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On the Cesàro and Riesz means of Fourier series[1]

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1. Let

$$(1.1)\qquad f(x) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

be a Fourier series with the partial sums

$$(1.2)\qquad s_0 = \frac{a_0}{2},\ s_n(x) = \frac{a_0}{2} + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x),$$
$$n = 1, 2, 3, \ldots,$$

and the Cesàro means of first order

$$(1.3)\qquad C_n^{(1)}(x) = \frac{s_0 + \ldots + s_n}{n+1},\ n = 0, 1, 2, \ldots.$$

It is well known that the series (1.1) represents the function $f(x)$ in the sense that $C_n^{(1)}(x)$ converges to $f(x)$ almost everywhere. Only recently, Fejér discovered that the Cesàro means of higher order follow the shape of the function even for every finite n. To quote a particular result:

Theorem I. Let $f(x)$ be convex upward in the interval $(0, \pi)$, then the Cesàro means of third order of its sine series are all convex. For the second order means this is not true in general.

Adopting the device used by Fejér we get some further results. For instance, in the case mentioned above, consider the Riesz means of second order:

$$(1.4)\qquad R_n^{(2)}(x) \equiv (n+1)^{-2} \sum_{\nu=1}^{n} (n-\nu+1)^2 (b_\nu \sin \nu x).$$

We find that under the assumption of theorem I the curves $y = R_n^{(2)}(x)$ are all convex in the interval $(0, \pi)$. From this result

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[1]) (Presented to the American Mathematical Society, December 29, 1938.)

theorem I follows easily. We shall extend similarly some other results of Fejér.

We quote the following definitions:

Given an infinite series $\sum_{\nu=0}^{\infty} u_\nu$, the Cesàro means (C, k) of order k $(k=0, 1, 2, \ldots)$ are

$$(1.5)\quad C_n^{(k)} \equiv \binom{n+k}{k}^{-1} \left\{ \binom{n+k}{k} u_0 + \binom{n+k-1}{k} u_1 + \cdots + \binom{k}{k} u_n \right\},$$

while the corresponding Riesz means (R, k) are (M. Riesz (5); see the list of references at the end of this paper)

$$(1.6)\quad R_n^{(k)} \equiv (n+1)^{-k}\{(n+1)^k u_0 + n^k u_1 + \cdots + 1^k u_n\}.$$

Putting

$$(1.7)\qquad (n+1)^k u_0 + n^k u_1 + \cdots + 1^k u_n = \varrho_n^{(k)},$$

we have

$$(1.8)\qquad R_n^{(k)} = (n+1)^{-k} \varrho_n^{(k)}, \quad n, k = 0, 1, 2, \ldots.$$

Obviously

$$R_0^{(k)} = \varrho_0^{(k)} = C_0^{(k)} = u_0.$$

2. Denote by $F(r)$ a formal power series $\sum_0^\infty u_\nu r^\nu$, not necessarily convergent. Let

$$\sum_0^n u_\nu = U_n = U_n^{(0)},\ \sum_0^n U_\nu = U_n^{(1)},\ \sum_0^n U_\nu^{(1)} = U_n^{(2)},\ \sum_0^n U_\nu^{(2)} = U_n^{(3)},$$

then, in the sense of formal Cauchy multiplication

$$(1-r)^{-k-1} F(r) = \sum_{\nu=0}^{\infty} U_\nu^{(k)} r^\nu, \qquad k = 0, 1, 2,$$

and

$$(2.1)\qquad (1+r)(1-r)^{-3} F(r) = u_0 + \sum_1^\infty (U_{\nu-1}^{(2)} + U_\nu^{(2)}) r^\nu.$$

On the other hand

$$(1+r)(1-r)^{-3} = \frac{1+r}{2} \sum_0^\infty (\nu+1)(\nu+2) r^\nu = \sum_0^\infty (\nu+1)^2 r^\nu,$$

and again by formal multiplication with the power series $F(r) \equiv \sum_0^\infty u_\nu r^\nu$

$$(2.2)\qquad (1+r)(1-r)^{-3}F(r) = \sum_0^\infty \varrho_\nu^{(2)} r^\nu.$$

(2.1) and (2.2) yield

$$(2.3)\qquad \boxed{\varrho_0^{(2)} = u_0,\ \varrho_n^{(2)} = U_{n-1}^{(2)} + U_n^{(2)}, \qquad n = 1, 2, 3, \ldots.}$$

But

$$(2.4)\qquad U_n^{(2)} = \binom{n+2}{2}u_0 + \binom{n+1}{2}u_1 + \cdots + \binom{2}{2}u_n = \binom{n+2}{2}C_n^{(2)}.$$

Whence from (1.8), (2.3) and (2.4)

$$R_n^{(2)} = (n+1)^{-2}\left\{\frac{n(n+1)}{2}C_{n-1}^{(2)} + \frac{(n+1)(n+2)}{2}C_n^{(2)}\right\}.$$

We thus have the following identity, connecting the $(C, 2)$ and $(R, 2)$ means

$$(2.5)\qquad \boxed{R_0^{(2)} = C_0^{(2)},\ R_n^{(2)} = \frac{nC_{n-1}^{(2)} + (n+2)C_n^{(2)}}{2(n+1)}, \qquad n = 1, 2, \ldots.}$$

Note also that

$$U_n^{(1)} = U_n^{(2)} - U_{n-1}^{(2)}, \qquad n = 1, 2, 3, \ldots.$$

Hence

$$U_n^{(2)} = U_{n-1}^{(2)} + U_n^{(1)}, \qquad n = 1, 2, 3, \ldots,$$

and from (2.3)

$$\varrho_n^{(2)} = 2U_{n-1}^{(2)} + U_n^{(1)}, \qquad n = 1, 2, 3, \ldots.$$

Combining this with (1.8) and (2.4), we get

$$R_n^{(2)} = (n+1)^{-2}\left\{2\binom{n+1}{2}C_{n-1}^{(2)} + (n+1)C_n^{(1)}\right\},$$

or

$$(2.6)\qquad \boxed{R_n^{(2)} = \frac{nC_{n-1}^{(2)} + C_n^{(1)}}{n+1}, \qquad n=1, 2, 3, \ldots.}$$

We now put

$$u_\nu = \nu \sin \nu x, \qquad \nu = 0, 1, 2, \ldots.$$

Hence

$$F(r) = \sum_0^\infty \nu r^\nu \sin \nu x = r \sin x \,\frac{1 - r^2}{(1-2r\cos x+r^2)^2}.$$

To (2.2) now corresponds

$$\sum_0^\infty \varrho_\nu^{(2)}(x)\, r^\nu = r \sin x \left\{ \frac{1-r^2}{(1-r)^2(1-2r\cos x+r^2)} \right\}^2$$

$$= r \sin x \left\{ \sum_0^\infty \left(\frac{\sin (\nu+1)\frac{x}{2}}{\sin \frac{x}{2}} \right)^2 r^\nu \right\}^2 .$$

Hence

$$\varrho_\nu^{(2)}(x) \geqq 0 \text{ for } 0 \leqq x \leqq \pi,\ \nu = 0, 1, 2, \ldots.$$

Moreover, denoting $\left(\dfrac{\sin (\nu+1)\frac{x}{2}}{\sin \frac{x}{2}} \right)^2$ by $k_\nu(x)$, $\nu = 0, 1, 2, \ldots$,

we have

$$\text{(2.7)} \qquad \boxed{\begin{array}{l} \varrho_{n+1}^{(2)}(x) = \sin x \sum\limits_0^n k_\nu(x)\, k_{n-\nu}(x) > 0 \\ \text{for } 0 < x < \pi, \qquad n = 0, 1, 2, \ldots \end{array}}\ [2].$$

We shall apply this result, using a device of Fejér, to prove that the $(R, 2)$ means of the series $\sum\limits_0^\infty \sin \nu a \sin \nu x$ are positive for $0 < a < \pi$, $0 < x < \pi$.

Denote the partial sums of this series by $P_n(a, x)$ and the $(R, 2)$ means by $(n+1)^{-2} Q_n(a, x)$, $n = 0, 1, 2, \ldots$.

Obviously

$$Q_n(a, x) = Q_n(x, a), \qquad Q_n(\pi-a, \pi-x) = Q_n(a, x).$$

Thus it is sufficient to prove our proposition in the triangle

$$\text{(2.8)} \qquad 0 \leqq a + x \leqq \pi,\ 0 \leqq a - x \leqq \pi.$$

But on the hypotenuse $x = 0$, $0 \leqq a \leqq \pi$

$$Q_n(a, 0) = 0.$$

Moreover, obviously

$$\frac{\partial}{\partial x} Q_n(a, x) = \frac{1}{2}\left\{ \varrho_n^{(2)}(a+x) + \varrho_n^{(2)}(a-x) \right\},$$

and by (2.7) this expression is positive inside the triangle (2.8). Thus on every vertical line in this triangle (i.e., for fixed a) $Q_n(a, x)$

[2] For $(C, 3)$ means, cf. Fejér [1,2].

is monotonically increasing; hence in this triangle $Q_n(a, x) > 0$ for $x > 0$. This yields

$$(2.9)\quad \boxed{Q_n(a, x) > 0 \text{ for } 0 < a < \pi,\ 0 < x < \pi,\ n = 1, 2, 3, \ldots,}$$

which proves our statement.

3. We shall now apply these results to Fourier sine series. Let

$$t_0 = 0,\ t_n(x) = \sum_1^n \frac{\sin \nu x}{\nu},\quad t_n^{(1)}(x) = \sum_0^n t_\nu(x),\ t_n^{(2)}(x) = \sum_0^n t_\nu^{(1)}(x).$$

Then from (2.7)

$$\frac{d^2}{dx^2}\left\{t_{n-1}^{(2)}(x) + t_n^{(2)}(x)\right\} < 0 \text{ for } 0 < x < \pi, \qquad n = 1, 2, 3, \ldots.$$

Hence the curves

$$(3.1)\qquad y = t_{n-1}^{(2)}(x) + t_n^{(2)}(x), \qquad n = 1, 2, 3, \ldots,$$

are convex upwards for $0 < x < \pi$. For $t_n^{(2)}(x)$ itself this is no longer true. For $n \geqq 0$, $t_n(x)$ are the partial sums of the series

$$\frac{1}{2}(\pi - x) = 0 + \sum_1^\infty \frac{\sin \nu x}{\nu},$$

and the $(R, 2)$ means of this series are

$$\sigma_n(x) = (n+1)^{-2}\left\{t_{n-1}^{(2)}(x) + t_n^{(2)}(x)\right\}, \qquad n = 1, 2, 3, \ldots.$$

It is known that $t_n(x) > 0$ for $0 < x < \pi$, and

$$0 < (n+1)^{-1} t_n^{(1)}(x) < \frac{1}{2}(\pi - x),\ 0 < x < \pi.$$

Suppose now

$M \geqq f(x) \geqq 0$ for $0 < x < \pi$, $f(-x) = -f(x)$, $f(x) \not\equiv 0$, and

$$(3.2)\qquad f(x) \sim \sum_1^\infty b_\nu \sin \nu x;$$

then $s_0 = 0$,

$$s_n(x) = \sum_1^n b_\nu \sin \nu x = \frac{2}{\pi}\int_0^\pi f(t)\left(\sum_1^n \sin \nu t \sin \nu x\right) dt$$

$$= \frac{2}{\pi}\int_0^\pi f(t) P_n(t, x)\, dt,$$

and

$$s_{n-1}^{(2)}(x) + s_n^{(2)}(x) = (n+1)^2 R_n^{(2)}(x) = \frac{2}{\pi}\int_0^{\pi} f(t)\, Q_n(t, x)\, dt.$$

If now $0 \leqq f(x) \leqq M$ for $0 < x < \pi$, then from (2.9)

$$0 \leqq R_n^{(2)}(x) \leqq \frac{2M}{\pi(n+1)^2}\int_0^{\pi} Q_n(t, x)\,dt,$$

where

$$\frac{2}{\pi}\int_0^{\pi} P_n(t, x)\,dt = \frac{2}{\pi}\int_0^{\pi}\left(\sum_1^n \sin \nu t \sin \nu x\right) dt = \frac{2}{\pi}\sum_1^n \frac{1-(-1)^{\nu}}{\nu}\sin \nu x.$$

This is the n^{th} partial sum of the development

$$1 = \frac{2}{\pi}\sum_1^{\infty}\{1-(-1)^{\nu}\}\frac{\sin \nu x}{\nu},\quad 0 < x < \pi.$$

Thus the $(R, 2)$ means of the series (3.2) lie below the corresponding means of the development

$$\text{(3.3)}\quad M = \frac{2M}{\pi}\sum_1^{\infty}\{1-(-1)^{\nu}\}\frac{\sin \nu x}{\nu} = 0 + \frac{4M}{\pi}\sin x + 0 + \frac{4M}{\pi}\frac{\sin 3x}{3} + \cdots.$$

For Cesàro means of the third order the corresponding result has been proved by Fejér (1, 2; cf. also 7, p. 57).

Summarizing, we have

Theorem 1. *If $0 \leqq f(x) \leqq M$ for $0 < x < \pi$, then for the sine series of $f(x)$*

$$0 \leqq R_n^{(2)}(x) \leqq B_n(x)$$

where $B_n(x)$ are the $(R, 2)$ means of the series (3.3).

4. We shall now prove the result announced in the introduction. First we consider the „roof-function"

$$\text{(4.1)}\quad 0 + \frac{2b}{a(\pi-a)}\sum_1^{\infty}\frac{\sin \nu a \sin \nu x}{\nu^2} = \begin{cases} \dfrac{b}{a}x & \text{for } 0 \leqq x \leqq a \\[2mm] b\dfrac{\pi - x}{\pi - a} & \text{for } a \leqq x \leqq \pi. \end{cases}$$

where $0 < a < \pi$, $0 < b$.

Denote the partial sums of this series by

$$\tau_0 = 0,\quad \tau_n(x) = \frac{2b}{a(\pi-a)}\sum_1^n \frac{\sin \nu a \sin \nu x}{\nu^2},\qquad n = 1, 2, 3, \ldots,$$

and let

$$\tau_n^{(1)}(x) = \sum_0^n \tau_\nu(x),\quad \tau_n^{(2)}(x) = \sum_0^n \tau_\nu^{(1)}(x).$$

Then, using (2.9)

$$\frac{d^2}{dx^2}\{\tau_{n-1}^{(2)}(x)+\tau_n^{(2)}(x)\} = -Q_n(a, x) < 0$$

for $0 < a < \pi$, $0 < x < \pi$. Hence the $(R, 2)$ means of the series (4.1) are convex upwards for $0 < x < \pi$.

The same is true for the limiting cases $a \to 0$ and $a \to \pi$. If $a \to 0$, then

$$\tau_n(x) \to \frac{2b}{\pi}\sum_1^n \frac{\sin \nu x}{\nu} = \frac{2b}{\pi} t_n(x).$$

If $a \to \pi$, then

$$\tau_n(x) \to \frac{2b}{\pi}\sum_1^n (-1)^\nu \frac{\sin \nu x}{-\nu} = \frac{2b}{\pi}\sum_1^n \frac{\sin \nu(\pi-x)}{\nu} = \frac{2b}{\pi} t_n(\pi-x).$$

Now every polygon convex upwards and lying above the axis of abscissae is expressible as a finite sum with positive coefficients of roof functions, and every finite sum with positive coefficients of functions convex upwards is again convex upwards. Finally, functions positive in $0 < x < \pi$ and convex upwards can be approximated uniformly by the aforementionned polygons. This gives

THEOREM 2. *If $f(x) > 0$ in $0 < x < \pi$ and is convex upwards, then the $(R, 2)$ means of the sine series of $f(x)$ are positive and convex upwards in $0 < x < \pi$.*

For the third Cesàro means cf. Fejér (1, 3, 4).

5. We now consider the cosine series of the step function

$$\frac{2b}{\pi}\left\{\frac{\pi-a}{2} - \sum_1^\infty \frac{\sin \nu a \cos \nu x}{\nu}\right\} = \begin{cases} 0 & \text{for } 0 \leqq x \leqq a \\ b & \text{for } a < x \leqq \pi \end{cases}$$

where $0 < a < \pi$, $b > 0$.

For the partial sums of this series we obviously have

$$s_n'(x) = \frac{b}{\pi}\sum_1^n \sin \nu a \sin \nu x = \frac{b}{\pi} P_n(a, x).$$

Hence, using (2.9), the curves $y = R_n^{(2)}(x)$ are monotone increasing. Using now the same argument as Fejér (1) used for the Cesàro means of third order we get

THEOREM 3. *If $f(x)$ is monotone in $0 < x < \pi$, then the $(R, 2)$ means of its cosine series are monotone in the same sense.*

6. We now turn to power series $\sum_0^\infty a_\nu z^\nu$ convergent for $|z| < 1$. We shall prove

THEOREM 4. *Suppose the power series* $\sum_0^\infty a_\nu z^\nu = f(z) = w = u + iv$ *is regular and univalent (i.e.,* $f(z_1) \neq f(z_2)$ *for* $z_1 \neq z_2$*) in* $|z| < 1$ *and all* a_ν *are real. Suppose further that the images* K_r *of the circles* $|z| = r$ *are convex for* $0 < r < 1$ *in the direction of the v-axis* [3]*. Let* $A_n(z) = \sum_0^n a_\nu z^\nu$, $A_n^{(1)}(z) = \sum_0^n A_\nu(z)$, $A_n^{(2)}(z) = \sum_0^n A_\nu^{(1)}(z)$, *then the polynomials of Riesz type of the second order* $W_n(z) = (n+1)^{-2}\{A_{n-1}^{(2)}(z) + A_n^{(2)}(z)\}$ *are univalent for* $|z| < 1$.

For the proof we may assume without loss of generality that the upper half of the circle $|z| < 1$ is mapped onto the upper half of the image in the w-plane. If we write

$$w(e^{ix}) = u(x) + iv(x) \sim \sum_0^\infty a_\nu \cos \nu x + i \sum_0^\infty a_\nu \sin \nu x,$$

then $v(x)$ is positive for $0 < x < \pi$ and $u(x)$ is decreasing in the same interval. Hence by theorem 1 the $(R, 2)$ means of the series $\sum a_\nu \sin \nu x$ are positive, and by theorem 3 the $(R, 2)$ means of the series $\sum a_\nu \cos \nu x$ are strictly increasing. Thus the curve $W_n(e^{ix})$ is a Jordan curve. This proves the theorem.

7. Formula (2.5) represents $R_n^{(2)}$ as an average of $C_{n-1}^{(2)}$ and $C_n^{(2)}$ with positive weights of sum 1. Hence

$$\liminf_{n\to\infty} C_n^{(2)} \leqq \liminf_{n\to\infty} R_n^{(2)} \leqq \limsup_{n\to\infty} R_n^{(2)} \leqq \limsup_{n\to\infty} C_n^{(2)}.$$

More generally, the domain of oscillation of the sequence $\{R_n^{(2)}\}$ is contained in the domain of oscillation of the sequence $\{C_n^{(2)}\}$. This is a special case of the „Kernsatz'' of Knopp.

We can also express $C_n^{(3)}$ as an average of the $R_n^{(2)}$ with positive weights. From (2.3)

$$\varrho_n^{(2)} = U_{n-1}^{(3)} - U_{n-2}^{(3)} + U_n^{(3)} - U_{n-1}^{(3)} = U_n^{(3)} - U_{n-2}^{(3)},\ n = 2, 3, \ldots,$$

hence if n is even, $n = 2m$, $m \geqq 1$,

$$\varrho_{2m}^{(2)} = U_{2m}^{(3)} - U_{2m-2}^{(3)},$$

and

$$\sum_{\nu=1}^m \varrho_{2\nu}^{(2)} = U_{2m}^{(3)} - U_0^{(3)},\ \text{or}\ U_{2m}^{(3)} = \sum_{\nu=0}^m \varrho_{2\nu}^{(2)},$$

[3]) That is, every parallel to the v-axis has at most two points in common with the image curve. In particular, if the domain (finite or infinite) in which the unit circle is mapped by $f(z)$ is convex in the usual sense, then our condition is satisfied. For $(C, 3)$ means cf. Fejér [1, 3, 4].

thus

$$(7.1)\qquad C_{2m}^{(3)} = \binom{2m+3}{3}^{-1} \sum_{0}^{m} (2\nu+1)^2 R_{2\nu}^{(2)},\ m = 1, 2, 3, \ldots.$$

If n is odd, $n = 2m+1$, $m \geqq 1$,

$$\varrho_{2m+1}^{(2)} = U_{2m+1}^{(3)} - U_{2m-1}^{(3)},$$

hence

$$\sum_{1}^{m} \varrho_{2\nu+1}^{(2)} = U_{2m+1}^{(3)} - U_{1}^{(3)} = U_{2m+1}^{(3)} - U_{0}^{(2)} - U_{1}^{(2)},$$

and

$$U_{2m+1}^{(3)} = \sum_{0}^{m} \varrho_{2\nu+1}^{(2)},$$

$$C_{2m+1}^{(3)} = \binom{2m+4}{3}^{-1} \sum_{0}^{m} (2\nu+2)^2 R_{2\nu+1}^{(2)} = \binom{2m+4}{3}^{-1} \sum_{1}^{m+1} (2\nu)^2 R_{2\nu-1}^{(2)},$$

or

$$(7.2)\qquad C_{2m-1}^{(3)} = \binom{2m+2}{3}^{-1} \sum_{1}^{m} (2\nu)^2 R_{2\nu-1}^{(2)},\ m = 2, 3, \ldots.$$

Now

$$\sum_{1}^{n} \nu^2 = \frac{n(n+1)(2n+1)}{6},\ \sum_{1}^{n} (2\nu)^2 = \frac{2}{3} n(n+1)(2n+1) = \binom{2n+2}{3},$$

and

$$\sum_{0}^{n} (2\nu+1)^2 = \frac{n+1}{3}(2n+1)(2n+3) = \binom{2n+3}{3},$$

hence

$$\binom{2m+3}{3}^{-1} \sum_{0}^{m} (2\nu+1)^2 = 1.$$

Also $\binom{2m+2}{3}^{-1} \sum_{1}^{m} (2\nu)^2 = 1$. Hence the sum of the weights in (7.1) and (7.2) is again 1. Finally

$$C_0^{(3)} = R_0^{(3)},\ C_1^{(3)} = \binom{4}{3}^{-1} U_1^{(3)} = \frac{1}{4}(U_0^{(2)} + U_1^{(2)});$$

hence $C_1^{(3)} = \frac{1}{4}\varrho_1^{(2)} = R_1^{(2)}$; $C_3^{(3)} = \frac{1}{20}\sum_{0}^{3} U_\nu^{(2)}$

$$= \frac{\varrho_1^{(2)} + \varrho_3^{(2)}}{20} = \frac{4R_1^{(2)} + 16R_3^{(2)}}{20}.$$

This proves that the domain of oscillation of the sequence $\{C_n^{(3)}\}$ is contained in the domain of oscillation of the sequence $\{R_n^{(2)}\}$.

In particular the results (2.7), (2.9) and the theorems 1, 2, 3, 4 yield the corresponding results of Fejér.

It is now obvious that the existence of the limit $\lim_{n\to\infty} C_n^{(2)} = l$ implies $\lim_{n\to\infty} R_n^{(2)} = l$. That the converse is not true has been observed by M. Riesz (6). Formula (2.3) suggests a simple example:

Let $$U_n^{(2)} = (-1)^n (n+1)(n+2), \qquad n = 0, 1, 2, \ldots,$$

then

$$\varrho_n^{(2)} = (-1)^n (n+1)(n+2-n) = (-1)^n 2(n+1).$$

Hence

$$R_n^{(2)} = 2(-1)^n (n+1)^{-1} \to 0 \text{ as } n \to \infty.$$

But $C_n^{(2)} = \binom{n+2}{2}^{-1} U_n^{(2)}$ obviously oscillates.

In this connection it is interesting to observe that the two methods are equivalent for Fourier series. This follows from

THEOREM 5. *If Σu_ν is summable $(R, 2)$, then a necessary and sufficient condition for its $(C, 2)$ summability is*

$$\liminf_{n\to\infty} n^{-3} \sum_{\nu=0}^{n} (n-\nu+1)\,\nu u_\nu \geqq 0. \tag{7.3}$$

In particular $u_n \to 0$ is a sufficient condition.

We have from (2.5)

$$R_n^{(2)} = C_n^{(2)} - \frac{n}{2(n+1)} \{C_n^{(2)} - C_{n-1}^{(2)}\}, \tag{7.4}$$

and an easy calculation yields

$$\frac{1}{2}\{C_n^{(2)} - C_{n-1}^{(2)}\} = \frac{1}{n(n+1)(n+2)} \sum_0^n (n-\nu+1)\,\nu u_\nu.$$

Let $R_n^{(2)} \to s$, and

$$\limsup_{n\to\infty} C_n^{(2)} = \bar{c}, \quad \liminf_{n\to\infty} C_n^{(2)} = \underline{c},$$

$$\liminf_{n\to\infty} n^{-3} \sum_0^n (n-\nu+1)\,\nu u_\nu = \underline{u}.$$

Then from (7.4)

$$\underline{c} = s + \underline{u}.$$

Again, from (2.5)

$$R_n^{(2)} = C_{n-1}^{(2)} + \frac{n+2}{2(n+1)}\{C_n^{(2)} - C_{n-1}^{(2)}\},$$

hence

$$\bar{c} = s - \underline{u}.$$

Since $\bar{c} \geqq \underline{c}$, we have $\underline{u} \leqq 0$.
Thus $\bar{c} = \underline{c} = s$ if and only if $\underline{u} \geqq 0$, and this, of course, actually implies $\underline{u} = 0$.

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