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# A theorem on Banach spaces

by

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1. Let  $E$  be a normed complete linear vector space, that is to say a space ( $B$ ) in the terminology of S. Banach<sup>1</sup>), let  $E_1, E_2, E_3, \dots, E_k$  ( $k \geq 1$ ) be linear subspaces of  $E$ , which are linearly independent.<sup>2</sup>) Let  $E_1 \dot{+} E_2 \dot{+} E_3 \dot{+} \dots \dot{+} E_k$  be the smallest linear subspace of  $E$ , which contains all of  $E_1, E_2, \dots, E_k$ . Of course every element  $\psi$  of  $E_1 \dot{+} E_2 \dot{+} \dots \dot{+} E_k$  can be represented uniquely in the form  $\psi = \varphi_1 + \varphi_2 + \dots + \varphi_k$  ( $\varphi_1 \in E_1, \varphi_2 \in E_2, \dots, \varphi_k \in E_k$ ).

**THEOREM 1.** *Let  $E$  be a ( $B$ ) space,  $E_1$  and  $E_2$  linear closed<sup>3</sup>) subspaces of  $E$  and linearly independent, then the space  $E_{12} = E_1 \dot{+} E_2$  is closed if, and only if, there exists some constant  $A$  such that, for all elements  $\varphi_1, \varphi_2$  ( $\varphi_1 \in E_1, \varphi_2 \in E_2$ )*

$$(1) \quad \|\varphi_1\| \leq A \|\varphi_1 + \varphi_2\|. \quad (4)$$

Of course both  $E_1$  and  $E_2$  are ( $B$ ) spaces and, if the condition (1) is satisfied, so is  $E_{12}$ .

The proof of the sufficiency of (1) is quite trivial. Let  $\{\psi^{(n)}\}$  ( $n=1, 2, \dots$ ) be any convergent sequence<sup>5</sup>) of  $E_{12}$ ; then we have to show only that it converges to an element  $\psi$  belonging to  $E_{12}$ . Since  $\psi^{(j)} = \varphi_1^{(j)} + \varphi_2^{(j)}$  ( $j=1, 2, \dots$ ),  $\varphi_i^{(m)} - \varphi_i^{(n)} \in E_i$  ( $i=1, 2$ ), it follows from (1) that

$$\|\varphi_1^{(m)} - \varphi_1^{(n)}\| \leq A \|(\varphi_1^{(m)} - \varphi_1^{(n)}) + (\varphi_2^{(m)} - \varphi_2^{(n)})\| = A \|\psi^{(m)} - \psi^{(n)}\| \rightarrow 0,$$

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<sup>1</sup>) Théorie des opérations linéaires, Warszawa 1932, 53; the norm of  $\varphi$  is  $\|\varphi\|$ .

<sup>2</sup>) This means: If  $\varphi_1 + \varphi_2 + \dots + \varphi_k = 0$ ,  $\varphi_i \in E_i$  ( $i=1, 2, \dots, k$ ), then all elements  $\varphi_i$  must be zéro. If  $k=2$ ,  $E_1$  and  $E_2$  are linearly independent if, and only if, they have no common element except the element zéro.

<sup>3</sup>) „fermé”, Banach l.c., 13.

<sup>4</sup>) Connected problems: H. KOBER [Proc. London Math. Soc. (2), 44 (1938), 453—65], Satz VI'b; see also a forthcoming paper in the Annals of Mathem., Satz IIIβ.

<sup>5</sup>) The sequence has to satisfy the condition of Cauchy  $\|\psi^{(m)} - \psi^{(n)}\| \rightarrow 0$  ( $m \geq n \rightarrow \infty$ ). Since  $\psi^{(j)} \in E$  and  $E$  is complete,  $\{\psi^{(n)}\}$  converges to an element  $\psi \in E$ ,  $\|\psi^{(n)} - \psi\| \rightarrow 0$ .

when  $m \geq n \rightarrow \infty$ . Now  $E_1$  is closed, so that the sequence  $\{\varphi_1^{(n)}\}$  converges to a limit point  $\varphi_1 \in E_1$ ; so also the sequence  $\{\varphi_2^{(m)}\}$  converges to a limit point  $\varphi_2 \in E_2$ , since

$$\begin{aligned} \|\varphi_2^{(m)} - \varphi_2^{(n)}\| &= \|(\psi^{(m)} - \psi^{(n)}) - (\varphi_1^{(m)} - \varphi_1^{(n)})\| \\ &\leq \|\psi^{(m)} - \psi^{(n)}\| + \|\varphi_1^{(m)} - \varphi_1^{(n)}\| \rightarrow 0 \quad (m \geq n \rightarrow \infty). \end{aligned}$$

Hence the sequence  $\{\psi^{(n)}\} \equiv \{\varphi_1^{(n)} + \varphi_2^{(n)}\}$  converges to  $\varphi_1 + \varphi_2 = \psi$  and plainly  $\varphi_1 + \varphi_2 = \psi$  belongs to  $E_1 \dot{+} E_2 = E_{12}$ .

The condition (1) is necessary. For to every element  $\psi \in E_1 \dot{+} E_2$  corresponds exactly one  $\varphi_1 \in E_1$  since  $\psi = \varphi_1 + \varphi_2$ ; hence  $T\psi = \varphi_1$  is an operation, which evidently is additive (Banach, 23); now let the sequences  $\{\psi^{(n)}\} \in E_1 \dot{+} E_2$  and  $\{\varphi_1^{(n)}\} \equiv \{T\psi^{(n)}\} \in E_1$  have the limits points  $\psi$  and  $\varphi_1$  respectively, and then plainly  $\psi \in E_1 \dot{+} E_2$ ,  $\varphi_1 \in E_1$ , since  $E_1 \dot{+} E_2$  and  $E_1$  are closed. We next show that  $T\psi = \varphi_1$ . Since  $\psi^{(j)} = \varphi_1^{(j)} + \varphi_2^{(j)}$ ,  $\varphi_1^{(j)} \in E_1$ ,  $\varphi_2^{(j)} \in E_2$  ( $j=1, 2, \dots$ ),

$$\|\varphi_2^{(m)} - \varphi_2^{(n)}\| \leq \|\psi^{(m)} - \psi^{(n)}\| + \|\varphi_1^{(m)} - \varphi_1^{(n)}\| \rightarrow 0 \quad (m \geq n \rightarrow \infty)$$

in consequence of the convergence of  $\{\psi^{(n)}\}$  and  $\{\varphi_1^{(n)}\}$ , so that  $\{\varphi_2^{(n)}\}$  also converges,  $\varphi_2^{(n)} \rightarrow \varphi_2 \in E_2$ . Since

$$\varphi_1^{(n)} \rightarrow \varphi_1, \quad \varphi_2^{(n)} \rightarrow \varphi_2, \quad \psi^{(n)} \rightarrow \psi \quad \text{and} \quad \psi^{(n)} = \varphi_1^{(n)} + \varphi_2^{(n)},$$

we have  $\psi = \varphi_1 + \varphi_2$ ,  $\varphi_1 = T\psi$ . Now an additive operation  $T$  is known to be linear and consequently bounded when it satisfies the condition that  $\psi^{(n)} \rightarrow \psi$  and  $T\psi^{(n)} \rightarrow \varphi$  imply  $\varphi = T\psi$  (Banach, 41 and 54). Then a number  $A$  exists with the property that

$$\|T\psi\| \leq A\|\psi\| \quad \text{for all admissible } \psi.$$

Putting  $\psi = \varphi_1 + \varphi_2$ ,  $T\psi = \varphi_1$ , we have (1), q.e.d.

From theorem 1 we can easily prove

**THEOREM 1a.** *Let  $E$  be a (B) space, let  $E_1, E_2, \dots, E_k$  be linear closed and linearly independent subspaces of  $E$ . Then a necessary and sufficient condition for all spaces  $E_1 \dot{+} E_2 \dot{+} \dots \dot{+} E_j$  ( $j=2, 3, \dots, k$ ) to be closed, and therefore (B) spaces, is the existence of some number  $A$  such that, for all  $\varphi_n \in E_n$  ( $n=1, 2, \dots, k$ )*

$$\|\varphi_j\| \leq A\|\varphi_1 + \varphi_2 + \dots + \varphi_k\| \quad (j=1, 2, \dots, k-1).$$

## 2. Hilbert space.

**THEOREM 2.** *Let  $\mathfrak{H}$  be a Hilbert space, let  $\mathfrak{H}_1$  and  $\mathfrak{H}_2$  be closed linear manifolds in  $\mathfrak{H}$  and linearly independent, and let  $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$  be closed. The best possible value of  $A$  (Theorem 1) is equal to unity if, and only if,  $\mathfrak{H}_1$  and  $\mathfrak{H}_2$  are mutually orthogonal.*

Let  $(\varphi, f)$  be the „inner product” of  $\varphi \in \mathfrak{H}$  and  $f \in \mathfrak{H}$ ;  $\mathfrak{H}_1$  and  $\mathfrak{H}_2$  are called orthogonal <sup>6)</sup> to each other, when  $(\varphi_1, \varphi_2) = 0$  for all  $\varphi_1 \in \mathfrak{H}_1$ ,  $\varphi_2 \in \mathfrak{H}_2$ . When this is the case we have

$$\|\varphi_1 + \varphi_2\|^2 = (\varphi_1 + \varphi_2, \varphi_1 + \varphi_2) = (\varphi_1, \varphi_1) + (\varphi_2, \varphi_2) = \|\varphi_1\|^2 + \|\varphi_2\|^2,$$

so that the condition (1) is satisfied, and it is permissible to take  $A = 1$ ; by theorem 1,  $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$  is closed (cf. Stone, Theorem 1.22). Conversely, if  $\|\varphi_1\| \leq \|\varphi_1 + \varphi_2\|$  for all  $\varphi_1 \in \mathfrak{H}_1$ ,  $\varphi_2 \in \mathfrak{H}_2$ , then, for all numbers  $\alpha$ , we plainly have  $\|\varphi_1\| \leq \|\varphi_1 + \alpha\varphi_2\|$ . If  $(\varphi_1, \varphi_2)$  were equal  $Re^{i\theta}$ ,  $R > 0$ , take  $\alpha = \delta \exp(i\pi + i\theta)$ ,  $\delta > 0$ . Then

$$\begin{aligned} \|\varphi_1\|^2 &\leq \|\varphi_1 + \alpha\varphi_2\|^2 = \|\varphi_1\|^2 + 2\Re\{\alpha(\varphi_2, \varphi_1)\} + |\alpha|^2\|\varphi_2\|^2 \\ &= \|\varphi_1\|^2 - 2R\delta + \delta^2\|\varphi_2\|^2, \end{aligned}$$

and hence  $2R \leq \delta\|\varphi_2\|^2$ ; if we now make  $\delta \rightarrow 0$  we get the contradiction  $2R \leq 0$ .

As a special case of theorem 1a it now easily follows that, if  $E$  is a Hilbert space, then the best possible value of  $A$  is unity if, and only if, the spaces  $E_1, E_2, \dots, E_k$  are mutually orthogonal; for instance, taking  $A = 1$ ,  $j = 1$ ,  $\varphi_3 = \varphi_4 = \dots = \varphi_k = 0$ , we have  $\|\varphi_1\| \leq \|\varphi_1 + \varphi_2\|$ , so that  $E_1$  is orthogonal to  $E_2$ ; the converse is evident, since

$$\|\varphi_1 + \varphi_2 + \dots + \varphi_k\|^2 = \|\varphi_1\|^2 + \dots + \|\varphi_k\|^2 \geq \|\varphi_j\|^2 \quad (j=1, 2, \dots, k)$$

when the spaces  $E_1, \dots, E_k$  are mutually orthogonal (cf. Stone, Theorem 1.22).

From the preceding theorems we can easily get a number of results such as the following:

If  $\mathfrak{H}$  is a Hilbert space, and  $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$  are linear, closed and linearly independent manifolds in  $\mathfrak{H}$ , if  $\mathfrak{H}_3$  is orthogonal to  $\mathfrak{H}_1$  and to  $\mathfrak{H}_2$ , and if  $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$  is closed, then  $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2 \dot{+} \mathfrak{H}_3$  is closed.

If  $E_1, E_2, E_3$  are linear, closed and linearly independent subspaces of a ( $B$ ) space  $E$ , and if  $E_1 \dot{+} E_2$ ,  $E_1 \dot{+} E_2 \dot{+} E_3$  are closed, then  $E_1 \dot{+} E_3$ ,  $E_2 \dot{+} E_3$  are also closed.

### 3. The space $L_p$ ( $p \geq 1$ ).

Let  $L_p(a, b)$  be the space of all functions  $f(t)$  such that  $|f(t)|^p$

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<sup>6)</sup> M. H. STONE, Linear transformations in Hilbert space and their applications to analysis [New York 1932], Chapter 1; J. v. NEUMANN [Mathem. Ann. 102 (1930), 49—131].

is integrable over  $(a, b)$ ,  $-\infty \leq a < b \leq \infty$ , with the norm

$$\|f\| = \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} \quad (\infty > p \geq 1).$$

Plainly  $L_p(a, b)$  is a  $(B)$  space.

**THEOREM 3.** *Let  $E_1$  and  $E_2$  be any subspaces of  $L_p(a, b)$  such that, for all  $\varphi_1 \in E_1$ ,  $\varphi_2 \in E_2$*

$$(2) \quad \int_a^b |\varphi_1(t)|^{p-2} \varphi_1(t) \overline{\varphi_2(t)} dt = 0.$$

*Then, for all  $\varphi_1 \in E_1$ ,  $\varphi_2 \in E_2$  we have  $\|\varphi_1\| \leq \|\varphi_1 + \varphi_2\|$ . When  $p = 1$ , the interval  $(a, b)$  in (2) is to be replaced by the subset  $F$  of  $(a, b)$  in which  $\varphi_1$  does not vanish.*

Evidently (2) implies that no common element of  $E_1$  and  $E_2$  exists, which is different from zéro.

We have to prove that, for all  $\varphi_1 \in E_1$ ,  $\varphi_2 \in E_2$ ,

$$\Delta(\varphi_1, \varphi_2) = \int_F |\varphi_1(t) + \varphi_2(t)|^p dt - \int_F |\varphi_1(t)|^p dt \geq 0.$$

When we put

$$\begin{aligned} |(\varphi_1(t))| &= \xi, |\varphi_2(t)| = \eta, \varphi_1(t)\overline{\varphi_2(t)} + \overline{\varphi_1(t)}\varphi_2(t) = u, \\ G(u) &= G(u; \xi, \eta) = (u + \xi^2 + \eta^2)^{\frac{p}{2}} - \xi^p - \frac{1}{2}pu \xi^{p-2}, \end{aligned}$$

then

$$(3) \quad -2\xi\eta \leq u \leq 2\xi\eta, \\ \Delta - \frac{p}{2} \int_F |\varphi_1|^{p-2} \{ \varphi_1 \overline{\varphi_2} + \overline{\varphi_1} \varphi_2 \} dt = \int_F G dt.$$

Now the function  $G$  takes no negative value:

When  $p > 2$ , then, for any fixed  $\xi \geq 0, \eta \geq 0$  and for  $u \geq -\xi^2 - \eta^2$ , the function has its minimum at  $u = -\eta^2$  while  $G(-\eta^2) = \frac{1}{2}p\xi^{p-2}\eta^2 \geq 0$ .

When  $p = 2$ , then  $G = \eta^2 \geq 0$ . When  $1 \leq p < 2$ , we can easily see that

$$G \geq \min \{ G(2\xi\eta), G(-2\xi\eta) \} \quad (-2\xi\eta \leq u \leq 2\xi\eta);$$

when we put  $w = \frac{\eta}{\xi}$ ,  $g(w) = |1 + w|^p - 1 - pw$ , then

$$G(\pm 2\xi\eta) = \xi^p g(\pm w) \geq 0,$$

since  $g(z) \geq g(0) = 0$  ( $-\infty < z < \infty$ ). Hence in any case  $G \geq 0$ , and from (3) and (2) it now easily follows that  $\Delta \geq 0$ , q.e.d.

4. *Examples.*

I. Let  $a > 0$ ,  $p \geq 1$ , let  $E_1$  and  $E_2$  be the subspaces of  $L_p(-a, a)$  consisting of all functions of  $L_p(-a, a)$  which are equivalent to any even or odd function respectively. It is evident that  $E_1$  and  $E_2$  are linear and linearly independent closed vector spaces, while  $E_1 \dot{+} E_2$  is  $L_p$  and therefore closed. Hence, by theorem 1,

$$\|\varphi_1\| \leq A \|\varphi_1 + \varphi_2\| \quad (\varphi_1 \in E_1, \varphi_2 \in E_2).$$

This result is trivial, since for  $j = 1, 2$

$$\|\varphi_j\| = \left\| \frac{\varphi_1 + \varphi_j}{2} \pm \frac{\varphi_1 - \varphi_j}{2} \right\| \leq \frac{1}{2} \|\varphi_1 + \varphi_2\| + \frac{1}{2} \|\varphi_1 - \varphi_2\|,$$

$$\|\varphi_1(t) - \varphi_2(t)\| = \|\varphi_1(-t) - \varphi_2(-t)\| = \|\varphi_1(t) + \varphi_2(t)\|,$$

and hence

$$(4) \quad \|\varphi_1\| \leq \|\varphi_1 + \varphi_2\|, \quad \|\varphi_2\| \leq \|\varphi_1 + \varphi_2\|;$$

we may therefore take  $A = 1$ . Since  $\varphi_1$  is even and  $\varphi_2$  odd, we evidently have

$$\int_{-a}^a |\varphi_1(t)|^{p-2} \varphi_1(t) \overline{\varphi_2(t)} dt = 0, \quad \int_{-a}^a |\varphi_2(t)|^{p-2} \varphi_2(t) \overline{\varphi_1(t)} dt = 0,$$

and hence (4) also follows from theorem 3.

When we take  $\varphi_1 = \alpha_0 + \alpha_1 \cos t + \dots + \alpha_M \cos Mt$ ,  $\varphi_2 = \beta_1 \sin t + \beta_2 \sin 2t + \dots + \beta_N \sin Nt$ , with  $M, N$  arbitrary integers,  $M \geq 0, N \geq 1$ ,  $\alpha_n, \beta_n$  arbitrary numbers, then (4) is also valid throughout the interval  $a, b$ , if  $\pi^{-1}(a + b)$  or  $\pi^{-1}(b - a)$  are even integers, as can easily be proved.

II. The following example, given by Stone <sup>7)</sup> without the condition (1), illustrates the necessity for the condition.

Let  $\{g_n\}$  ( $n=0, 1, \dots$ ) be a complete orthonormal system in a Hilbert space  $\mathfrak{H}$ , let  $\vartheta_n$  be any sequence of numbers which contains a subsequence with the limit point  $\frac{1}{2}\pi$ , let the Hilbert spaces  $\mathfrak{H}_1$  and  $\mathfrak{H}_2$  be determined by the orthonormal sets  $\{\psi_n\}$  and  $\{\chi_n\}$  respectively,  $\psi_n = g_{2n}$ ,  $\chi_n = g_{2n-1} \cos \vartheta_n + g_{2n} \sin \vartheta_n$ . Stone has proved that  $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$  is not closed. In fact the condition (2) is not satisfied. To prove this, we put

$$\varphi_1 = -\psi_n \sin \vartheta_n \in \mathfrak{H}_1, \quad \varphi_2 = \chi_n \in \mathfrak{H}_2; \quad \text{then since } \|g_n\| = 1,$$

we have

$$\frac{\|\varphi_1 + \varphi_2\|}{\|\varphi_1\|} = \frac{\|g_{2n-1} \cos \vartheta_n\|}{\|g_{2n} \sin \vartheta_n\|} = |\cot \vartheta_n|,$$

<sup>7)</sup> Stone l.c., theorem 1.22.

and there exists no positive number  $A$  such that  $|\cot \vartheta_n| \geq A^{-1}$ .

III. Let  $L_n^{(\alpha)}(z)$  be the generalised Laguerre polynomial,

$$\Phi_n^{(\alpha)}(x) = \left\{ \frac{2 \cdot n! e^{-x^2}}{\Gamma(n+\alpha+1)} \right\}^{\frac{1}{2}} x^{\alpha+\frac{1}{2}} L_n^{(\alpha)}(x^2), \quad L_n^{(\alpha)}(z) = \sum_{r=0}^n \binom{n+\alpha}{n-r} \frac{(-z)^r}{r!},$$

$\Re(\alpha) > -1$ . When  $\alpha$  is real, then  $\{\Phi_n^{(\alpha)}\}$ ,  $n = 0, 1, \dots$  is a complete orthonormal set of  $L_2(0, \infty)$ ; otherwise the set  $\{\Phi_n^{(\alpha)}\}$  determines <sup>8)</sup> the closed linear manifold  $L_2(0, \infty)$ . Now, for all numbers  $a_0, a_1, \dots, a_m$ ,  $m \geq 0$ , and all real  $r$ , in  $L_2(0, \infty)$

$$\left\| \sum_{n=0}^m a_n \Phi_n^{(\alpha)}(x) e^{2i\pi r n} \right\| \leq A \left\| \sum_{n=0}^m a_n \Phi_n^{(\alpha)}(x) \right\|,$$

where  $A$  depends on  $\alpha$  only and  $A \geq 1$  <sup>9)</sup>. Take  $r = \frac{1}{2}$ ,

$$\varphi_1 = \sum_{n=0}^{[\frac{1}{2}m]} a_{2n} \Phi_{2n}^{(\alpha)}, \quad \varphi_2 = \sum_{n=0}^{[\frac{1}{2}m-\frac{1}{2}]} a_{2n+1} \Phi_{2n+1}^{(\alpha)},$$

then

$$\|\varphi_1 - \varphi_2\| \leq A \|\varphi_1 + \varphi_2\|,$$

$$\|\varphi_1\| \leq \frac{1}{2} \|(\varphi_1 - \varphi_2) + (\varphi_1 + \varphi_2)\| \leq \frac{1}{2} (A+1) \|\varphi_1 + \varphi_2\|,$$

$$(5) \quad \|\varphi_1\| \leq A \|\varphi_1 + \varphi_2\|, \quad \varphi_2 \leq A \|\varphi_1 + \varphi_2\|.$$

Let  $\mathfrak{H}_1$  and  $\mathfrak{H}_2$  be the closed linear manifolds determined by the sets  $\{\Phi_{2n}^{(\alpha)}\}$  and  $\{\Phi_{2n+1}^{(\alpha)}\}$  respectively,  $n = 0, 1, \dots$ . Then, from (5) and theorem 1, it follows easily, that  $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$  is closed; since  $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$  contains the set  $\{\Phi_n^{(\alpha)}\}$ ,  $n = 0, 1, \dots$ , it must be identical with  $L_2(0, \infty)$  <sup>10)</sup>. The result is self-evident, when  $\alpha$  is real.

By the same reasoning we may see that, when  $k \geq 2$ ,  $0 \leq a < k$ ,  $0 \leq b < k$ ,  $a \neq b$ , and  $\mathfrak{H}_1$  and  $\mathfrak{H}_2$  are the closed linear manifolds determined by the sets  $\{\Phi_{a+sk}^{(\alpha)}\}$ ,  $\{\Phi_{b+sk}^{(\alpha)}\}$  respectively ( $s=0, 1, \dots$ ), then  $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$  is also closed.

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<sup>8)</sup> This means: The smallest closed linear manifold which contains all  $\Phi_n^{(\alpha)}$  is  $L_2(0, \infty)$ .

<sup>9)</sup> H. KOBER [Quart. J. of Math. (Oxford) 10 (1939), 45—59], sections 7, 8, 9.

<sup>10)</sup> Added in proof, 14.7.39: This no longer holds in the space  $L_p$ ,  $1 \leq p < 2$  [ $\Re(\alpha) > \frac{1}{p} - \frac{3}{2}$ , when  $1 < p < 2$ ,  $\Re(\alpha) \geq -\frac{1}{2}$ , when  $p = 1$ ].