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by

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1. *Introduction.* The well-known theorem that a quadratic form $Q = a_{ij}x_ix_j$ $[a_{ij}=a_{ji}]$ of rank r is equivalent to a form $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_r y_r^2$ with diagonal matrix is the same as the statement that the matrix $A = (a_{ij})$ of Q can be „factored" into $B'DB$, where D is the diagonal matrix

$$\left\|\begin{matrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_r \end{matrix}\right\|,$$

B' denotes the transpose of B, and B is a matrix of rank r with r rows. If we write $B = (b_{\alpha i}) = (b_{\alpha j})$, we have

$$A = \left(\sum_{\alpha=1}^{r} \lambda_\alpha b_{\alpha i} b_{\alpha j}\right).$$

In the present paper *we are concerned with the problem of „factorability" of a general symmetric matrix* $(a_{ij\ldots m})$ *into a form*

$$\left(\sum_{\alpha=1}^{\sigma} \lambda_\alpha b_{\alpha i} b_{\alpha j} \cdots b_{\alpha m}\right), \tag{1.1}$$

where σ is finite. If A factors as in (1.1) the associated form $a_{ij\ldots m}\, x_i x_j \cdots x_m$ can be written as a linear combination of powers of linear forms. Such linear combinations are useful in treating some of the classical problems of algebra [1]).

2. *Definitions.* We shall say that a matrix $A = (a_{ij\ldots m})$ is *p-way* if it has p indices $i, j, \ldots, m$. If each index ranges over $1, 2, \ldots, n$, we say that A is of *order n.* In the introduction and in what follows the term symmetric matrix refers to a matrix

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[1]) R. Oldenburger, Representation and equivalence of forms [Proceedings Nat. Acad. Sci. 24 (1938), 193—198].

for which the values of the elements are unchanged under permutation of the subscripts. If a matrix A can be written as (1.1) with elements in a field K, we shall say that A is *factorable with respect to* K.

3. *Factorability.* In the following theorem, the term „order" of K refers to the number of elements in the field K.

THEOREM 3.1. *The class of symmetric p-way matrices factorable with respect to a field K is identical with the class of all symmetric p-way matrices if and only if K is of order p or more.*

We shall sketch the proof of Theorem 3.1 leaving out some of the more complicated details.

A p-way matrix $A = (a_{ij\ldots m})$ of order n is factorable if and only if there exist elements $\lambda_\alpha, b_{\alpha i}$ $[\alpha = 1, 2, \ldots, \sigma;$ $i = 1, 2, \ldots, n]$ such that the following equations are satisfied:

$$\sum_{\alpha=1}^{\sigma} \lambda_\alpha b_{\alpha i} b_{\alpha j} \cdots b_{\alpha m} = a_{ij\ldots m}. \tag{3.1}$$

This is a system of linear equations in the λ's. Due to the symmetry of A many equations are repeated in (3.1). When we expand $(x_1+x_2+\ldots+x_n)^p$ we obtain a sum

$$\sum_{i=1}^{N} a_i f_i(x),$$

where the a_i are integers, and the f_i are distinct power products of degree p in the x_j $[j=1, 2, \ldots, n]$. We let b_i denote the set of elements $(b_{i1}, b_{i2}, \ldots, b_{in})$ for each i in the set $1, 2, \ldots, \sigma$. The system of equations (3.1) for $\sigma = N$ is then equivalent to the set

$$\sum_{\alpha=1}^{N} f_\beta(b_\alpha)\lambda_\alpha = y_\beta \qquad (\beta=1, 2, \ldots, N), \tag{3.2}$$

where $y_1, y_2, \ldots, y_n$ are equal in some order to the elements of A. We assume that $(y_1, \ldots, y_n)$ is not the zero vector, since then A is trivial. If we can prove that we can choose the b_α in K so that the determinant

$$| D | = | f_\beta(b_\alpha)|$$

is not zero, there exist solutions for the λ's in (3.2), and A is factorable.

We write the matrix D as the matrix $(M_{\varrho\alpha})$ $[\varrho=1, 2, \ldots, n;$ $\alpha=1, 2, \ldots, N]$ where $M_{\varrho\alpha}$ is the minor of D composed of power products $f_\beta(b_\alpha)$ which contain $b_{\alpha\varrho}$ as a factor, and no $b_{\alpha\sigma}$ where $\sigma > \varrho$. The $M_{\varrho\alpha}$ are minors with one column only. We let t_ϱ denote the number of elements (rows) in $M_{\varrho\alpha}$. We construct minors $N_{\varrho\sigma}$ of D $[\varrho, \sigma=1, 2, \ldots, n]$ such that $N_{\varrho\sigma}$ is the matrix $(M_{\varrho\alpha})$ composed of the columns $M_{\varrho,\alpha}$ where α ranges over the values $g_\sigma + 1, g_\sigma + 2, \ldots, g_{\sigma+1}$, and g_σ is given by

$$g_1 = 0; \; g_\sigma = \sum_{i=1}^{\sigma-1} t_i.$$

The matrix D is then given by $(N_{\varrho\sigma})$ $[\varrho, \sigma=1, 2, \ldots, n]$. We set $b_{\alpha i} = 0$ in D when α is in the range $g_\sigma + 1, g_\sigma + 2, \ldots, g_{\sigma+1}$, and i in the range $\sigma + 1, \sigma + 2, \ldots, n$. That is, we set each $b_{\alpha i}$ equal to zero that occurs in $N_{\sigma+1,\sigma}, N_{\sigma+2,\sigma}, \ldots, N_{n\sigma}$ and not in $N_{1\sigma}, N_{2\sigma}, \ldots, N_{\sigma\sigma}$, so that we obtain

$$D = \left\| \begin{matrix} N_{11} & N_{12} & \cdots & N_{1,n-1} & N_{1n} \\ O & N_{22} & \cdots & N_{2,n-1} & N_{2n} \\ . & . & \cdots & . & . \\ O & O & \cdots & O & N_{nn} \end{matrix} \right\|.$$

The minor $N_{\sigma\sigma}$ is square and contains only elements $b_{\alpha\lambda}$, where $\lambda \leqq \sigma$. We take $b_{\alpha\sigma} = 1$ for α in the range $g_\sigma + 1, g_\sigma + 2, \ldots, g_{\sigma+1}$. The minor $N_{\sigma\sigma}$ is now, with possibly a rearrangement of rows, of the form

$$\| c_h^g \, d_h^r \cdots f_h^s \| \qquad \text{(column index is } h\text{)},$$

where $h = 1, 2, \ldots, t_\sigma$, and $g, r, \ldots, s$ are $\sigma - 1$ non-negative integral exponents satisfying the inequality

$$g + r + \cdots + s \leqq p - 1. \tag{3.3}$$

It is understood that $c_h^0, d_h^0, \ldots, f_h^0$ denote 1 for each h. The distinct sets of exponents $(g, r, \ldots, s)$ satisfying (3.3) are evidently in 1-1 correspondence with the integers in the range of h. We set h in 1-1 correspondence with sets $(i, j, \ldots, m)$ of $\sigma - 1$ non-negative integers $i, j, \ldots, m$ subject to the restriction.

$$i + j + \ldots + m \leqq p - 1. \tag{3.4}$$

For each set $(i, j, \ldots, m)$ and corresponding h we write

$$c_h = \alpha_i, \; d_h = \alpha_j, \ldots, f_h = \alpha_m,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{p-1}$ are indeterminates over K and $\alpha_0 = 1$. By this choice of the $c_h, \ldots, f_h$ the minor $N_{\sigma\sigma}$ takes on the form

$$(\alpha_i^g \, \alpha_j^r \ldots \alpha_m^s), \tag{3.5}$$

where the exponents satisfy (3.3) and (3.4). We remark that the exponents in (3.5) form a multipartite row index of $N_{\sigma\sigma}$, and the subscripts form a multipartite column index of $N_{\sigma\sigma}$. We shall need the following lemma.

LEMMA 3.1. *The matrix* (3.5) *is non-singular if* $\alpha_0(=1), \alpha_1, \alpha_2, \ldots, \alpha_{p-1}$ *are distinct elements in* K.

Lemma 3.1 can be proved by showing that the matrix (3.5) is equivalent to a triangular matrix with diagonal minors of the same form as (3.5) with p replaced by smaller integers. Since (3.5) is non-singular if it is of order 1 [that is, $p = 1$ in (3.3) and (3.4)], it follows by induction that Lemma 3.1 holds. Thus A is factorable if K is of order p or more.

To complete the proof of the theorem we assume that K is of order $\psi < p$. It is obviously necessary to consider only p-way matrices where $p \geqq 3$. We shall exhibit a p-way matrix A of order two which is not factorable with respect to K. We define A to be a p-way symmetric matrix $(a_{ij\ldots m})$ of order 2 whose non vanishing elements are those which have exactly ψ subscripts equal to 1; the non-vanishing elements of A are taken equal to one. We let S denote the subset of the equations (3.1) for which $(i, j, \ldots, m)$ range over the sets of values $(2, 2, \ldots, 2)$, $(2, 2, \ldots, 2, 1)$, $(2, 2, \ldots, 2, 1, 1)$, $\ldots$, $(2, 2, \ldots, 2, 1, \ldots, 1)$, where there are ψ 1's in the last set. If there is no solution for the λ's in the set S there is no solution for the λ's in (3.1). We assume that there is a positive integer σ, and that there are values λ_α, $b_{\alpha i}$, in K so that S is satisfied. The matrix $T = (b_{\alpha i} b_{\alpha j} \ldots b_{\alpha m})$ of coefficients of the λ's in S is the following $(\psi+1)$ by σ rectangular matrix:

$$\left\|\begin{matrix} b_{12}^{p} & b_{22}^{p} & \cdots & b_{\sigma 2}^{p} \\ b_{12}^{p-1} b_{11} & b_{22}^{p-1} b_{21} & \cdots & b_{\sigma 2}^{p-1} b_{\sigma 1} \\ \cdot & \cdot & \cdots & \cdot \\ b_{12}^{p-\psi} b_{11}^{\psi} & b_{22}^{p-\psi} b_{21}^{\psi} & \cdots & b_{\sigma 2}^{p-\psi} b_{\sigma 1}^{\psi} \end{matrix}\right\|.$$

Since K is of order ψ, it follows from the theory of Vandermonian determinants that each possible $(\psi+1)$-st order minor of T vanishes for each choice of the b's. Thus for a choice of the b's the rank of T is r, where $r < \psi + 1$. The matrix

$$T' = \left\|\begin{matrix} & 0 \\ T & \vdots \\ & 0 \\ & 1 \end{matrix}\right\|$$

obtained by adjoining the column of elements $(a_{2\ldots 2})$, $(a_{2\ldots 21})$, $\ldots$, $(a_{2\ldots 21\ldots 1})$ of A occurring in S, is the augmented matrix of the set S. Since $r \leqq \psi$, the rank of T is $r+1$. The ranks of T and T' are thus unequal. By the well-known theorem that a system of linear equations has a solution if and only if the rank of the matrix of coefficients equals the rank of the augmented matrix, the set S has no solution for the λ's. Thus A is not factorable. The proof of Theorem 3.1 is now complete.

4. *Example.* Let $A = (a_{ij})$ be a symmetric matrix of order 2. Equations (3.2) now become

$$\sum_{\alpha=1}^{3} \lambda_\alpha b_{\alpha 1}^2 = a_{11},$$

$$\sum_{\alpha=1}^{3} \lambda_\alpha b_{\alpha 1} b_{\alpha 2} = a_{12},$$

$$\sum_{\alpha=1}^{3} \lambda_\alpha b_{\alpha 2}^2 = a_{22}.$$

The matrix D is

$$\begin{Vmatrix} b_{11}^2 & b_{21}^2 & b_{31}^2 \\ b_{11}\,b_{12} & b_{21}\,b_{22} & b_{31}\,b_{32} \\ b_{12}^2 & b_{22}^2 & b_{32}^2 \end{Vmatrix}.$$

Now

$$D = \begin{Vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \end{Vmatrix},$$

where $M_{1i} = b_{i1}^2$ for $i = 1, 2, 3$, and

$$M_{2i} = \begin{Vmatrix} b_{i1} & b_{i2} \\ & b_{i2}^2 \end{Vmatrix}.$$

We write $N_{11} = M_{11}$; $N_{21} = M_{21}$, $N_{12} = (M_{12} M_{13})$, $N_{22} = (M_{22} M_{23})$, whence

$$D = \begin{Vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{Vmatrix},$$

where N_{11}, N_{22} are square minors of orders 1 and 2, respectively. Setting $b_{12} = 0$, we get

$$D = \begin{Vmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{Vmatrix}.$$

Taking $b_{11} = b_{22} = b_{32} = 1$, we obtain

$$N_{11} = 1,\ N_{22} = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} c_1^0 & c_2^0 \\ c_1^1 & c_2^1 \end{Vmatrix}.$$

We write $c_1 = \alpha_0$, $c_2 = \alpha_1$, whence the last matrix above becomes

$$\begin{Vmatrix} \alpha_0^0 & \alpha_1^0 \\ \alpha_0^1 & \alpha_1^1 \end{Vmatrix}.$$

Taking $\alpha_0 = 1$, and $\alpha_1 \neq 1$, we arrive at a non-singular specialization of D.

5. *Note on the matrix* (3.5). The non-singularity of the matrix (3.5) for distinct α's may be used to give a new proof of the following theorem. The proof is not shorter than existing proofs, but is merely given to illustrate a use of (3.5).

THEOREM 5.1. *Let P be a polynomial of degree p with coefficients in a field K of order $p+1$ or more. If P is zero for all values of the variables in K, then P is identically zero* (*that is, all coefficients of P vanish*).

The polynomial $P = P(x, y, \ldots, z)$ can be written as

$$\sum_{r, s, \ldots, t} a_{rs\ldots t}\, x^r y^s \cdots z^t, \tag{5.1}$$

where $x, y, \ldots, z$ are the variables in P, say n in all, and the summation is over all admissible values of $r, s, \ldots, t$. Let $\alpha_0 = 1$, and $\alpha_0, \alpha_1, \ldots, a_p$ be $p+1$ distinct elements in K. Let the set $S = (\alpha_i, \alpha_j, \ldots, \alpha_m)$ correspond to the term $a_{ij\ldots m}\, x^i y^j \cdots z^m$ in (5.1). This correspondence is unique. Substitute the sets of values S for $(x, y, \ldots, z)$ in the equation $P = 0$. We thus obtain the set of linear equations

$$\sum_{r, s, \ldots, t} a_{rs\ldots t}\, \alpha_i^r \alpha_j^s \cdots \alpha_m^t = 0$$

homogeneous in the a's. Since by Lemma 3.1 the matrix $(\alpha_i^r a_j^s \cdots a_m^t)$ of coefficients is non-singular, the a's vanish.

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