S. GOLAB, quoted in this latter paper.

in the  $k$ -direction of the rigging. If we use  $\xi^\alpha$  ( $\alpha, \dots, \delta=1, \dots, m$ ) as coordinates in the  $X_m$  we have for that system

$$(35) \quad B_\beta^\alpha = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases} \quad B_\beta^{m+1} = \dots = B_\beta^n = 0,$$

$$(36) \quad t_{m+1, \dots, n} = 1; \quad e^{m+1, \dots, n} = 1$$

and all components of  $t_{\lambda_1 \dots \lambda_k}$  and  $e^{\lambda_1 \dots \lambda_k}$  with a suffix  $\leq m$  vanish.  $\omega = +1$  if  $e^{\lambda_1 \dots \lambda_k}$  in this order determine the exterior orientation of the tangent  $E_m$  and  $-1$  in the other case.

CASE 2'. *Rigged imbedding with exterior orientation.*

This case is a combination of 1' and 2 and requires  $\omega$  as well as  $B_\lambda^\lambda$  to be given.

With respect to the special coordinate system mentioned above the condition (30) for the non oriented quantity  $\pi^{\lambda_1 \dots \lambda_k}$  is

$$(37) \quad \pi^{1+m, \dots, n} > 0$$

and the equation (31) takes the form

$$(38) \quad \pi^{\lambda_1 \dots \lambda_k} = p^{\lambda_1 \dots \lambda_k} \text{ sign } (p^{m+1, \dots, n}).$$

CASE 3. *Imbedding with normalisation and exterior orientation.*

In every point of the  $X_m$  a simple covariant  $k$ -vector  $t_{\lambda_1 \dots \lambda_k}$  is given whose  $m$ -direction lies in the tangent  $E_m$ . It determines uniquely and is uniquely determined by the scalar density

$$(39) \quad \mathfrak{z} = \frac{t_{\lambda_1 \dots \lambda_k}}{t_{\lambda_1 \dots \lambda_k}}$$

of weights  $-1$  and  $+1$  with respect to transformations in  $X_n$  and  $X_m$  respectively. Obviously

$$(40) \quad \omega = \frac{\mathfrak{z}}{|\mathfrak{z}|}.$$

In each  $E_k$  in the local  $E_n$ , having no direction in common with the tangent  $E_m$ ,  $t_{\lambda_1 \dots \lambda_k}$  determines (by section) uniquely a contravariant  $k$ -vector  $p^{\lambda_1 \dots \lambda_k}$  satisfying the equation

$$(41) \quad \frac{1}{k!} t_{\lambda_1 \dots \lambda_k} p^{\lambda_1 \dots \lambda_k} = 1.$$

With respect to the special coordinate system mentioned above we have  $t_{m+1, \dots, n} = \mathfrak{z}^{-1}$  and all components with a suffix  $\leq m$  vanish.

CASE 3'. *Imbedding with normalisation without orientation.*

We get this case by giving  $|\mathfrak{z}|$  instead of  $\mathfrak{z}$ . Then instead of  $t_{\lambda_1 \dots \lambda_k}$  only the quantity

$$(42) \quad \tau_{\lambda_1 \dots \lambda_k} = \omega t_{\lambda_1 \dots \lambda_k} = |\mathfrak{z}|^{-1} t_{\lambda_1 \dots \lambda_k}$$

is determined, and v.v.  $\tau_{\lambda_1 \dots \lambda_k}$  determines  $|\mathfrak{z}|$ . The geometric representation is a cylinder consisting of  $\infty^{k-1}$  (2 for  $k=1$ ) totally parallel  $E_m$ 's, all parallel with the tangent  $E_m$ , but having neither interior nor exterior orientation. With respect to the special system of coordinates mentioned before,  $\tau_{m+1, \dots, n} = |\mathfrak{z}|^{-1}$ , and all components with a suffix  $\leq m$  vanish. In each  $E_k$ , having no direction in common with the tangent  $E_m$ ,  $\tau_{\lambda_1 \dots \lambda_k}$  determines a non oriented contravariant quantity as considered under case 1' and determined by the equation

$$(43) \quad \frac{1}{k!} \tau_{\lambda_1 \dots \lambda_k} \pi^{\lambda_1 \dots \lambda_k} = 1.$$

CASE 4. *Rigged imbedding with normalisation and exterior orientation.*

This case is a combination of 2' and 3 and requires  $B_\lambda^\mathfrak{z}$  and  $\mathfrak{z}$  to be given. In the  $k$ -direction of the rigging a simple contravariant  $k$ -vector  $n^{\mathfrak{x}_1 \dots \mathfrak{x}_k} = \mathfrak{z} e^{\mathfrak{x}_1 \dots \mathfrak{x}_k}$  exists, which is uniquely determined by the conditions

$$(44) \quad \frac{1}{k!} n^{\mathfrak{x}_1 \dots \mathfrak{x}_k} t_{\mathfrak{x}_1 \dots \mathfrak{x}_k} = 1;$$

$$(45) \quad B_{\mathfrak{x}_1}^\mathfrak{z} n^{\mathfrak{x}_1 \dots \mathfrak{x}_k} = 0,$$

and for which the equation

$$(46) \quad t_{\lambda_1 \dots \lambda_k} n^{\mathfrak{x}_1 \dots \mathfrak{x}_k} = k! C_{[\lambda_1}^{[\mathfrak{x}_1} \dots C_{\lambda_k]}^{\mathfrak{x}_k]}; \quad C_\lambda^\mathfrak{z} = A_\lambda^\mathfrak{z} - B_\lambda^\mathfrak{z}$$

holds. With respect to the special system of coordinates mentioned before  $n^{m+1, \dots, n} = \mathfrak{z}$  and all components with a suffix  $\leq m$  vanish.

CASE 4'. *Rigged imbedding with normalisation without orientation.*

This case is a combination of 2 and 3' and requires  $B_\lambda^\mathfrak{z}$  and  $|\mathfrak{z}|$  to be given. In the  $k$ -direction of the rigging a non oriented contravariant quantity  $v^{\mathfrak{x}_1 \dots \mathfrak{x}_k} = |\mathfrak{z}| e^{\mathfrak{x}_1 \dots \mathfrak{x}_k}$  exists, which is uniquely determined by the conditions

$$(47) \quad \frac{1}{k!} v^{\mathfrak{x}_1 \dots \mathfrak{x}_k} \tau_{\mathfrak{x}_1 \dots \mathfrak{x}_k} = 1$$

$$(48) \quad B_{\mathfrak{x}_1}^\mathfrak{z} v^{\mathfrak{x}_1 \dots \mathfrak{x}_k} = 0$$

and for which the equation

$$(49) \quad \tau_{\lambda_1 \dots \lambda_k} v^{\mathfrak{x}_1 \dots \mathfrak{x}_k} = k! C_{[\lambda_1}^{[\mathfrak{x}_1} \dots C_{\lambda_k]}^{\mathfrak{x}_k]}$$

holds. With respect to the special system of coordinates men-

tioned before  $\nu^{m+1, \dots, n} = |\delta|$  and all components with a suffix  $\leq m$  vanish.

The following table shows the different cases and the quantities involved <sup>12)</sup>.

Table 4.

Imbedding:	1	1'	2	2'	3'	3	4'	4	quantities occurring besides $B_b^x$ and $t$
included cases:		1	1	1, 1', 2	1	1, 1', 3'	1, 2, 3'	all	
orientation:	-	+	-	+	-	+	-	+	$\omega$
rigging:	-	-	+	+	-	-	+	+	$B_\lambda^x, (B_\lambda^a), (n)$
normalisation:	-	-	-	-	+	+	+	+	$ \delta , (\tau)$
quantities occurring besides $B_b^x$	$t$	$t$	$t$	$t$	$t, (\tau)$	$t, (\tau, t)$	$t, (\tau)$	$t, (\tau, t)$	
		$\omega$	$n$	$n$		$\omega$	$n, (\nu)$	$n, (\nu, n)$	
			$(B_\lambda^x, B_\lambda^a)$	$(B_\lambda^x, B_\lambda^a)$	$ \delta $	$ \delta , (\delta)$	$ \delta $	$ \delta , (\delta)$	
							$(B_\lambda^x, B_\lambda^a)$	$(B_\lambda^x, B_\lambda^a)$	

Fig. 3 shows the eight different cases for the imbedding of an  $X_2$  in ordinary space.

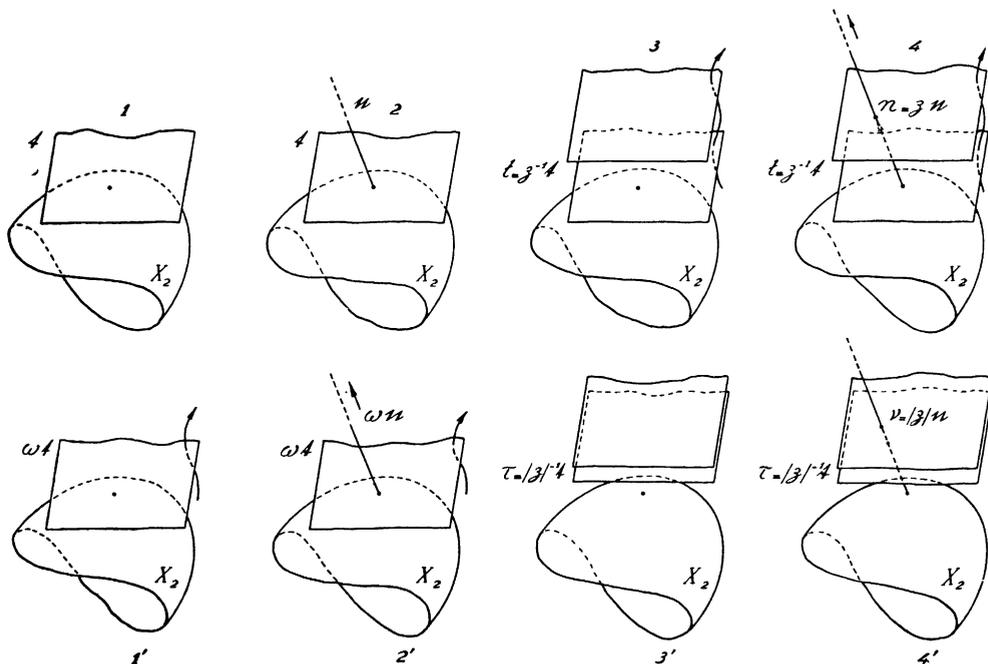


Fig. 3.

<sup>12)</sup> The quantities with  $k$  indices are denoted by their central letter. Quantities in brackets in the last row or column can be derived from the other quantities in

**8. Relations between quantities in  $X_m$  and  $X_n$ .<sup>13)</sup>**

**A. Ordinary contravariant  $p$ -vectors.**

A contravariant  $p$ -vector  $'v^{a_1 \dots a_p}$  ( $p \leq m$ ) in a point of  $X_m$  determines uniquely a contravariant  $p$ -vector  $'v^{\kappa_1 \dots \kappa_p}$  in the corresponding point of  $X_m$  according to the equation

$$(50) \quad 'v^{\kappa_1 \dots \kappa_p} = B_{a_1}^{\kappa_1} \dots B_{a_p}^{\kappa_p} 'v^{a_1 \dots a_p}.$$

Because of (26) we have

$$(51) \quad 'v^{\kappa_1 \dots \kappa_p} t_{\kappa_1 \lambda_2 \dots \lambda_p} = 0.$$

At the other hand each  $p$ -vector  $'v^{\kappa_1 \dots \kappa_p}$  ( $p \leq m$ ) with respect to transformations in  $X_n$ , defined in a point of  $X_m$  and satisfying (51)<sup>14)</sup> determines *uniquely* a  $p$ -vector  $'v^{a_1 \dots a_p}$  defined by (50) as these equations have a unique solution. It can be written in the form

$$(52) \quad 'v^{a_1 \dots a_p} = \frac{1}{p!} \beta_{\lambda_1 \dots \lambda_p}^{a_1 \dots a_p} 'v^{\kappa_1 \dots \kappa_p}$$

where  $\beta_{\lambda_1 \dots \lambda_p}^{a_1 \dots a_p}$  is determined except for the product of an *arbitrary* alternating quantity  $w^{a_1 \dots a_p \lambda_{p+1} \dots \lambda_m}$  with  $t_{\lambda_1 \dots \lambda_m}$ . These quantities can all be derived from  $\beta_{\lambda_1 \dots \lambda_m}^{a_1 \dots a_m}$ , viz.

$$(53) \quad \beta_{\lambda_1 \dots \lambda_p}^{a_1 \dots a_p} = \beta_{\lambda_1 \dots \lambda_p \kappa_{p+1} \dots \kappa_m}^{a_1 \dots a_p b_{p+1} \dots b_m} B_{b_{p+1}}^{\kappa_{p+1}} \dots B_{b_m}^{\kappa_m}.$$

In case 2 (rigging; existence of  $B_{\lambda}^a$ ) the solution has the form

$$(54) \quad 'v^{a_1 \dots a_p} = B_{\kappa_p}^{a_1} \dots B_{\kappa_p}^{a_p} 'v^{\kappa_1 \dots \kappa_p},$$

i.e. a particular choice of  $\beta_{\lambda_1 \dots \lambda_p}^{a_1 \dots a_p}$  is  $p! B_{[\lambda_1}^{[a_1} \dots B_{\lambda_p]}^{a_p]}$ . Moreover in this case an *arbitrary*  $p$ -vector  $v^{\kappa_1 \dots \kappa_p}$  defined in a point of  $X_m$  also determines a  $p$ -vector in  $X_m$  according to

$$(55) \quad 'v^{a_1 \dots a_p} = B_{\kappa_1}^{a_1} \dots B_{\kappa_p}^{a_p} v^{\kappa_1 \dots \kappa_p},$$

but is not itself uniquely determined by  $'v^{a_1 \dots a_p}$  except for

the same row and column. + stands for „given”, — for „not given”. All cases can be obtained by giving none, one, two or three of the three independent quantities  $n$ ,  $\omega$  and  $|\beta|$ , which determine independently the rigging the orientation and the normalisation respectively.

<sup>13)</sup> The relations between ordinary quantities have been treated by J. A. SCHOUTEN [i.e. in note <sup>8)</sup>].

<sup>14)</sup> Such a  $p$ -vector in  $X_n$  is said to „lie in  $X_m$ ”.

$p = 0$ , where (55) simply becomes  $'v = v$ .  $'v^{\alpha_1 \dots \alpha_p}$  can be called the *projection* of  $v^{\alpha_1 \dots \alpha_p}$  in the direction of the rigging upon the local  $E_n$  tangent to  $X_m$ .

By means of (9) and the analogous equation in  $X_m$  the equations corresponding with (50), (54) and (55) are found. They contain a covariant  $(n-p)$ -vectordensity of weight  $-1$  in  $X_n$  and a covariant  $(m-p)$ -vectordensity of weight  $+1$  in  $X_m$ .

In case 3 (normalisation; existence of  $t_{\lambda_1 \dots \lambda_k}$ ) for  $p \geq k = n - m$  still another quantity in  $X_m$  is determined by an *arbitrary*  $p$ -vector  $v^{\alpha_1 \dots \alpha_p}$ , viz.

$$(56) \quad ''v^{\alpha_1 \dots \alpha_{p-k}} = \frac{1}{k!} B_{\alpha_1}^{\alpha_1} \dots B_{\alpha_{p-k}}^{\alpha_{p-k}} v^{\alpha_1 \dots \alpha_p} t_{\alpha_{p-k+1} \dots \alpha_p}.$$

If  $v^{\alpha_1 \dots \alpha_p}$  is simple, the  $(p-k)$ -direction of  $v^{\alpha_1 \dots \alpha_p} t_{\alpha_{p-k+1} \dots \alpha_p}$  and of  $''v^{\alpha_1 \dots \alpha_{p-k}}$  is contained in the  $(p+m-n)$ -dimensional intersection of the  $p$ -direction of  $v^{\alpha_1 \dots \alpha_p}$  and the local  $m$ -direction. By (56)  $v^{\alpha_1 \dots \alpha_p}$  is not uniquely determined for a given  $''v^{\alpha_1 \dots \alpha_{p-k}}$  (except for  $p=n, p-k=m$ ). In case 4 however (and in case 3 for  $p=n$ ) each  $q$ -vector  $''v^{\alpha_1 \dots \alpha_q}$  in  $X_m$  determines a  $(q+k)$ -vector  $''v^{\alpha_1 \dots \alpha_{q+k}}$  in  $X_n$ , according to

$$(57) \quad ''v^{\alpha_1 \dots \alpha_{q+k}} = \binom{q+k}{k} ''v^{\alpha_1 \dots \alpha_q} B_{\alpha_1}^{[\alpha_1} \dots B_{\alpha_q}^{\alpha_q} v^{\alpha_{q+1} \dots \alpha_{q+k}},$$

which is uniquely determined by  $''v^{\alpha_1 \dots \alpha_q}$ , the solution of (57) being

$$(58) \quad ''v^{\alpha_1 \dots \alpha_q} = \frac{1}{k!} B_{\alpha_1}^{\alpha_1} \dots B_{\alpha_q}^{\alpha_q} v^{\alpha_1 \dots \alpha_{q+k}} t_{\alpha_{q+1} \dots \alpha_{q+k}}.$$

By means of (9) and the analogous equation in  $X_m$  the equations corresponding with (56), (57) and (58) are found, containing a covariant  $(n-p)$ -vectordensity of weight  $-1$  in  $X_n$  and a covariant  $(n-p)$ -vectordensity of weight  $-1$  in  $X_m$ :

$$(59) \quad ''v_{b_1 \dots b_l} = \delta^{-1} B_{b_1}^{\lambda_1} \dots B_{b_l}^{\lambda_l} v_{\lambda_1 \dots \lambda_l}; \quad (\text{case 3})$$

$$(60) \quad ''v_{\lambda_1 \dots \lambda_l} = \delta B_{\lambda_1}^{b_1} \dots B_{\lambda_l}^{b_l} ''v_{b_1 \dots b_l} \left. \vphantom{''v_{\lambda_1 \dots \lambda_l}} \right\} (\text{case 4, for } l = 0$$

$$(61) \quad ''v_{b_1 \dots b_l} = \delta^{-1} B_{b_1}^{\lambda_1} \dots B_{b_l}^{\lambda_l} ''v_{\lambda_1 \dots \lambda_l} \left. \vphantom{''v_{b_1 \dots b_l}} \right\} \text{also case 3)}$$

where  $l = n - p$ .

With respect to the special coordinate-system the equations (55) and (56) take the very simple form

$$(62) \quad 'v^{\alpha_1 \dots \alpha_p} = v^{\alpha_1 \dots \alpha_p}; \quad (p \leq m, \text{ case 2, for } p = 0 \text{ also case 1})$$

$$(63) \quad ''v^{\alpha_1 \dots \alpha_{p-k}} = \delta^{-1} v^{\alpha_1 \dots \alpha_{p-k}, m+1, \dots, n} \quad (p \geq k = n - m, \text{ case 3}),$$

e.g. for  $m = n - 1, p = 1, (k=1, q=0), n \geq 2$ :

$$\left. \begin{aligned} (64) \quad 'v^\alpha &= v^\alpha; & (\text{case 2}) \\ (65) \quad ''v &= \delta^{-1} v^n; & (\text{case 3}) \end{aligned} \right\} \text{together valid only in case 4.}$$

The equations of the corresponding densities are ( $l=n-p, k=n-m$ )

$$(66) \quad 'v_{\beta_1 \dots \beta_{l-k}} = v_{\beta_1 \dots \beta_{l-k}, m+1, \dots, n}$$

$$(67) \quad ''v_{\beta_1 \dots \beta_l} = \delta^{-1} v_{\beta_1 \dots \beta_l},$$

e.g. for  $m = n - 1, p = n - 1, (k=1, l=1)$

$$(68) \quad 'v = v_n$$

$$(69) \quad ''v = \delta^{-1} v.$$

It is remarkable, that in case 4, if  $m \geq p \geq k = n - m$  (which is only possible if  $m \geq \frac{1}{2}n$ ) both quantities  $'v^{\alpha_1 \dots \alpha_p}$  and  $''v^{\alpha_1 \dots \alpha_{p-k}}$  in  $X_m$  exist and *that these two quantities together for  $m = n - 1$  determine completely the  $p$ -vector  $v^{\alpha_1 \dots \alpha_p}$  in  $X_n$  as follows from (62) and (63).*

**B. Ordinary covariant  $p$ -vectors.**

For  $p \leq m$  a covariant  $p$ -vector  $w_{\lambda_1 \dots \lambda_p}$  in  $X_n$ , defined in a point of  $X_m$ , always (i.e. in case 1) determines a covariant  $p$ -vector in  $X_m$ , viz.

$$(70) \quad 'w_{b_1 \dots b_p} = B_{b_1}^{\lambda_1} \dots B_{b_p}^{\lambda_p} w_{\lambda_1 \dots \lambda_p}.$$

If  $w_{\lambda_1 \dots \lambda_p}$  is simple this quantity is the intersection of  $w_{\lambda_1 \dots \lambda_p}$  with the local  $E_m$ . By equation (70)  $w_{\lambda_1 \dots \lambda_p}$  is not uniquely determined. In case 2 (rigging) however (and in case 1 for  $p=0$ ) each covariant  $p$ -vector  $'w_{b_1 \dots b_p}$  in  $X_m$  determines such a quantity in  $X_n$  according to

$$(71) \quad 'w_{\lambda_1 \dots \lambda_p} = B_{\lambda_1}^{b_1} \dots B_{\lambda_p}^{b_p} 'w_{b_1 \dots b_p},$$

which is uniquely determined by  $'w_{b_1 \dots b_p}$ , the solution of (71) being

$$(72) \quad 'w_{b_1 \dots b_p} = B_{b_1}^{\lambda_1} \dots B_{b_p}^{\lambda_p} 'w_{\lambda_1 \dots \lambda_p}.$$

If  $'w_{b_1 \dots b_p}$  is simple, the  $(n-p)$ -direction of  $'w_{\lambda_1 \dots \lambda_p}$  contains and is composed of the  $(m-p)$ -direction of  $'w_{b_1 \dots b_p}$  and the

$(n-m)$ -direction of the rigging. By means of (10) and the analogous equations in  $X_m$  the equations corresponding with (70), (71) and (72) are found, containing a contravariant  $(n-p)$ -vector-density of weight  $+1$  in  $X_n$  and a contravariant  $(m-p)$ -vector-density of weight  $+1$  in  $X_m$ .

In case 3 (normalisation, existence of  $t_{\lambda_1 \dots \lambda_k}$ ) a covariant  $q$ -vector  $''w_{b_1 \dots b_q}$  in  $X_m$  determines also a covariant  $(q+k)$ -vector  $''w_{\lambda_1 \dots \lambda_{q+k}}$  in  $X_n$ , according to

$$(73) \quad ''w_{\lambda_1 \dots \lambda_{q+k}} = \binom{q+k}{k} ''w_{b_1 \dots b_q} \beta^{b_1 \dots b_q}_{\lambda_1 \dots \lambda_q} t_{\lambda_{q+1} \dots \lambda_{q+k}},$$

as the alternated product of  $\beta^{b_1 \dots b_q}_{\lambda_1 \dots \lambda_q}$  with  $t_{\lambda_{q+1} \dots \lambda_{q+k}}$  is not affected by the ambiguity of  $\beta^{b_1 \dots b_q}_{\lambda_1 \dots \lambda_q}$ .

If  $''w_{b_1 \dots b_q}$  is simple,  $''w_{\lambda_1 \dots \lambda_{q+k}}$  also is; its  $(n-q-k) = (m-q)$ -direction is the same as the  $(m-q)$ -direction of  $''w_{b_1 \dots b_q}$  and is contained in the local  $m$ -direction of  $X_m$ .

In case 4 equations (73) have the solution

$$(74) \quad ''w_{b_1 \dots b_q} = \frac{1}{k!} B_{b_1}^{\lambda_1} \dots B_{b_q}^{\lambda_q} ''w_{\lambda_1 \dots \lambda_{q+k}} n^{\lambda_{q+1} \dots \lambda_{q+k}}.$$

This solution however is valid already in case 3 though  $n^{\lambda_1 \dots \lambda_k}$  is not uniquely determined then. In fact, as the ambiguity of  $n^{\lambda_1 \dots \lambda_k}$  consists in alternated products containing a factor  $B$ , and as the transvection of  $''w_{\lambda_1 \dots \lambda_{q+k}}$  with  $q+1$  or more factors  $B$  vanishes, this ambiguity bears no influence upon the left side of (74). Hence in this case  $''w_{\lambda_1 \dots \lambda_{q+k}}$  and  $''w_{b_1 \dots b_q}$  can be considered to represent the same object.

Moreover in case 4 an arbitrary covariant  $(q+k)$ -vector  $w_{\lambda_1 \dots \lambda_{q+k}}$  defined in a point of  $X_m$  determines a covariant  $q$ -vector in  $X_m$  according to

$$(75) \quad ''w_{b_1 \dots b_q} = \frac{1}{k!} B_{b_1}^{\lambda_1} \dots B_{b_q}^{\lambda_q} w_{\lambda_1 \dots \lambda_{q+k}} n^{\lambda_{q+1} \dots \lambda_{q+k}},$$

but evidently is not uniquely determined by it (except for  $q = m$ ,  $q+k = n$ ).  $''w_{b_1 \dots b_q}$  can be called the *projection* of  $w_{\lambda_1 \dots \lambda_{q+k}}$  in the direction of the rigging upon the local  $E_m$  tangent to  $X_m$ . If we take for  $m = n - 1$  the special coordinate-system mentioned above, the equation (75) takes the very simple form

$$(76) \quad ''w_{\beta_1 \dots \beta_q} = \delta w_{\beta_1 \dots \beta_q} n.$$

By means of (10) and the analogous equation in  $X_m$  the equations corresponding with (73), (74) and (75) are found, containing a contravariant  $(n-p)$ -vectordensity of weight  $+1$  in  $X_n$  and a contravariant  $(n-p)$ -vectordensity of weight  $+1$  in  $X_m$

$$(77) \quad ''w^{a_1 \dots a_l} = \delta B_{\kappa_1 \dots \kappa_l}^{a_1 \dots a_l} w^{\kappa_1 \dots \kappa_l} \quad (\text{case 4, for } l=0 \text{ also in case 3})$$

$$(78) \quad ''w^{\kappa_1 \dots \kappa_l} = \delta^{-1} B_{a_1 \dots a_l}^{\kappa_1 \dots \kappa_l} ''w^{a_1 \dots a_l} \left. \vphantom{''w^{\kappa_1 \dots \kappa_l}} \right\} (\text{case 3}).$$

$$(79) \quad ''w^{a_1 \dots a_l} = \delta B_{\kappa_1 \dots \kappa_l}^{a_1 \dots a_l} ''w^{\kappa_1 \dots \kappa_l} \left. \vphantom{''w^{a_1 \dots a_l}} \right\}$$

With respect to the special coordinatesystem the equations (70) and (75) get the very simple form

$$(80) \quad 'w_{\beta_1 \dots \beta_p} = w_{\beta_1 \dots \beta_p}; \quad (p \leq m, \text{ case 1})$$

$$(81) \quad ''w_{\beta_1 \dots \beta_{p-k}} = \delta w_{\beta_1 \dots \beta_{p-k}, m+1, \dots, n};$$

$$(p \geq k, \text{ case 4, for } p = n \text{ also in case 3}),$$

f.i. for  $m = n - 1, p = 1, n \geq 2$

$$(82) \quad 'w_{\beta} = w_{\beta}; \quad (\text{case 1})$$

$$(83) \quad ''w = \delta w_n; \quad (\text{case 4}).$$

The equations of the corresponding densities are ( $l=n-p$ )

$$(84) \quad 'w^{\alpha_1 \dots \alpha_{l-k}} = w^{\alpha_1 \dots \alpha_{l-k}, m+1, \dots, n}$$

$$(85) \quad ''w^{\alpha_1 \dots \alpha_l} = \delta w^{\alpha_1 \dots \alpha_l}$$

f.i. for  $m = n - 1, p = n - 1$

$$(86) \quad 'w = w_n$$

$$(87) \quad ''w = \delta^{-1} w.$$

It is remarkable, that in case 4, if  $m \geq p \geq k$  (only possible if  $m \geq \frac{1}{2}n$ ) both quantities  $'w_{b_1 \dots b_p}$  and  $w_{b_1 \dots b_{p-k}}$  in  $X_m$  exist and that these two quantities together determine for  $m = n - 1$  completely the  $p$ -vector  $w_{\lambda_1 \dots \lambda_p}$  in  $X_n$  as follows from (80) and (81).

We collect the results in the following table <sup>15)</sup>:

Table 5.

$X_n$ $p \leq m$	Case		$X_m$	$X_n$ $p \geq k$	Case		$X_m$
	1	2			3	4	
$v^{x_1 \dots x_p}$	$\begin{cases} \xrightarrow{\leftarrow} p = 0 \\ \xleftarrow{\rightarrow} \end{cases}$	$B_\lambda^a$	$\begin{cases} v^{a_1 \dots a_p} \\ v_{b_1 \dots b_{m-p}} \end{cases}$	$v^{x_1 \dots x_p}$	$\begin{cases} (B_\lambda^a), t \\ \xrightarrow{\leftarrow} \end{cases}$	$B_\lambda^a, t$	$\begin{cases} v^{a_1 \dots a_{p-k}} \\ v_{b_1 \dots b_l} \end{cases}$
$v_{\lambda_1 \dots \lambda_l}$		$B_b^x$	$B_b^x$	$\begin{cases} v^{a_1 \dots a_p} \\ v_{b_1 \dots b_{m-p}} \end{cases}$	$v^{x_1 \dots x_p}$	$B_b^x, (n)$	$B_b^x, n$
$v^{x_1 \dots x_p}$	$\begin{cases} \xrightarrow{\leftarrow} \\ \xleftarrow{\rightarrow} p = 0 \end{cases}$	$(B_\lambda^a)$	$\begin{cases} v^{a_1 \dots a_p} \\ v_{b_1 \dots b_{m-p}} \end{cases}$	$v_{\lambda_1 \dots \lambda_l}$	$\begin{cases} (B_\lambda^a), t \\ \xrightarrow{\leftarrow} \end{cases}$	$B_\lambda^a, t$	$\begin{cases} v_{b_1 \dots b_l} \end{cases}$
$v_{\lambda_1 \dots \lambda_l}$		$B_b^x$	$B_b^x$	$\begin{cases} v_{b_1 \dots b_p} \\ w^{a_1 \dots a_{m-p}} \end{cases}$	$w_{\lambda_1 \dots \lambda_p}$	$\begin{cases} B_b^x, (n) \\ \xrightarrow{\leftarrow} p = n \end{cases}$	$B_b^x, n$
$w_{\lambda_1 \dots \lambda_p}$	$\begin{cases} \xrightarrow{\leftarrow} \\ \xleftarrow{\rightarrow} p = 0 \end{cases}$	$B_b^x$	$\begin{cases} w_{b_1 \dots b_p} \\ w^{a_1 \dots a_{m-p}} \end{cases}$	$w^{x_1 \dots x_l}$	$\begin{cases} (B_\lambda^a), t \\ \xrightarrow{\leftarrow} \end{cases}$	$B_\lambda^a, t$	$\begin{cases} w_{b_1 \dots b_{p-k}} \\ w^{a_1 \dots a_l} \end{cases}$
$w_{\lambda_1 \dots \lambda_p}$		$B_b^x$	$B_b^x$	$\begin{cases} w_{b_1 \dots b_p} \\ w^{a_1 \dots a_{m-p}} \end{cases}$	$w^{x_1 \dots x_l}$	$B_b^x, (n)$	$B_b^x, n$

$$k = n - m; \quad l = n - p.$$

C.  $W$ -Quantities.

For  $W$ - $p$ -vectors and their corresponding  $W$ - $(n-p)$ -vector-densities the equations are:

Contravariant  $W$ - $p$ -vectors,  $p \leq m$ :

$$\left. \begin{aligned} (50W) \quad v^{x_1 \dots x_p} &= \omega B_{a_1}^{x_1} \dots B_{a_p}^{x_p} v^{a_1 \dots a_p} \\ (54W) \quad v^{a_1 \dots a_p} &= \omega B_{x_1}^{a_1} \dots B_{x_p}^{a_p} v^{x_1 \dots x_p} \end{aligned} \right\} \text{(case 1')}.$$

$$(55W) \quad v^{a_1 \dots a_p} = \omega B_{x_1}^{a_1} \dots B_{x_p}^{a_p} v^{x_1 \dots x_p} \text{ (case 2', for } p = 0 \text{ also case 1')};$$

Contravariant  $W$ - $p$ -vectors,  $p \geq k$ :

$$(56W) \quad v^{a_1 \dots a_{p-k}} = \frac{1}{k!} B_{x_1}^{a_1} \dots B_{x_{p-k}}^{a_{p-k}} v^{x_1 \dots x_p} \tau_{x_{p-k+1} \dots x_p} \text{ (case 3')};$$

<sup>15)</sup> The arrows in the 2nd and 3rd column show which of the quantities in the 1st and 4th column is determined by which, and analogously in the last four columns. Near the arrows the quantities are written, used for the determination. Quantities, which are not uniquely determined but can be used because their ambiguity does not affect the result, are written in brackets.

$$\left. \begin{aligned}
 (57W) \quad \overline{\overline{v}}^{\alpha_1 \dots \alpha_p} &= \binom{p}{k} \overline{\overline{v}}^{\alpha_1 \dots \alpha_{p-k}} B_{\alpha_1}^{\lambda_1} \dots B_{\alpha_{p-k}}^{\lambda_{p-k}} v^{\alpha_{p-k} \dots \alpha_p} \\
 (58W) \quad \overline{\overline{v}}^{\alpha_1 \dots \alpha_{p-k}} &= \frac{1}{k!} B_{\alpha_1}^{\lambda_1} \dots B_{\alpha_{p-k}}^{\lambda_{p-k}} \overline{\overline{v}}^{\alpha_1 \dots \alpha_p} \tau_{\lambda_{p-k+1} \dots \lambda_p}
 \end{aligned} \right\} \begin{array}{l} \text{(case 4', for} \\ p = n \text{ also} \\ \text{case 3').} \end{array}$$

*Covariant W-(n-p)-vectordensities of weight - 1, p ≥ k:*

$$(59W) \quad \overline{\overline{v}}_{b_1 \dots b_i} = |\delta|^{-1} B_{b_1}^{\lambda_1} \dots B_{b_i}^{\lambda_i} \overline{\overline{v}}_{\lambda_1 \dots \lambda_i} \quad \text{(case 3')};$$

$$(60W) \quad \overline{\overline{v}}_{\lambda_1 \dots \lambda_i} = |\delta| B_{\lambda_1}^{b_1} \dots B_{\lambda_i}^{b_i} \overline{\overline{v}}_{b_1 \dots b_i} \quad \left\{ \begin{array}{l} \text{(case 4', for } l=0 \\ \text{also case 3').} \end{array} \right.$$

$$(61W) \quad \overline{\overline{v}}_{b_1 \dots b_i} = |\delta|^{-1} B_{b_1}^{\lambda_1} \dots B_{b_i}^{\lambda_i} \overline{\overline{v}}_{\lambda_1 \dots \lambda_i} \quad \left. \right\}$$

*Contravariant W-p-vectors w.r. to special coordinatesystem:*

$$(62W) \quad \overline{\overline{v}}^{\alpha_1 \dots \alpha_p} = \omega \overline{\overline{v}}^{\alpha_1 \dots \alpha_p}; \quad (p \leq m, \text{ case 2', for } p=0 \text{ also case 1'})$$

$$(63W) \quad \overline{\overline{v}}^{\alpha_1 \dots \alpha_{p-k}} = |\delta|^{-1} \overline{\overline{v}}^{\alpha_1 \dots \alpha_{p-k}, m+1, \dots, n} \quad (p \geq k, \text{ case 3'});$$

e.g. for  $m = n - 1, p = 1, n \geq 2$ :

$$\left. \begin{aligned}
 (64W) \quad \overline{\overline{v}}^\alpha &= \omega \overline{\overline{v}}^\alpha; \quad \text{(case 2')} \\
 (65W) \quad \overline{\overline{v}} &= |\delta|^{-1} \overline{\overline{v}}^n; \quad \text{(case 3')}
 \end{aligned} \right\} \begin{array}{l} \text{together valid only in case} \\ \text{4 (not in case 4').} \end{array}$$

*Covariant W-(n-p)-vectordensities of weight - 1 w.r. to special coordinatesystem:*

$$(66W) \quad \overline{\overline{v}}_{\beta_1 \dots \beta_{i-k}} = \overline{\overline{v}}_{\beta_1 \dots \beta_{i-k}, m+1, \dots, n} \quad \left\{ \begin{array}{l} (p \leq m, \text{ case 2', for} \\ p = 0 \text{ also case 1')} \end{array} \right.$$

$$(67W) \quad \overline{\overline{v}}_{\beta_1 \dots \beta_i} = \delta^{-1} \overline{\overline{v}}_{\beta_1 \dots \beta_i} \quad (p \geq k, \text{ case 3'});$$

e.g. for  $m = n - 1, p = n - 1, n \geq 2$ :

$$\left. \begin{aligned}
 (68W) \quad \overline{\overline{v}} &= \omega \overline{\overline{v}}_n \quad \text{(case 2');} \\
 (69W) \quad \overline{\overline{v}}_\beta &= |\delta|^{-1} \overline{\overline{v}}_\beta \quad \text{(case 3');}
 \end{aligned} \right\} \begin{array}{l} \text{together only valid in} \\ \text{case 4 (not in case 4').} \end{array}$$

*Covariant W-p-vectors, p ≤ m:*

$$(70W) \quad \overline{\overline{w}}_{b_1 \dots b_p} = \omega B_{b_1}^{\lambda_1} \dots B_{b_p}^{\lambda_p} \overline{\overline{w}}_{\lambda_1 \dots \lambda_p} \quad \text{(case 1')};$$

$$(71W) \quad \overline{\overline{w}}_{\lambda_1 \dots \lambda_p} = \omega B_{\lambda_1}^{b_1} \dots B_{\lambda_p}^{b_p} \overline{\overline{w}}_{b_1 \dots b_p} \quad \left\{ \begin{array}{l} \text{(case 2', for } p = 0 \\ \text{also case 1')} \end{array} \right.$$

$$(72W) \quad \overline{\overline{w}}_{b_1 \dots b_p} = \omega B_{b_1}^{\lambda_1} \dots B_{b_p}^{\lambda_p} \overline{\overline{w}}_{\lambda_1 \dots \lambda_p} \quad \left. \right\}$$

*Covariant W-p-vectors, p ≥ k:*

$$(73W) \quad \overline{\overline{w}}_{\lambda_1 \dots \lambda_p} = \binom{p}{k} \overline{\overline{w}}_{b_1 \dots b_{p-k}} B_{\lambda_1}^{b_1} \dots B_{\lambda_{p-k}}^{b_{p-k}} \tau_{\lambda_{p-k+1} \dots \lambda_p} \quad \text{(case 3')}$$

$$(74W) \quad ''\bar{w}_{b_1 \dots b_{p-k}} = \frac{1}{k!} B_{b_1}^{\lambda_1} \dots B_{b_{p-k}}^{\lambda_{p-k}} ''\bar{w}_{\lambda_1 \dots \lambda_p} \nu^{\lambda_{p-k+1} \dots \lambda_p} \quad (\text{case } 3')$$

$$(75W) \quad ''\bar{w}_{b_1 \dots b_{p-k}} = \frac{1}{k!} B_{b_1}^{\lambda_1} \dots B_{b_{p-k}}^{\lambda_{p-k}} \bar{w}_{\lambda_1 \dots \lambda_p} \nu^{\lambda_{p-k+1} \dots \lambda_p} \quad (\text{case } 4', \text{ for } p = n \text{ also case } 3').$$

*Contravariant  $W$ -( $n-p$ )-vectordensities of weight + 1,  $p \geq k$ :*

$$(77W) \quad ''\bar{w}^{a_1 \dots a_l} = |\bar{\mathfrak{g}}| B_{x_1}^{a_1} \dots B_{x_l}^{a_l} \bar{w}^{x_1 \dots x_l} \quad (\text{case } 4', \text{ for } l = 0; \text{ also case } 3')$$

$$(78W) \quad ''\bar{w}^{x_1 \dots x_l} = |\bar{\mathfrak{g}}|^{-1} B_{a_1}^{x_1} \dots B_{a_l}^{x_l} ''\bar{w}^{a_1 \dots a_l} \quad \left. \vphantom{(78W)} \right\} (\text{case } 3').$$

$$(79W) \quad ''\bar{w}^{a_1 \dots a_l} = |\bar{\mathfrak{g}}| B_{x_1}^{a_1} \dots B_{x_l}^{a_l} ''\bar{w}^{x_1 \dots x_l} \quad \left. \vphantom{(79W)} \right\}$$

*Covariant  $W$ - $p$ -vectors w.r. to special coordinatesystem:*

$$(80W) \quad '\bar{w}_{\beta_1 \dots \beta_p} = \omega \bar{w}_{\beta_1 \dots \beta_p} \quad (p \leq m, \text{ case } 1');$$

$$(81W) \quad ''\bar{w}_{\beta_1 \dots \beta_{p-k}} = |\bar{\mathfrak{g}}| \bar{w}_{\beta_1 \dots \beta_{p-k} m+1, \dots, n} \quad (p \geq k, \text{ case } 4', \text{ for } p = n \text{ also in case } 3');$$

f.i. for  $m = n - 1, p = 1, n \geq 2$

$$(82W) \quad '\bar{w}_\beta = \omega \bar{w}_\beta; (\text{case } 1') \quad \left. \vphantom{(82W)} \right\} \text{together valid only in case } 4$$

$$(83W) \quad ''\bar{w} = |\bar{\mathfrak{g}}| \bar{w}_n; (\text{case } 4') \quad \left. \vphantom{(83W)} \right\} (\text{not in case } 4').$$

*Contravariant  $W$ -( $n-p$ )-vectordensities of weight + 1 w.r. to special coordinatesystem:*

$$(84W) \quad '\bar{w}^{\alpha_1 \dots \alpha_{l-k}} = \omega \bar{w}^{\alpha_1 \dots \alpha_{l-k} m+1, \dots, n}; \quad (p \leq m, \text{ case } 1')$$

$$(85W) \quad ''\bar{w}^{\alpha_1 \dots \alpha_l} = |\bar{\mathfrak{g}}| \bar{w}^{\alpha_1 \dots \alpha_l} \quad \left\{ \begin{array}{l} (p \geq k, \text{ case } 4', \text{ for } p = n \\ \text{also case } 3'), \end{array} \right.$$

f.i. for  $m = n - 1, p = n - 1, n \geq 2$ .

$$(86W) \quad '\bar{w} = \omega \bar{w}^n (\text{case } 1') \quad \left. \vphantom{(86W)} \right\} \text{together valid only in case } 4$$

$$(87W) \quad ''\bar{w}^\alpha = |\bar{\mathfrak{g}}| \bar{w}^\alpha (\text{case } 4') \quad \left. \vphantom{(87W)} \right\} (\text{not case } 4').$$

From (62W) and (63W) follows that for  $m = n - 1$  the  $W$ - $p$ -vector  $\bar{v}^{x_1 \dots x_p}$  is completely determined by  $'\bar{v}^{a_1 \dots a_p}$  and  $''\bar{v}^{a_1 \dots a_{p-1}}$ . The same holds for  $\bar{w}_{\lambda_1 \dots \lambda_p}, '\bar{w}_{b_1 \dots b_p}$  and  $''\bar{w}_{b_1 \dots b_{p-1}}$  in (80W) and (81W). All formulae differ from the corresponding ones for ordinary quantities only by a factor  $\omega$  at the right. From this follows that the cases 1', 2', 3', 4' play now the same role

as the cases 1, 2, 3, 4 for ordinary quantities <sup>16)</sup>. We collect the results in the following table:

Table 6.

$X_n$ $p \leq m$	Case		$X_m$	$X_n$ $p \geq k$	Case		$X_n$
	1'	2'			3'	4'	
$\overline{v}^{x_1 \dots x_p}$	$\rightleftarrows p = 0$	$B_\lambda^a, \omega$	$\left\{ \begin{array}{l} \overline{v}^{a_1 \dots a_p} \\ \overline{v}_{b_1 \dots b_{m-p}} \end{array} \right.$	$\overline{v}^{x_1 \dots x_p}$	$\left\{ \begin{array}{l} (B_\lambda^a), \tau \\ \rightarrow \end{array} \right.$	$B_\lambda^a, \tau$	$\left\{ \begin{array}{l} ''v^{a_1 \dots a_{p-k}} \\ ''v_{b_1 \dots b_k} \end{array} \right.$
$\overline{v}_{\lambda_1 \dots \lambda_l}$							
$\overline{v}^{x_1 \dots x_p}$	$B_b^x, \omega$	$B_b^x, \omega$	$\left\{ \begin{array}{l} \overline{v}^{a_1 \dots a_p} \\ \overline{v}_{b_1 \dots b_{m-p}} \end{array} \right.$	$''v^{x_1 \dots x_p}$	$\left\{ \begin{array}{l} B_b^x, (\nu) \\ \rightleftarrows p = n \\ (B_\lambda^a), \tau \end{array} \right.$	$B_b^x, \nu$	$\left\{ \begin{array}{l} ''v \dots a^{-k} \\ ''v_{b_1 \dots b_i} \end{array} \right.$
$\overline{v}_{\lambda_1 \dots \lambda_l}$							
$\overline{w}_{\lambda_1 \dots \lambda_p}$	$B_b^x, \omega$	$B_b^x, \omega$	$\left\{ \begin{array}{l} \overline{w}_{b_1 \dots b_p} \\ \overline{w}^{a_1 \dots a_{m-p}} \end{array} \right.$	$\overline{w}_{\lambda_1 \dots \lambda_p}$	$\left\{ \begin{array}{l} B_b^x, (\nu) \\ \rightleftarrows p = n \\ (B_\lambda^a), \tau \end{array} \right.$	$B_b^x, \nu$	$\left\{ \begin{array}{l} ''w_{b_1 \dots b_{p-k}} \\ ''w^{a_1 \dots a_i} \end{array} \right.$
$\overline{w}^{x_1 \dots x_l}$							
$\overline{w}_{\lambda_1 \dots \lambda_p}$	$\rightleftarrows p = 0$	$B_\lambda^a, \omega$	$\left\{ \begin{array}{l} \overline{w}_{b_1 \dots b_p} \\ \overline{w}^{a_1 \dots a_{m-p}} \end{array} \right.$	$''\overline{w}_{\lambda_1 \dots \lambda_p}$	$\left\{ \begin{array}{l} (B_\lambda^a), \tau \\ \rightleftarrows \\ B_b^x, (\nu) \end{array} \right.$	$B_\lambda^a, \tau$	$\left\{ \begin{array}{l} ''w_{b_1 \dots b_{p-k}} \\ ''w^{a_1 \dots a_i} \end{array} \right.$
$\overline{w}^{x_1 \dots x_l}$							

From (64, 65, 68, 69, 82, 83, 86, 87) and (64W, 65W, 68W, 69W, 82W, 83W, 86W, 87W) it is easy to deduce the quantities in  $X_{n-1}$  derived from an affinor or affinordensity in  $X_n$ , which can be written as a sum of products of vectors and vectordensities of suitable weigths. E.g. the  $W$ -affinordensity  $\overline{\mathfrak{P}}_\lambda^x$  of weight + 1 can be written as a sum of products of a covariant vector and a contravariant  $W$ -vectordensity of weight + 1 and therefore, according to (82, 83, 86W, 87W) gives rise to the following quantities in  $X_{n-1}$ :

$$(88) \left\{ \begin{array}{l} |\delta| \overline{\mathfrak{P}}_\beta^\alpha ; W\text{-affinordensity of weight } + 1, (82, 87W, \text{ case } 4'). \\ \delta | \delta | \overline{\mathfrak{P}}_n^\alpha ; \text{ contr. } W\text{-vectordensity of weight } + 1, (83, 87W, \text{ case } 4). \\ \omega \overline{\mathfrak{P}}_\beta^n ; \text{ cov. } W\text{-vectordensity of weight } + 1, (82, 86W, \text{ case } 1'). \\ |\delta| \overline{\mathfrak{P}}_n^n ; W\text{-density of weight } + 1, (83, 86W, \text{ case } 4'). \end{array} \right.$$

<sup>16)</sup> But it must be remarked that 4' does not include 1', 2' and 3' but only 1, 2 and 3'.

### 9. Generalisation of Stokes formulae for ordinary quantities and $W$ -quantities.

In  $X_n$  we consider an orientable and closed  $X_m$  called  $\tau_m$ , bounding a simply connected part  $\tau_{m+1}$  of an orientable  $X_{m+1}$ . The  $m$ -dimensional element with an interior orientation be  $df^{\varkappa_1 \dots \varkappa_m}$ , the  $(m+1)$ -dimensional element with an interior orientation of  $X_{m+1}$  be  $df^{\varkappa_1 \dots \varkappa_{m+1}}$ . The orientations are chosen in such a way that the direction from an interior point of  $\tau_{m+1}$  towards the boundary followed by the orientation of  $df^{\varkappa_1 \dots \varkappa_m}$  gives the orientation of  $df^{\varkappa_1 \dots \varkappa_{m+1}}$ . Be now  $v_{\lambda_1 \dots \lambda_m}$  an  $m$ -vector field that satisfies the ordinary conditions of continuity. Then in a well known way we may derive <sup>17)</sup>

$$(89) \quad \int_{\tau_{m+1}} (\partial_\mu v_{\lambda_1 \dots \lambda_m}) d f^{\mu \lambda_1 \dots \lambda_m} = \int_{\tau_m} v_{\lambda_1 \dots \lambda_m} d f^{\lambda_1 \dots \lambda_m}.$$

If we introduce in this formula  $v^{\varkappa_{m+1} \dots \varkappa_n}$ ,  $f_{\lambda_{m+1} \dots \lambda_n}$  and  $f_{\lambda_{m+1} \dots \lambda_{n-1}}$  we get four other forms: <sup>18)</sup>

$$(90) \quad \frac{1}{h!} \int_{\tau_{m+1}} d f_{\lambda_1 \dots \lambda_h} \partial_\mu v^{\lambda_1 \dots \lambda_h \mu} = \frac{1}{(h+1)!} \int_{\tau_m} d f_{\lambda_1 \dots \lambda_{h+1}} v^{\lambda_1 \dots \lambda_{h+1}}; \quad h = n - m - 1$$

$$(91) \quad \frac{1}{h!} \int_{\tau_{m+1}} (\partial_{[\mu} v_{\lambda_1 \dots \lambda_m}) d f_{\varkappa_1 \dots \varkappa_h]} = \frac{1}{(h+1)!} \int_{\tau_m} v_{[\lambda_1 \dots \lambda_m} d f_{\varkappa_1 \dots \varkappa_h \mu]}$$

$$(92) \quad \int_{\tau_{m+1}} d f^{\mu [\lambda_1 \dots \lambda_m} \partial_\mu v^{\varkappa_1 \dots \varkappa_h]} = \int_{\tau_m} d f^{[\lambda_1 \dots \lambda_m} v^{\varkappa_1 \dots \varkappa_h]}$$

<sup>17)</sup> SCHOUTEN-STRIJK, Einführung I, p. 130; Ricci-Kalkül, p. 97, (204); earlier literature is mentioned there.  $df^{\varkappa_1 \dots \varkappa_m}$  and  $df^{\varkappa_1 \dots \varkappa_{m+1}}$  differ by a factor  $m!$  and  $(m+1)!$  resp. from  $f^{\varkappa_1 \dots \varkappa_m} d\tau_m$  and  $f^{\varkappa_1 \dots \varkappa_{m+1}} d\tau_{m+1}$  used in R.K.

<sup>18)</sup> The formulae (90), (91) and (92) correspond with (211), (210) and (208) in R.K. p. 98 where still multivectors in stead of multivector densities were used. (90) occurs for the special but typical case of Maxwell's equations in D. VAN DANTZIG, The fundamental equations of electromagnetism independent of metrical geometry [Proc. Camb. Phil. Soc. 30 (1934), 421—427]. The formulae (89—93) occur as (I), (II), (IV), (III') and (III) in J. VAN WEYSENHOFF, Duale Größen, Großrotation, Großdivergenz und die Stokes-Gaußschen Sätze in allgemeinen Räumen [Ann. de la Soc. Pol. de Math. 16 (1937), 127—144], 141, 142. In III there is a misprint, the index  $\varkappa$  being not excluded from the alternation.

$$(93) \quad \int_{\tau_{m+1}} \binom{n}{m+1} d f^{[\nu \lambda_1 \dots \lambda_m \partial_\mu \bar{v}^{\alpha_1 \dots \alpha_n] \mu}} = \binom{n}{m} \int_{\tau_m} d f^{[\lambda_1 \dots \lambda_m \bar{v}^{\alpha_1 \dots \alpha_n \nu]}.$$

The formulae (89) and (90) are valid for an  $X_m$  in  $X_n$  with an interior orientation. But they cannot be used in the case more frequently occurring in physical applications of an  $X_m$  with an exterior orientation (inducing also an exterior orientation in  $\tau_{m+1}$ ). In order to derive the formulae of Stokes for this case we introduce the following  $W$ -quantities

$$(94) \quad d \bar{f}^{\alpha_1 \dots \alpha_{m+1}} = \bar{k} d f^{\alpha_1 \dots \alpha_{m+1}}; \quad d \bar{f}_{\lambda_1 \dots \lambda_h} = \bar{k} d f_{\lambda_1 \dots \lambda_h}$$

$$(95) \quad d \bar{f}^{\alpha_1 \dots \alpha_m} = \bar{k} d f^{\alpha_1 \dots \alpha_m}; \quad d \bar{f}_{\lambda_1 \dots \lambda_{h+1}} = \bar{k} d f_{\lambda_1 \dots \lambda_{h+1}}$$

$$(96) \quad \bar{v}_{\lambda_1 \dots \lambda_m} = \bar{k} v_{\lambda_1 \dots \lambda_m}; \quad \bar{v}^{\alpha_1 \dots \alpha_{h+1}} = \bar{k} v^{\alpha_1 \dots \alpha_{h+1}},$$

where  $\bar{k}$  is a  $W$ -scalar which has with respect to some given coordinatesystem e.g.  $(x)$  the constant value  $+1$  and transforms according to the formula

$$(97) \quad \bar{k}^{(x')} = \frac{\Delta}{|\Delta|} \bar{k}^{(x)}.$$

Then we get

$$(98) \quad \int_{\tau_{m+1}} (\partial_{\lambda_1} \bar{v}_{\lambda_2 \dots \lambda_{m+1}}) d \bar{f}^{\lambda_1 \dots \lambda_{m+1}} = \int_{\tau_m} \bar{v}_{\lambda_1 \dots \lambda_m} d \bar{f}^{\lambda_1 \dots \lambda_m}$$

$$(99) \quad \frac{1}{h!} \int_{\tau_{m+1}} d \bar{f}_{\lambda_1 \dots \lambda_h} \partial_\mu \bar{v}^{\lambda_1 \dots \lambda_h \mu} = \frac{1}{(h+1)!} \int_{\tau_m} d \bar{f}_{\lambda_1 \dots \lambda_{h+1}} \bar{v}^{\lambda_1 \dots \lambda_{h+1}}$$

$$(100) \quad \frac{1}{h!} \int_{\tau_{m+1}} (\partial_{[\mu} \bar{v}_{\lambda_1 \dots \lambda_m]}) d \bar{f}_{\alpha_1 \dots \alpha_h} = \frac{1}{(h+1)!} \int_{\tau_m} \bar{v}_{[\lambda_1 \dots \lambda_m} d \bar{f}_{\alpha_1 \dots \alpha_h \mu]}$$

$$(101) \quad \int_{\tau_{m+1}} d \bar{f}^{\mu [\lambda_1 \dots \lambda_m \partial_\mu \bar{v}^{\alpha_1 \dots \alpha_h}]} = \int_{\tau_m} d \bar{f}^{[\lambda_1 \dots \lambda_m \bar{v}^{\alpha_1 \dots \alpha_h \mu]}$$

$$(102) \quad \binom{n}{m+1} \int_{\tau_{m+1}} d \bar{f}^{[\nu \lambda_1 \dots \lambda_m \partial_\mu \bar{v}^{\alpha_1 \dots \alpha_n] \mu}} = \binom{n}{m} \int_{\tau_m} d \bar{f}^{[\lambda_1 \dots \lambda_m \bar{v}^{\alpha_1 \dots \alpha_n \nu]}.$$

They evidently are independent of the choice of  $\bar{k}$  and therefore invariant under arbitrary transformations of coordinates.

If we consider only changes of the coordinatesystem for which  $\Delta$  in  $\tau_{m+1}$  and on  $\tau_m$  has everywhere the same sign, also formulae hold like

$$(103) \quad \frac{1}{h!} \int_{\tau_{m+1}} d\bar{f}_{\lambda_1 \dots \lambda_h} \partial_\mu v^{\lambda_1 \dots \lambda_h \mu} = \frac{1}{(h+1)!} \int_{\tau_m} d\bar{f}_{\lambda_1 \dots \lambda_{h+1}} v^{\lambda_1 \dots \lambda_{h+1}}.$$

They express equalities between two  $W$ -scalars, which in this special case can be defined for the whole region concerned.

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