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A theorem on Banach spaces

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1. Let E be a normed complete linear vector space, that is to say a space (B) in the terminology of S. Banach [1]), let $E_1, E_2, E_3, \ldots, E_k$ $(k \geqq 1)$ be linear subspaces of E, which are linearly independent. [2]) Let $E_1 \dotplus E_2 \dotplus E_3 \dotplus \ldots \dotplus E_k$ be the smallest linear subspace of E, which contains all of $E_1, E_2, \ldots, E_k$. Of course every element ψ of $E_1 \dotplus E_2 \dotplus \ldots \dotplus E_k$ can be represented uniquely in the form $\psi = \varphi_1 + \varphi_2 + \ldots \varphi_k$ $(\varphi_1 \in E_1, \varphi_2 \in E_2, \ldots, \varphi_k \in E_k)$.

THEOREM 1. *Let E be a (B) space, E_1 and E_2 linear closed* [3]) *subspaces of E and linearly independent, then the space $E_{12} = E_1 \dotplus E_2$ is closed if, and only if, there exists some constant A such that, for all elements φ_1, φ_2 $(\varphi_1 \in E_1, \varphi_2 \in E_2)$*

$$(1) \qquad \|\varphi_1\| \leqq A\, \|\varphi_1 + \varphi_2\|. \text{ [4])}$$

Of course both E_1 and E_2 are (B) spaces and, if the condition (1) is satisfied, so is E_{12}.

The proof of the sufficiency of (1) is quite trivial. Let $\{\psi^{(n)}\}$ $(n=1, 2, \ldots)$ be any convergent sequence [5]) of E_{12}; then we have to show only that it converges to an element ψ belonging to E_{12}. Since $\psi^{(j)} = \varphi_1^{(j)} + \varphi_2^{(j)}$ $(j=1, 2, \ldots)$, $\varphi_i^{(m)} - \varphi_i^{(n)} \in E_i$ $(i=1, 2)$, it follows from (1) that

$$\|\varphi_1^{(m)} - \varphi_1^{(n)}\| \leqq A \|(\varphi_1^{(m)} - \varphi_1^{(n)}) + (\varphi_2^{(m)} - \varphi_2^{(n)})\| = A \|\psi^{(m)} - \psi^{(n)}\| \to 0,$$

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1) Théorie des opérations linéaires, Warszawa 1932, 53; the norm of φ is $\|\varphi\|$.

2) This means: If $\varphi_1 + \varphi_2 + \ldots + \varphi_k = 0$, $\varphi_i \in E_i$ $(i=1, 2, \ldots, k)$, then all elements φ_i must be zéro. If $k = 2$, E_1 and E_2 are linearly independent if, and only if, they have no common element except the element zéro.

3) „fermé", Banach l.c., 13.

4) Connected problems: H. KOBER [Proc. London Math. Soc. (2), 44 (1938), 453—65], Satz VI'*b*; see also a forthcoming paper in the Annals of Mathem., Satz IIIβ.

5) The sequence has to satisfy the condition of Cauchy $\|\psi^{(m)} - \psi^{(n)}\| \to 0$ $(m \geqq n \to \infty)$. Since $\psi^{(j)} \in E$ and E is complete, $\{\psi^{(n)}\}$ converges to an element $\psi \in E$, $\|\psi^{(n)} - \psi\| \to 0$.

when $m \geqq n \to \infty$. Now E_1 is closed, so that the sequence $\{\varphi_1^{(n)}\}$ converges to a limit point $\varphi_1 \in E_1$; so also the sequence $\{\varphi_2^{(m)}\}$ converges to a limit point $\varphi_2 \in E_2$, since

$$\|\varphi_2^{(m)} - \varphi_2^{(n)}\| = \|(\psi^{(m)} - \psi^{(n)}) - (\varphi_1^{(m)} - \varphi_1^{(n)})\|$$
$$\leqq \|\psi^{(m)} - \psi^{(n)}\| + \|\varphi_1^{(m)} - \varphi_1^{(n)}\| \to 0 \quad (m \geqq n \to \infty).$$

Hence the sequence $\{\psi^{(n)}\} \equiv \{\varphi_1^{(n)} + \varphi_2^{(n)}\}$ converges to $\varphi_1 + \varphi_2 = \psi$ and plainly $\varphi_1 + \varphi_2 = \psi$ belongs to $E_1 \dot{+} E_2 = E_{12}$.

The condition (1) is necessary. For to every element $\psi \in E_1 \dot{+} E_2$ corresponds exactly one $\varphi_1 \in E_1$ since $\psi = \varphi_1 + \varphi_2$; hence $T\psi = \varphi_1$ is an operation, which evidently is additive (Banach, 23); now let the sequences $\{\psi^{(n)}\} \in E_1 \dot{+} E_2$ and $\{\varphi_1^{(n)}\} \equiv \{T\psi^{(n)}\} \in E_1$ have the limits points ψ and φ_1 respectively, and then plainly $\psi \in E_1 \dot{+} E_2$, $\varphi_1 \in E_1$, since $E_1 \dot{+} E_2$ and E_1 are closed. We next show that $T\psi = \varphi_1$. Since $\psi^{(j)} = \varphi_1^{(j)} + \varphi_2^{(j)}$, $\varphi_1^{(j)} \in E_1$, $\varphi_2^{(j)} \in E_2$ $(j=1, 2, \ldots)$,

$$\|\varphi_2^{(m)} - \varphi_2^{(n)}\| \leqq \|\psi^{(m)} - \psi^{(n)}\| + \|\varphi_1^{(m)} - \varphi_2^{(n)}\| \to 0 \quad (m \geqq n \to \infty)$$

in consequence of the convergence of $\{\psi^{(n)}\}$ and $\{\varphi_1^{(n)}\}$, so that $\{\varphi_2^{(n)}\}$ also converges, $\varphi_2^{(n)} \to \varphi_2 \in E_2$. Since

$$\varphi_1^{(n)} \to \varphi_1,\ \varphi_2^{(n)} \to \varphi_2,\ \psi^{(n)} \to \psi \text{ and } \psi^{(n)} = \varphi_1^{(n)} + \varphi_2^{(n)},$$

we have $\psi = \varphi_1 + \varphi_2$, $\varphi_1 = T\psi$. Now an additive operation T is known to be linear and consequently bounded when it satisfies the condition that $\psi^{(n)} \to \psi$ and $T\psi^{(n)} \to \varphi$ imply $\varphi = T\psi$ (Banach, 41 and 54). Then a number A exists with the property that

$$\|T\psi\| \leqq A\|\psi\| \text{ for all admissible } \psi.$$

Putting $\psi = \varphi_1 + \varphi_2$, $T\psi = \varphi_1$, we have (1), q.e.d.,

From theorem 1 we can easily prove

THEOREM 1a. *Let E be a (B) space, let $E_1, E_2, \ldots, E_k$ be linear closed and linearly independent subspaces of E. Then a necessary and sufficient condition for all spaces $E_1 \dot{+} E_2 \dot{+} \ldots \dot{+} E_j$ $(j=2, 3, \ldots, k)$ to be closed, and therefore (B) spaces, is the existence of some number A such that, for all $\varphi_n \in E_n$ $(n=1, 2, \ldots, k)$*

$$\|\varphi_j\| \leqq A\|\varphi_1+\varphi_2 + \ldots + \varphi_k\| \qquad (j=1, 2, \ldots, k-1).$$

2. *Hilbert space.*

THEOREM 2. *Let $\mathfrak{H}$ be a Hilbert space, let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be closed linear manifolds in $\mathfrak{H}$ and linearly independent, and let $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$ be closed. The best possible value of A (Theorem 1) is equal to unity if, and only if, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are mutually orthogonal.*

Let (φ, f) be the „inner product" of $\varphi \in \mathfrak{H}$ and $f \in \mathfrak{H}$; $\mathfrak{H}_1$ and $\mathfrak{H}_2$, are called orthogonal [6]) to each other, when $(\varphi_1, \varphi_2) = 0$ for all $\varphi_1 \in \mathfrak{H}_1$, $\varphi_2 \in \mathfrak{H}_2$. When this is the case we have

$$\|\varphi_1+\varphi_2\|^2=(\varphi_1+\varphi_2, \varphi_1+\varphi_2) = (\varphi_1, \varphi_1) + (\varphi_2, \varphi_2) = \|\varphi_1\|^2 + \|\varphi_2\|^2,$$

so that the condition (1) is satisfied, and it is permissible to take $A = 1$; by theorem 1, $\mathfrak{H}_1 \dotplus \mathfrak{H}_2$ is closed (cf. Stone, Theorem 1.22). Conversely, if $\|\varphi_1\| \leqq \|\varphi_1 + \varphi_2\|$ for all $\varphi_1 \in \mathfrak{H}_1$, $\varphi_2 \in \mathfrak{H}_2$, then, for all numbers α, we plainly have $\|\varphi_1\| \leqq \|\varphi_1 + \alpha\varphi_2\|$. If (φ_1, φ_2) were equal $Re^{i\vartheta}$, $R > 0$, take $\alpha = \delta \exp (i\pi+i\vartheta)$, $\delta > 0$. Then

$$\begin{aligned}\|\varphi_1\|^2 \leqq \|\varphi_1+\alpha\varphi_2\|^2 &= \|\varphi_1\|^2 + 2\Re\{\alpha(\varphi_2, \varphi_1)\} + |\alpha|^2\|\varphi_2\|^2 \\ &= \|\varphi_1\|^2 - 2R\delta + \delta^2\|\varphi_2\|^2,\end{aligned}$$

and hence $2R \leqq \delta\|\varphi_2\|^2$; if we now make $\delta \to 0$ we get the contradiction $2R \leqq 0$.

As a special case of theorem 1a it now easily follows that, if E is a Hilbert space, then the best possible value of A is unity if, and only if, the spaces $E_1, E_2, \ldots, E_k$ are mutually orthogonal; for instance, taking $A = 1$, $j = 1$, $\varphi_3 = \varphi_4 = \ldots = \varphi_k = 0$, we have $\|\varphi_1\| \leqq \|\varphi_1 + \varphi_2\|$, so that E_1 is orthogonal to E_2; the converse is evident, since

$$\|\varphi_1 + \varphi_2 + \cdots + \varphi_k\|^2 = \|\varphi_1\|^2 + \cdots + \|\varphi_k\|^2 \geqq \|\varphi_j\|^2 \quad (j=1, 2, \ldots, k)$$

when the spaces $E_1, \ldots, E_k$ are mutually orthogonal (cf. Stone, Theorem 1.22).

From the preceding theorems we can easily get a number of results such as the following:

If $\mathfrak{H}$ is a Hilbert space, and $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ are linear, closed and linearly independent manifolds in $\mathfrak{H}$, if $\mathfrak{H}_3$ is orthogonal to $\mathfrak{H}_1$ and to $\mathfrak{H}_2$, and if $\mathfrak{H}_1 \dotplus \mathfrak{H}_2$ is closed, then $\mathfrak{H}_1 \dotplus \mathfrak{H}_2 \dotplus \mathfrak{H}_3$ is closed.

If E_1, E_2, E_3 are linear, closed and linearly independent subspaces of a (B) space E, and if $E_1 \dotplus E_2$, $E_1 \dotplus E_2 \dotplus E_3$ are closed, then $E_1 \dotplus E_3$, $E_2 \dotplus E_3$ are also closed.

3. *The space* L_p $(p \geqq 1)$.

Let $L_p(a, b)$ be the space of all functions $f(t)$ such that $|f(t)|^p$

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[6]) M. H. Stone, Linear transformations in Hilbert space and their applications to analysis [New York 1932], Chapter 1; J. v. Neumann [Mathem. Ann. **102** (1930), 49—131].

is integrable over (a, b), $-\infty \leqq a < b \leqq \infty$, with the norm

$$\|f\| = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \qquad (\infty > p \geqq 1).$$

Plainly $L_p(a, b)$ is a (B) space.

THEOREM 3. *Let E_1 and E_2 be any subspaces of $L_p(a, b)$ such that, for all $\varphi_1 \in E_1$, $\varphi_2 \in E_2$*

$$\int_a^b |\varphi_1(t)|^{p-2}\varphi_1(t)\,\overline{\varphi_2}(t)dt = 0. \tag{2}$$

Then, for all $\varphi_1 \in E_1$, $\varphi_2 \in E_2$ we have $\|\varphi_1\| \leqq \|\varphi_1 + \varphi_2\|$. *When $p = 1$, the interval (a, b) in (2) is to be replaced by the subset F of (a, b) in which φ_1 does not vanish.*

Evidently (2) implies that no common element of E_1 and E_2 exists, which is different from zéro.

We have to prove that, for all $\varphi_1 \in E_1$, $\varphi_2 \in E_2$,

$$\Delta(\varphi_1, \varphi_2) = \int_F |\varphi_1(t) + \varphi_2(t)|^p dt - \int_F |\varphi_1(t)|^p dt \geqq 0.$$

When we put

$$|(\varphi_1(t)| = \xi, |\varphi_2(t)| = \eta, \ \varphi_1(t)\overline{\varphi_2}(t) + \overline{\varphi_1}(t)\varphi_2(t) = u,$$

$$G(u) = G(u; \xi, \eta) = (u+\xi^2+\eta^2)^{\frac{p}{2}} - \xi^p - \tfrac{1}{2}pu\,\xi^{p-2},$$

then

$$-2\xi\eta \leqq u \leqq 2\xi\eta,$$

$$\Delta - \frac{p}{2}\int_F |\varphi_1|^{p-2}\{\varphi_1\overline{\varphi_2} + \overline{\varphi_1}\varphi_2\}dt = \int_F G\,dt. \tag{3}$$

Now the function G takes no negative value:

When $p>2$, then, for any fixed $\xi \geqq 0, \eta \geqq 0$ and for $u \geqq -\xi^2-\eta^2$, the function has its minimum at $u=-\eta^2$ while $G(-\eta^2)=\frac{1}{2}p\xi^{p-2}\eta^2 \geqq 0$. When $p = 2$, then $G = \eta^2 \geqq 0$. When $1 \leqq p < 2$, we can easily see that

$$G \geqq \min\{G(2\xi\eta), G(-2\xi\eta)\} \quad (-2\xi\eta \leqq u \leqq 2\xi\eta);$$

when we put $w = \frac{\eta}{\xi}$, $g(w) = |1 + w|^p - 1 - pw$, then

$$G(\pm 2\xi\eta) = \xi^p g(\pm w) \geqq 0,$$

since $g(z) \geqq g(0) = 0$ $(-\infty < z < \infty)$. Hence in any case $G \geqq 0$, and from (3) and (2) it now easily follows that $\Delta \geqq 0$, q.e.d.

4. *Examples.*

I. Let $a > 0$, $p \geqq 1$, let E_1 and E_2 be the subspaces of $L_p(-a, a)$ consisting of all functions of $L_p(-a, a)$ which are equivalent to any even or odd function respectively. It is evident that E_1 and E_2 are linear and linearly independent closed vector spaces, while $E_1 \dot{+} E_2$ is L_p and therefore closed. Hence, by theorem 1,

$$\|\varphi_1\| \leqq A \,\|\varphi_1 + \varphi_2\| \qquad (\varphi_1 \in E_1, \varphi_2 \in E_2).$$

This result is trivial, since for $j = 1, 2$

$$\|\varphi_j\| = \left\| \frac{\varphi_1 + \varphi}{2} \pm \frac{\varphi_1 - \varphi_2}{2} \right\| \leqq \frac{1}{2} \|\varphi_1 + \varphi_2\| + \frac{1}{2} \|\varphi_1 - \varphi_2\|,$$

$$\|\varphi_1(t) - \varphi_2(t)\| = \|\varphi_1(-t) - \varphi_2(-t)\| = \|\varphi_1(t) + \varphi_2(t)\|,$$

and hence

$$\text{(4)} \qquad \|\varphi_1\| \leqq \|\varphi_1 + \varphi_2\|, \ \|\varphi_2\| \leqq \|\varphi_1 + \varphi_2\|;$$

we may therefore take $A = 1$. Since φ_1 is even and φ_2 odd, we evidently have

$$\int_{-a}^{a} |\varphi_1(t)|^{p-2} \varphi_1(t) \overline{\varphi_2}(t) dt = 0, \int_{-a}^{a} |\varphi_2(t)|^{p-2} \varphi_2(t) \overline{\varphi_1}(t) dt = 0,$$

and hence (4) also follows from theorem 3.

When we take $\varphi_1 = \alpha_0 + \alpha_1 \cos t + \dots + \alpha_M \cos Mt$, $\varphi_2 = \beta_1 \sin t + \beta_2 \sin 2t + \dots + \beta_N \sin Nt$, with M, N arbitrary integers, $M \geqq 0$, $N \geqq 1$, α_n, β_n arbitrary numbers, then (4) is also valid throughout the interval a, b, if $\pi^{-1}(a + b)$ or $\pi^{-1}(b - a)$ are even integers, as can easily be proved.

II. The following example, given by Stone [7]) without the condition (1), illustrates the necessity for the condition.

Let $\{g_n\}$ $(n = 0, 1, \dots)$ be a complete orthonormal system in a Hilbert space $\mathfrak{H}$, let ϑ_n be any sequence of numbers which contains a subsequence with the limit point $\frac{1}{2}\pi$, let the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be determined by the orthonormal sets $\{\psi_n\}$ and $\{\chi_n\}$ respectively, $\psi_n = g_{2n}$, $\chi_n = g_{2n-1} \cos \vartheta_n + g_{2n} \sin \vartheta_n$. Stone has proved that $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$ is not closed. In fact the condition (2) is not satisfied. To prove this, we put

$$\varphi_1 = -\psi_n \sin \vartheta_n \in \mathfrak{H}_1, \ \varphi_2 = \chi_n \in \mathfrak{H}_2; \text{ then since } \|g_n\| = 1,$$

we have

$$\frac{\|\varphi_1 + \varphi_2\|}{\|\varphi_1\|} = \frac{\|g_{2n-1} \cos \vartheta_n\|}{\|g_{2n} \sin \vartheta_n\|} = |\cot \vartheta_n|,$$

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[7]) Stone l.c., theorem 1.22.

and there exists no positive number A such that $|\cot \vartheta_n| \geqq A^{-1}$.

III. Let $L_n^{(\alpha)}(z)$ be the generalised Laguerre polynomial,

$$\Phi_n^{(\alpha)}(x) = \left\{\frac{2\cdot n!e^{-x^2}}{\Gamma(n+\alpha+1)}\right\}^{\frac{1}{2}} x^{\alpha+\frac{1}{2}} L_n^{(\alpha)}(x^2),\ L_n^{(\alpha)}(z) = \sum_{r=0}^{n} \binom{n+\alpha}{n-r} \frac{(-z)^r}{r!},$$

$\Re(\alpha) > -1$. When α is real, then $\{\Phi_n^{(\alpha)}\}$, $n = 0, 1, \dots$ is a complete orthonormal set of $L_2(0, \infty)$; otherwise the set $\{\Phi_n^{(\alpha)}\}$ determines [8]) the closed linear manifold $L_2(0, \infty)$. Now, for all numbers $a_0, a_1, \dots, a_m$, $m \geqq 0$, and all real r, in $L_2(0, \infty)$

$$\left\| \sum_{n=0}^{m} a_n \Phi_n^{(\alpha)}(x) e^{2i\pi rn} \right\| \leqq A \left\| \sum_{n=0}^{m} a_n \Phi_n^{(\alpha)}(x) \right\|,$$

where A depends on α anly and $A \geqq 1$ [9]). Take $r = \frac{1}{2}$,

$$\varphi_1 = \sum_{n=0}^{[\frac{1}{2}m]} a_{2n} \Phi_{2n}^{(\alpha)}, \qquad \varphi_2 = \sum_{n=0}^{[\frac{1}{2}m-\frac{1}{2}]} a_{2n+1} \Phi_{2n+1}^{(\alpha)},$$

then

$$\|\varphi_1 - \varphi_2\| \leqq A\,\|\varphi_1 + \varphi_2\|,$$

$$\|\varphi_1\| \leqq \frac{1}{2}\,\|(\varphi_1-\varphi_2)+(\varphi_1+\varphi_2)\| \leqq \frac{1}{2}\,(A+1)\,\|\varphi_1+\varphi_2\|,$$

$$(5) \qquad \|\varphi_1\| \leqq A\,\|\varphi_1+\varphi_2\|,\ \varphi_2 \leqq A\,\|\varphi_1+\varphi_2\|.$$

Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be the closed linear manifolds determined by the sets $\{\Phi_{2n}^{(\alpha)}\}$ and $\{\Phi_{2n+1}^{(\alpha)}\}$ respectively, $n = 0, 1, \dots$. Then, from (5) and theorem 1, it follows easily, that $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$ is closed; since $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$ contains the set $\{\Phi_n^{(\alpha)}\}$, $n = 0, 1, \dots$, it must be identical with $L_2(0, \infty)$ [10]). The result is self-evident, when α is real.

By the same reasoning we may see that, when $k \geqq 2$, $0 \leqq a < k$, $0 \leqq b < k$, $a \neq b$, and $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are the closed linear manifolds determined by the sets $\{\Phi_{a+sk}^{(\alpha)}\}$, $\{\Phi_{b+sk}^{(\alpha)}\}$ respectively ($s=0, 1, \dots$), then $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2$ is also closed.

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[8]) This means: The smallest closed linear manifold which contains all $\Phi_n^{(\alpha)}$ is $L_2(0, \infty)$.

[9]) H. Kober [Quart. J. of Math. (Oxford) **10** (1939), 45—59], sections 7, 8, 9.

[10]) Added in proof, 14.7.39: This no longer holds in the space L_p, $1 \leqq p < 2$ $[\Re(\alpha) > \frac{1}{p} - \frac{3}{2}$, when $1 < p < 2$, $\Re(\alpha) \geqq -\frac{1}{2}$, when $p = 1]$.