

COMPOSITIO MATHEMATICA

JAKOB LEVITZKI

On rings which satisfy the minimum condition for the right-hand ideals

Compositio Mathematica, tome 7 (1940), p. 214-222

http://www.numdam.org/item?id=CM_1940__7__214_0

© Foundation Compositio Mathematica, 1940, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

On rings which satisfy the minimum condition for the right-hand ideals

by

Jakob Levitzki

Jerusalem

Introduction.

A ring S which satisfies the double-chain-condition (or the equivalent maximum and minimum condition) for the right-hand (in short: r.h.) ideals, possesses according to E. Artin ¹⁾ a nilpotent radical R , and the quotient ring S/R is semi-simple. This fact, as well as the results obtained by Artin concerning the „primary” and the „completely primary” rings attached to S are valid for a wider class of rings. *In the present note it is shown that the maximum condition can be omitted without affecting the results achieved by Artin.*

The method used in the present note is partly an improvement of one used by the author in a previous paper ²⁾. On the other hand, the results obtained presently yield a generalisation of the principal theorem proved in L , which can be stated now as follows: *Each nil-subring of a ring which satisfies the minimum condition for the r.h. ideals, is nilpotent.* This statement is in particular a solution of a problem raised by G. Köthe ³⁾, whether or not there exist potent nil-rings which satisfy the maximum or the *minimum* condition for the r.h. ideals.

The importance of the nil-rings was first emphasized by Köthe ³⁾, who considers rings of a more general type but of similar structure, and which actually might contain potent (i.e. non-nilpotent) nil-subrings, and even nil-ideals.

¹⁾ E. ARTIN, Zur Theorie der hyperkomplexen Zahlen [Hamb. Abh. 5 (1927), 251—260], referred to as A.

²⁾ J. LEVITZKI, Über nilpotente Unterringe [Math. Ann. 105 (1931), 620—627], referred to as L.

³⁾ G. KÖTHE, Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist [Math. Zeitschr. 32 (1930), 161—186], referred to as K.

I. Notations and preliminary remarks.

1. As it is well known, a ring S is said to satisfy the minimum (maximum) condition for the r.h. ideals, if each non empty set of r.h. ideals contains a r.h. ideal which is not a proper subset of any other r.h. ideal of the set (respectively, which does not contain any other r.h. ideal of the set as a proper subset). This is equivalent with the condition that for each infinite „descending chain” of r.h. ideals $\mathfrak{R}_1 \supseteq \mathfrak{R}_2 \supseteq \dots$ (resp. for each „ascending chain” $\mathfrak{R}_1 \subseteq \mathfrak{R}_2 \subseteq \dots$) an index m exists such that $\mathfrak{R}_i = \mathfrak{R}_k$ if $m < i, m < k$. We say in short: S is a r.h.m.c.-ring.

2. If A is a finite or an infinite subset of a ring S , then let \bar{S} be the set containing A and all the finite sums and differences of all the finite products which can be derived from the elements of A . Then \bar{S} is evidently a subring of S , and we say: \bar{S} is generated by A . Evidently \bar{S} is the minimal subring (in the usual sense) of S containing A .

3. If A and B are subsets of a ring S , then we denote by AB the set of all the finite sums of elements which have the form ab , where $a \in A, b \in B$. Thus if B is a r.h. ideal or A is a l.h. ideal, then also AB is a r.h. ideal, or respectively a l.h. ideal.

4. If \mathfrak{R} is a r.h. ideal of a ring S , and \bar{S} is a subset of S , then we denote by $(\bar{S}, \bar{S}\mathfrak{R})$ the minimal r.h. ideal containing \bar{S} as well as $\bar{S}\mathfrak{R}$, i.e. the set of all elements of the form $\sum \pm \lambda_i s_i + \sum s'_q r_q$ where the λ_i are positive integers, $s_i, s'_q \in \bar{S}$ and $r_q \in \mathfrak{R}$.

5. If $\mathfrak{R}_1, \dots, \mathfrak{R}_m$ are r.h. ideals of a ring S , then their sum, i.e. the set of all elements of the form $\sum_{i=1}^m r_i, r_i \in \mathfrak{R}_i$ is denoted by $(\mathfrak{R}_1, \dots, \mathfrak{R}_m)$.

6. If S' and S'' are subsets of a ring S , then we denote by $[S', S'']$ the set of all elements of S which belong to S' as well as to S'' . If S'' is a r.h. ideal in S and S' is a subring of S , then evidently $[S', S'']$ is a r.h. ideal in S' .

7. An element a of a ring S is called nilpotent of index n if $a^n = 0, a^{n-1} \neq 0$.

A nilring is a ring which contains nilpotent elements only.

A ring N is called nilpotent of index n , if $N^n = 0, N^{n-1} \neq 0$.

An element a of a ring S is called properly nilpotent, if the r.h. ideal aS is nilpotent.

8. A r.h. ideal \mathfrak{R} of a ring S is called primitive, if \mathfrak{R} is potent (i.e. not nilpotent) and does not contain a potent r.h. ideal as a proper subset. It follows easily, that if S is a r.h.m.c.-ring, then each potent r.h. ideal of S contains a primitive r.h. ideal.

II. On subrings generated by finite subsets.

LEMMA 1. If S is a r.h.m.c.-ring (see I, 1), and if a finite set of elements a_1, \dots, a_m of S generates a potent ring N (see I, 2), then for a certain positive integer λ the r.h. ideal $\mathfrak{R} = N^\lambda S$ which is evidently different from zero (since N is potent), satisfies the relations

$$(1) \quad (a_1 \mathfrak{R}, \dots, a_m \mathfrak{R}) = \mathfrak{R}; \quad \mathfrak{R} = N^\lambda S = N^{\lambda+1} S = \dots$$

Proof. Since evidently $NS \supseteq N^2 S \supseteq N^3 S \supseteq \dots$, it follows by I, 1 that for a certain positive integer λ the relations $N^\lambda S = N^{\lambda+1} S = \dots$ are true. From the definition of N it follows further that $N^2 = (a_1 N, \dots, a_m N)$, and hence $N^\lambda S = N^{\lambda+2} S = N^2(N^\lambda S) = (a_1 N, \dots, a_m N)N^\lambda S = (a_1 N^{\lambda+1} S, \dots, a_m N^{\lambda+1} S) = (a_1 N^\lambda S, \dots, a_m N^\lambda S)$; by setting $\mathfrak{R} = N^\lambda S$ we have $\mathfrak{R} = (a_1 \mathfrak{R}, \dots, a_m \mathfrak{R})$, which completes the proof.

LEMMA 2⁴). If S is a r.h.m.c.-ring, and if for a certain r.h. ideal \mathfrak{R} , $\mathfrak{R} \neq 0$, and a finite set of elements a_1, \dots, a_m of S the relation $\mathfrak{R} = (a_1 \mathfrak{R}, \dots, a_m \mathfrak{R})$ holds, then there exists an infinite sequence b_1, b_2, \dots each b_λ being a certain a_j , such that the relation $0 \subset b_\lambda b_{\lambda+1} \cdots b_{\lambda+\sigma} \mathfrak{R} \subseteq b_\lambda \mathfrak{R}$ is satisfied for arbitrary positive integers λ and σ .

Proof. From $\mathfrak{R} \neq 0$ it follows that not all the $a_i \mathfrak{R}$ are zero. Let i_1 be the minimal index such that $a_{i_1} \mathfrak{R} \neq 0$. Since $a_{i_1} \mathfrak{R} = (a_{i_1} a_1 \mathfrak{R}, \dots, a_{i_1} a_m \mathfrak{R}) \neq 0$, it follows that not all the $a_{i_1} a_\sigma \mathfrak{R}$ are zero, and again let i_2 be the minimal index such that $a_{i_1} a_{i_2} \mathfrak{R} \neq 0$. This process can be infinitely repeated, and setting $b_j = a_{i_j}$ we obtain (by induction) the required infinite sequence. The lemma follows now from

$$0 \subset b_\lambda b_{\lambda+1} \cdots b_{\lambda+\sigma} \mathfrak{R} \subseteq b_\lambda \cdots b_{\lambda+\sigma-1} \mathfrak{R} \subseteq \dots \subseteq b_\lambda \mathfrak{R}.$$

THEOREM 1. If S is a r.h.m.c.-ring and N is a subring of S which is generated by a finite subset a_1, a_2, \dots, a_m , then N is potent if and only if S contains a r.h. ideal \mathfrak{R} such that $\mathfrak{R} \neq 0$ and $(a_1 \mathfrak{R}, \dots, a_m \mathfrak{R}) = \mathfrak{R}$.

Proof. This is an immediate consequence of lemmas 1 and 2.

THEOREM 2. Each nil-subring of a r.h.m.c.-ring S , which is generated by a finite subset of S is nilpotent. Moreover: Let N be a potent subring of S which is generated by the finite set

⁴) The proof of this lemma is included in the proof of theorem 2 in L.

d_1, d_2, \dots, d_m ; then N contains potent elements of the form $c_1c_2 \cdots c_j$, where each c_i is a certain d_j .

Proof. Let a_1, \dots, a_s be a subset of the d_j so that the ring N which is generated by the a_i is still potent and s is of the least possible value. By lemma 1 follows the existence of a r.h. ideal \mathfrak{R} so that $\mathfrak{R} \neq 0$, and the relations (1) are satisfied. By lemma 2 we deduce the existence of an infinite sequence b_1, b_2, \dots such that each b_j is a certain a_i and the relation

$$(2) \quad 0 \subset b_\lambda b_{\lambda+i} \cdots b_{\lambda+\varrho} \mathfrak{R} \subseteq b_\lambda \mathfrak{R}$$

is satisfied for each λ and ϱ . If now $s = 1$ then obviously a_1 is a potent element and the theorem is proved. If $s > 1$ then by assumption each proper subset of the a_i generates a nilpotent ring, in particular each a_i is nilpotent. In this case let t denote the index of the nilpotent element a_1 and let u be the index of the nilpotent ring which is generated by the finite set a_2, \dots, a_s . It follows easily that in each product of the form $p_{\lambda, \varrho} = b_{\lambda+1} b_{\lambda+2} \cdots b_{\lambda+\varrho}$ where $\varrho > t$, $\varrho > u$ at least one of the factors must be equal to a_1 and at least one of them must be different from a_1 , hence, for a certain positive integer k (which can be chosen so that $k \leq u$) the elements of the infinite sequence b_k, b_{k+1}, \dots can be joined to finite subsequences $b_k, b_{k+1}, \dots, b_{k+\varrho_1}; b_{k+\varrho_1+1}, b_{k+\varrho_1+2}, \dots, b_{k+\varrho_1+\varrho_2}; \dots$ which have the following properties: The product of the elements of each subsequence, the factors being taken in the written order, has the form $a_1^\sigma a_{i_1}, a_{i_2} \cdots a_{i_\varrho}$, where $i_\lambda \neq 1$, $\lambda = 1, \dots, \varrho$, hence

$$(3) \quad \sigma + \varrho < t + u.$$

We denote by p_λ the product which belongs to the λ^{th} subsequence, and thus obtain the infinite sequence

$$(4) \quad p_1, p_2, \dots$$

Since the p_λ are of the form $a_1^\sigma a_{i_1} \cdots a_{i_\varrho}$, where the a_{i_λ} are taken from the finite set a_2, \dots, a_s , it follows by (3) that also p_λ belong to a certain finite set which we denote by $\bar{d}_1, \dots, \bar{d}_m$. Since, further, from the definition of the p_λ it follows easily that each product of the form $p_\lambda p_{\lambda+1} \cdots p_{\lambda+\varrho}$ is equal to a certain $p_{i, k}$, we have

$$(5) \quad 0 \subset p_\lambda p_{\lambda+1} \cdots p_{\lambda+\varrho} \mathfrak{R} \subseteq P_\lambda \mathfrak{R} \subseteq a_1 \mathfrak{R}$$

for arbitrary positive λ and ϱ . The ring generated by the \bar{d}_λ is in virtue of (5) potent and evidently a subring of N . Again let

$\bar{a}_1, \dots, \bar{a}_s$ be a subset of the \bar{d}_i such that the ring \bar{N} which is generated by the \bar{a}_i — and is therefore a subring of N — is still potent, while s' is of the least possible value. As before follows the existence of a r.h. ideal $\bar{\mathfrak{R}}$ such that the relations (1), in which λ , \mathfrak{R} and N are replaced respectively by λ' $\bar{\mathfrak{R}}$ and \bar{N} where λ' is a suitably chosen integer, are again satisfied. Since $\bar{N} \subseteq N$ we have $\bar{\mathfrak{R}} = \bar{N}^{\lambda'} S = \bar{N}^{\lambda+\lambda'} S \subseteq N^{\lambda+\lambda'} S = N^{\lambda} S = \mathfrak{R}$, hence $\bar{a}_i \bar{\mathfrak{R}} \subseteq \bar{a}_i \mathfrak{R}$; on the other hand, from the definition of the \bar{a}_i follows by (5) that $\bar{a}_i \mathfrak{R} \subseteq a_1 \mathfrak{R}$, and hence we have

$$(6) \quad \bar{\mathfrak{R}} = (\bar{a}_1 \bar{\mathfrak{R}}, \dots, \bar{a}_s \bar{\mathfrak{R}}) \subseteq (\bar{a}_1 \mathfrak{R}, \dots, \bar{a}_s \mathfrak{R}) \subseteq a_1 \mathfrak{R}.$$

Since evidently $a_1 \mathfrak{R} \subset \mathfrak{R}$ (otherwise $a_1 \mathfrak{R} = \mathfrak{R}$, i.e. the element a_1 is potent, which contradicts $s > 1$) it follows by (6) that

$$(7) \quad \bar{\mathfrak{R}} \subset \mathfrak{R}.$$

If now $\bar{s} = 1$, then $\bar{a}_1 \bar{\mathfrak{R}} = \bar{\mathfrak{R}}$, i.e. the element \bar{a}_1 is potent and has according to its definition the required form, hence in this case the theorem is proved. In case $\bar{s} > 1$, the process which applied on the r.h. ideal \mathfrak{R} leads to the construction of the r.h. ideal $\bar{\mathfrak{R}}$ such that $\bar{\mathfrak{R}} \subset \mathfrak{R}$, can be now repeatedly applied on $\bar{\mathfrak{R}}$ and thus in a similar way the r.h. ideal $\bar{\bar{\mathfrak{R}}}$ can be found so that

$$(8) \quad \mathfrak{R} \supset \bar{\mathfrak{R}} \supset \bar{\bar{\mathfrak{R}}}.$$

Thus each new step either provides a potent element which has the required form, or adds a further r.h. ideal which is a proper subset of the preceding. By the minimum condition it follows therefore that after a finite number of steps a potent element can be found which has the required form.

III. On the divisors of zero in a r.h.m.c.-ring.

THEOREM 3. Let S be a r.h.m.c.-ring, S' a subring of S and \mathfrak{R} a r.h. ideal in S ; let further T denote the maximal subset of S' satisfying the relation $S'T \subseteq \mathfrak{R}$ ⁵⁾. Then a finite subset \bar{S} of S' can be found so that T is also the maximal subset of S' satisfying the relation $\bar{S}T \subseteq \mathfrak{R}$.

Proof. Let S_0 be an arbitrary finite subset of S' and T_0 the maximal subset of S' satisfying the relation $S_0 T_0 \subseteq \mathfrak{R}$. We consider now the set of all r.h. ideals of the form $\mathfrak{R}_0 = (T_0, T_0 S)$,

⁵⁾ i.e. T is the set of all elements t of S which satisfy the relative $St \subseteq \mathfrak{R}$.

then in virtue of the minimum condition follows the existence of a certain finite subset $\bar{S} = (a_1, \dots, a_\lambda)$ of S so that if \bar{T} is the maximal subset of S' satisfying the relation $\bar{S}\bar{T} \subseteq \mathfrak{K}$, the r.h. ideal $\bar{\mathfrak{R}} = (\bar{T}, \bar{T}S)$ is minimal (i.e. for any r.h. ideal \mathfrak{R}_0 of the set we have $\bar{\mathfrak{R}} \supset \mathfrak{R}_0$). From $S'T \subseteq \mathfrak{K}$ follows in particular $\bar{S}T \subseteq \mathfrak{K}$, i.e. $T \subseteq \bar{T}$; the theorem will be evidently proved if we show that $T = \bar{T}$. To this end it is evidently sufficient to prove that each element a of S' satisfies the relation $a\bar{T} \subseteq \mathfrak{K}$. In fact, setting $\bar{\bar{S}} = (a_1, \dots, a_\lambda, a)$ and denoting by $\bar{\bar{T}}$ the maximal subset of S' satisfying the relation $\bar{\bar{S}}\bar{\bar{T}} \subseteq \mathfrak{K}$, it follows from $\bar{\bar{S}} \supseteq \bar{S}$ that $\bar{\bar{T}} \subseteq \bar{T}$; hence by setting $\bar{\bar{\mathfrak{R}}} = (\bar{\bar{T}}, \bar{\bar{T}}S)$ we have $\bar{\bar{\mathfrak{R}}} \subseteq \bar{\mathfrak{R}}$ which in view of the minimality of $\bar{\mathfrak{R}}$ implies $\bar{\bar{\mathfrak{R}}} = \bar{\mathfrak{R}}$. Since $\bar{\mathfrak{R}} \supseteq \bar{T}$, $\bar{\bar{\mathfrak{R}}} \supseteq \bar{\bar{T}}$, and evidently $\bar{S}\bar{\mathfrak{R}} \subseteq \mathfrak{K}$, $\bar{\bar{S}}\bar{\bar{\mathfrak{R}}} \subseteq \mathfrak{K}$ it follows that $\bar{T} = [\bar{\mathfrak{R}}, S']$, $\bar{\bar{T}} = [\bar{\bar{\mathfrak{R}}}, S']$ which implies $\bar{T} = \bar{\bar{T}}$, i.e. $a\bar{T} \subseteq \mathfrak{K}$, q.e.d.

COROLLARY. Evidently the set \bar{S} can be replaced by any finite subset of S' containing \bar{S} ; in case $S' \neq 0$ it may be therefore assumed that also $\bar{S} \neq 0$.

THEOREM 4. Let S be a r.h.m.c.-ring, $S' \neq 0$ a nil-subring of S and T the maximal subset of S' satisfying the relation $S'T = 0$; then $T \neq 0$.

Proof. Applying theorem 3 to the special case $\mathfrak{K} = 0$ we deduce the existence of a finite subset \bar{S} of S' such that the maximal subset T of S' which satisfies the relation $S'T = 0$ is also the maximal subset of S' satisfying the relation $\bar{S}T = 0$ and furthermore (corollary to theorem 3) $\bar{S} \neq 0$. If now N denotes the nil-subring of S which is generated by \bar{S} , then (by theorem 2) N is nilpotent and hence denoting by m the index of N we have $N^{m-1} \neq 0$ and $0 = N^m = NN^{m-1} \supseteq \bar{S}N$, i.e. $N^{m-1} \subseteq T$ which implies $T \neq 0$, q.e.d.

THEOREM 5. Let S be a r.h.m.c.-ring, S' a nil-subring of S , T the maximal subset of S' satisfying the relation $S'T = 0$ and \bar{T} the maximal subset of S' satisfying the relation $S'\bar{T} \subseteq (T, TS)$. Then if $S' \supset T$ also $\bar{T} \supset T$.

Proof. Applying theorem 3 to the special case $\mathfrak{K} = (T, TS)$ we deduce the existence of a finite subset $\bar{S} = (a_1, \dots, a_m)$ of S' such that \bar{T} is the maximal subset of S' satisfying the relation $\bar{S}\bar{T} \subseteq T$. Since for $S' = 0$ the theorem is self evident, we may assume $S' \neq 0$ and hence (by the corollary to theorem 3) also $\bar{S} \neq 0$. From $S' \supset T$ follows the existence of an element a of S

such that $a \in S'$ but $a \notin T$; by the corollary to theorem 3 we may now replace \bar{S} by the set $S^* = (a_1, \dots, a_m, a)$. Let now N^* be the ring generated by S^* then (by theorem 2) N^* is nilpotent, and if λ is the index of N^* , it follows from $N^{*\lambda} = 0$ that $S^* N^{*\lambda-1} = 0 \subseteq (T, TS)$, and hence $N^{*\lambda-1} \subseteq \bar{T}$. Let now r be the smallest positive integer for which $N^{*r} \subseteq \bar{T}$; then in case $r = 1$ the theorem is true, since then $N^* \subseteq \bar{T}$ but N^* not $\subseteq T$ in virtue of $a \in N$, $a \notin T$; in case $r > 1$ we prove that N^r not $\subseteq T$ and thus complete the proof of the theorem. In fact, from $N^r \subseteq T$ would follow $NN^{r-1} \subseteq T$ and hence $N^{r-1} \subseteq \bar{T}$ in contradiction to the minimality of r .

COROLLARY. If S' is a nil-subring of S such that $S'^2 = S'$, then $S' = 0$.

Proof. In fact, from $S'\bar{T} \subseteq (T, TS)$ follows $S'^2 T \subseteq (S'T, S'TS) = 0$. Further in virtue of $S'^2 = S'$ we have $S'\bar{T} = 0$, which implies $\bar{T} \subseteq T$. Since in general $T \subseteq \bar{T}$, we obtain $T = \bar{T}$, and hence (by the theorem just proved) $S' = T$. From $S'\bar{T} = 0$ follows $S'^2 = 0$, q.e.d.

IV. On the structure of a r.h.m.c.-ring.

The proof of the theorems concerning the structure of a r.h.m.c.-ring which were announced in the introduction follow now easily from the results obtained in the previous sections.

THEOREM 6. Each r.h.m.c.-ring S possesses a nilpotent radical.

Proof. In fact, let \mathfrak{R} be the radical of S , i.e. the set of all properly nilpotent elements (see I, 7) of S , then, as it is well known, \mathfrak{R} is a nil-subring (moreover: a nil-ideal) of S . Applying the minimum condition we deduce from $\mathfrak{R} \supseteq \mathfrak{R}^2 \supseteq \mathfrak{R}^3 \supseteq \dots$ the existence of a positive integer λ such that $\mathfrak{R}^\lambda = \mathfrak{R}^{\lambda+1}$, and hence $\mathfrak{R}^\lambda = (\mathfrak{R}^\lambda)^2$ which by the corollary to theorem 5 implies $\mathfrak{R}^\lambda = 0$, i.e. \mathfrak{R} is nilpotent.

THEOREM 7. Each potent r.h. ideal \mathfrak{R} of a r.h.m.c.-ring S possesses a potent element.

Proof. By the minimum condition follows from $\mathfrak{R} \supseteq \mathfrak{R}^2 \supseteq \mathfrak{R}^3 \supseteq \dots$ as in the proof of theorem 6 the existence of a positive integer λ which satisfies the relation $(\mathfrak{R}^\lambda)^2 = \mathfrak{R}^\lambda$; suppose \mathfrak{R}^* , and hence also \mathfrak{R}^λ were nil-rings, then by the corollary to theorem 5, \mathfrak{R}^λ and hence also \mathfrak{R} were nilpotent, which contradicts the assumption of the theorem.

THEOREM 8. Each potent r.h. ideal \mathfrak{R} of a r.h.m.c.-ring S possesses an idempotent element e (i.e. $e^2=e$, $e \neq 0$).

Proof. By the minimum condition follows the existence of a primitive r.h. ideal \mathfrak{R}' such that $\mathfrak{R}' \subseteq \mathfrak{R}$. Let r be a potent element of \mathfrak{R}' (theorem 7), then evidently $r\mathfrak{R}' = \mathfrak{R}'$. The argument which now leads to the proof of the theorem is exactly the same as that of theorem 7 in \mathbf{K} , we merely have to replace the regular r.h. ideals of \mathbf{K} by the potent r.h. ideals of the present note.

THEOREM 9. If S is a r.h.m.c.-ring, then S is either nilpotent, or it can be represented as a direct sum of primitive r.h. ideals and a nilpotent r.h. ideal.

Proof. If S is nilpotent then the theorem is self evident. If S is potent, then let R_1 be a primitive r.h. ideal of S and e_1 an idempotent element of R_1 (theorem 8); then evidently $R_1 = e_1S$. If $S = \mathfrak{R}_1$ then the theorem is proved. If $S \neq \mathfrak{R}_1$ then the set of all elements of the form $s - e_1s$, where $s \in S$, is a r.h. ideal S_1 which is different from zero and we obtain the representation $S = S_1 + \mathfrak{R}_1$ (direct sum) where $S_1 \subseteq S$. If S_1 is nilpotent then the theorem is proved: if S_1 is potent, then by similar argument we obtain $S_1 = S_2 + \mathfrak{R}_2$, where \mathfrak{R}_2 is a primitive r.h. ideal, S_2 is a potent or a nilpotent r.h. ideal and $S = S_2 + \mathfrak{R}_1 + \mathfrak{R}_2$, $S \supseteq S_1 \supseteq S_2$. Hence by the minimum condition we finally obtain the desired representation.

The usual methods lead now from theorems 6 and 9 to

THEOREM 10. Let S be a r.h.m.c.-ring and \mathfrak{R} the nilpotent radical of S , then the quotient-ring S/\mathfrak{R} is semi-simple.

The further theorems describing the structural type of a r.h.m.c.-ring can be now obtained exactly as by Artin, and may be omitted here.

As a generalisation of the principal theorem of \mathbf{L} we finally state

THEOREM 12. Each nil-subring S' of a r.h.m.c.-ring S is nilpotent. If further m is the index of the radical R of S and l is the length of the semi-simple ring S/R , then $m + l + 1$ is an upper bound for the indices of the nilpotent subrings of S .

Proof. In fact, let S^* be the homomorphic image of S' in S/R , then S^* is evidently also a nilring. In \mathbf{L} it was proved that each nil-subring of a ring S which satisfies the minimum and the maximum condition is nilpotent, and if l is the length of S then $l + 1$ is an upper bound for the indices of the nilpotent subrings

of S . Since S/R is semi-simple, we may apply the theorem just stated and find that $S^{*^{t+1}}$ is zero in S/R , i.e. $S'^{t+1} \subseteq R$ and hence $(S'^{t+1})^m = S'^{(t+1)m} = 0$, q.e.d. ⁶⁾

(Received May 10th, 1939.)

⁶⁾ The present note was sent for publication in October 1938. In December 1938 a note by Charles Hopkins was published on „Nilrings with minimal condition for admissible left ideals” (Duke Math. Journ. 4 (1938), 664—667) in which some of the main results of the present note are proved by a different method. Nevertheless I trust that the present note might be still of some interest since the method used here can be applied also to other interesting classes of rings as I hope to show in a following communication.
