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Introduction.

This paper is concerned with certain limit groups and spaces in bounded open manifolds and regions. The theory of regions and of open manifolds generally is one of the least developed parts of topology, despite the fact that the notion of a region is one of the simplest and most frequently used topological notions in various mathematical subjects. In particular the theory of prime ends, which is concerned with the structure of the boundaries, has so far evaded modern topological methods ¹). This theory has remained an isolated subject depending on „direct” (and often complicated) methods. We give now an interpretation and an extension of its foundations in the light of modern topology.

§ 1 contains the definition of systems of boundary divisors or ends. The notion of a divisor is quite elementary; a divisor is essentially a decreasing sequence of part regions tending to the boundary and represented by its limit set. Divisors are denoted by \( E_G \). In particular the boundary \( F \) of a region \( G \) can be considered as a divisor which is denoted by \( \Gamma_G \).

In § 2 we define the limit groups of a region. These are groups of 0-dimensional infinite cycles tending to the boundary ²). The most important of these is the group \( \Sigma \) of pure cycles which is defined as the direct sum \( A(\Gamma_G) + B(\Gamma_G) \) of the groups of „convergent” (or \( \alpha \)-) and „divergent” (or \( \beta \)-) cycles. The former are

¹) See my thesis in Math. Annalen 103 (1930), also my papers in Math. Ann. 106 (1932) and Math. Zeitschrift 36 (1932). It is little known that this theory is valid for all bounded open manifolds embedded in Euclidean spaces (and not only for 3-dimensional regions). In this paper we can confine ourselves to (\( n \)-dimensional) regions, as the extension of its results to bounded manifolds is obvious.

²) Limit cycles are denoted by \( z_f = (z_k) = z_1, z_2, \ldots, z_k, \ldots \) or similarly.
limit cycles lying on finite systems of paths with (accessible) endpoints on the boundary, the latter are cycles of the opposite type (which „carry” no α-cycles). Similarly we define two fundamentally different types of boundary relations, the α- and β-homologies.

In § 3 we consider the group Σ as an abstract space Σ*. Closures in this group are defined by means of local homologies. The closure S of an aggregate S is defined as a sum of two aggregates (S)1 and (S)2, which we call closures of the first and second kind. While (S)1 is essentially induced by the topology of the region, it is the introduction of (S)2 which makes the whole theory effective 3). The limit space Σ* is a neighbourhood space (in the sense of H. Weyl). It satisfies all axioms of a topological space, except the distributive law of closures, which in our case however can be replaced by an equivalent condition.

Similarly, the fact that the „points” are not closed and the corresponding separation axiom is not fulfilled is practically unimportant, since the closures of points are shown to be very simple aggregates.

In § 4 we consider α- and β-homology groups. With the aid of these groups we define the prime ends in the space F of all α- and β-limit sequences of points, which forms a subspace of the space Σ*. This definition is essentially on the lines of my thesis. We also state some theorems and mention some general problems. But by far the most interesting problems arise in connection with the theory of conformal representation (of regions of arbitrary connectivity) and the theory of automorphic functions, which we shall consider elsewhere.

§ 1. Divisors of the boundary.

1. Let G be a region in the n-dimensional Euclidean space \( R^n \), and let \( \Gamma \) be its boundary. (We assume G to be bounded.) Let \( \Psi^n \) be an arbitrary subdivision of \( R^n \) into n-dimensional convex cells (simplices, cubes, etc.) forming a cell complex of some arbitrary mesh. The sum of all cells in \( \Psi^n \) we call an n-dimensional (infinite) polyedron, and denote it by \( | \Psi^n | \). The complex of all (n - 1)-dimensional faces of \( \Psi^n \) we call the (n - 1)-dimen-

\[ \text{3) This shows that the topology of the limit space depends not merely on „distances” between infinite chains („points”) } z_f \text{ but also essentially on the „distances” between finite chains } z \text{ and the infinite chains } z_f. \]
sional skeleton of $\Psi^n$, and denote it by $\Psi^{n-1}$. The sum of all cells in $\Psi^{n-1}$ we denote by $\mid \Psi^{n-1} \mid$.

A subset $Q$ of $\mid \Psi^{n-1} \mid$ we call a cut of $G$, if each point of $Q$ is a boundary point of at least two components of $G - Q$. A cut $Q$ of $G$ is called irreducible if the boundaries $\Gamma'$ and $\Gamma''$ of any two components $G'$ and $G''$ of $G - Q$ coincide inside $G$, i.e. if $\Gamma' G = \Gamma'' G$. An irreducible cut $Q$ is called regular, if $G - Q$ consists of precisely two components. If $Q$ is a cut of $G$, and $G'$ is an arbitrary component of $G - Q$, we speak also of a cut $Q$ corresponding to $G'$, and of a region $G'$ defined by $Q$.

2. Let $$(G_n) = G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots$$ be a decreasing sequence of regions defined by a sequence $$(Q_n) = Q_1, Q_2, \ldots, Q_n, \ldots$$ of cuts of $G$ such that the closures $\overline{Q}_\lambda$ and $\overline{Q}_\mu$ of any two cuts $Q_\lambda$ and $Q_\mu$ of the sequence have no common points. The product $$\mathcal{E}_G = \prod_{n=1}^{\infty} (\overline{G}_n)$$ is obviously a continuum or a point. It is obvious that the set $\mathcal{E}_G$ is uniquely defined by the sequence of cuts $(Q_n)$. We call $\mathcal{E}_G$ a divisor of the boundary $\Gamma$ or the end of the region $G$ if $\mathcal{E}_G$ is a part of $\Gamma$. A divisor $\mathcal{E}_G$ is called regular if all cuts $Q_n$ of the sequence $(Q_n)$ are regular.

It is easy to see that there exists a regular divisor $$\mathcal{E}_G = \Gamma_G$$ coinciding with the boundary $\Gamma$ itself; in particular the divisor $\Gamma_G$ can be defined by a sequence of regular cuts $(Q_n)$ such that each cut $Q_n$ lies entirely in $G$.

A divisor $G'$ is said to be contained in a divisor $G$, $$G' \subset G,$$ if $G'$ and $G$ are defined by sequences $(G'_m)$ and $(G_n)$ such that almost all regions $G'_m$ of the first sequence are contained in each region $G'_m$ of the second sequence. In particular the boundary $\Gamma$ of $G$, considered as a divisor, must obviously contain all divisors of $\Gamma$.

If $G'_G$ and $G''_G$ are two divisors such that the relations $$G'_G \subset G''_G,$$
and
\[ \mathcal{G}_G'' \subseteq \mathcal{G}_G' \]
are fulfilled simultaneously, we say that \( \mathcal{G}_G' \) and \( \mathcal{G}_G'' \) are equivalent and we write
\[ \mathcal{G}_G' = \mathcal{G}_G''. \]
The set of points in a divisor \( \mathcal{G}_G \) considered independently from the defining sequence \( (G_n) \) we denote by \( |\mathcal{G}_G| \). Thus from \( \mathcal{G}_G' = \mathcal{G}_G'' \) it follows that
\[ |\mathcal{G}_G'| = |\mathcal{G}_G''|, \]
but not vice versa.

Two divisors \( \mathcal{G}_G' \) and \( \mathcal{G}_G'' \) defined by the sequences \( (G_n') \) and \( (G_m'') \) are said to be distinct if there exist two integers \( n = \lambda \) and \( m = \mu \) such that
\[ G'_\lambda G''_\mu = 0. \]
It is obvious that \( \mathcal{G}_G' \) and \( \mathcal{G}_G'' \) are distinct if, and only if, there exists a pair of integers \( \lambda, \mu \) such that
\[ G'_{\lambda+v} G''_{\mu+v} = 0, \]
for each \( v = 1, 2, \ldots \).

3. Limit sequences in a divisor. A sequence of points of the region \( G \)
\[ (P_\lambda) = P_1, P_2, \ldots, P_\lambda, \ldots, \]
such that each limit point of \( (P_\lambda) \) lies on the boundary \( \Gamma \) of \( G \) we call a limit sequence of points in \( G \). A limit sequence of points with precisely one limit point we call a convergent sequence. A divisor \( \mathcal{G}_G \) defined by a sequence \( (G_n) \) contains a limit sequence \( (P_\lambda) \) if each region \( G_n \) \( (n = 1, 2, \ldots) \) contains almost all points of the sequence \( (P_\lambda) \). The aggregate of all convergent limit sequences \( (P_\lambda) \) contained in \( \mathcal{G}_G \) we denote by \( f(\mathcal{G}_G) \).

It is easy to see that two divisors \( \mathcal{G}_G' \) and \( \mathcal{G}_G'' \) are equivalent if, and only if, the aggregates \( f(\mathcal{G}_G') \) and \( f(\mathcal{G}_G'') \) are identical. In other words, from \( \mathcal{G}_G' = \mathcal{G}_G'' \) it follows that
\[ f(\mathcal{G}_G') = f(\mathcal{G}_G''). \]

4) From the above definition it is clear that a divisor \( \mathcal{G}_G \) is not merely a subset of \( \Gamma \), but a subset of \( \Gamma \) defined by and associated with a sequence \( (G_n) \). One could indeed define the sequence \( (G_n) \) itself as a divisor of \( \Gamma \), but it is more convenient to represent it by the product set \( \mathcal{E}_G \).

5) Instead of aggregates of convergent limit sequences in \( \mathcal{G}_G \) we could consider here the aggregates of arbitrary limit sequences in \( \mathcal{G}_G \), but this is not essential.
Thus the aggregate \( f(\mathcal{G}_G) \) of convergent limit sequences in \( \mathcal{G}_G \) is independent of the sequences \( (G_n) \) defining \( \mathcal{G}_G \).

On the other hand, two divisors \( \mathcal{G}'_G \) and \( \mathcal{G}''_G \) are distinct if, and only if, there exists a convergent sequence of points contained simultaneously in \( \mathcal{G}'_G \) and in \( \mathcal{G}''_G \), i.e., from \( \mathcal{G}'_G \mathcal{G}''_G = 0 \) it follows that

\[
f(\mathcal{G}'_G) f(\mathcal{G}''_G) = 0.
\]

If \( \mathcal{G}'_G \) and \( \mathcal{G}''_G \) are arbitrary, then the aggregate

\[
f(\mathcal{G}'_G) f(\mathcal{G}''_G)
\]

of all convergent limit sequences which are contained in both \( \mathcal{G}'_G \) and \( \mathcal{G}''_G \) is called the common part or the product of \( \mathcal{G}'_G \) and \( \mathcal{G}''_G \).

4. Full systems of distinct divisors. A system \( M(\mathcal{G}_G) \) of divisors of \( \Gamma \) we call a system of distinct divisors if any two divisors in \( M(\mathcal{G}_G) \) are distinct. A full system of distinct divisors of the boundary \( \Gamma \) is a system \( M(\mathcal{G}_G) \) of distinct divisors such that each convergent limit sequence \( (P') \) contains at least one subsequence

\[
(P'_1) = P'_1, P'_2, \ldots, P'_\lambda, \ldots
\]

contained in a divisor \( \mathcal{G}_G \) of the system \( M(\mathcal{G}_G) \).

A system \( M(\mathcal{G}_G) \) consisting of a single divisor \( \mathcal{G}_G = \Gamma_G \) is obviously a full system. It is not difficult to show that in any region \( G \) the totality of all full systems of distinct divisors is infinite, and that it has virtually the power \( 2^\aleph \).

§ 2. Limit groups of the region.

5. The aggregate of all inner points of the region \( G \) can be considered as a field of vertices in which abstract, geometrical ("flat") and topological complexes can be defined. Let \([x^1]\) be the system of all closed topological 1-cells (arcs) lying entirely in \( G \). Each 1-cell of \([x^1]\) is defined as a topological transformation of a segment \( 0 \leq x \leq 1 \). By \([x^0]\) we denote the aggregate of all 0-cells (points) in \( G \).

A finite system \( K^1 \) of 1-cells in \( G \) we call a 1-dimensional complex (1-complex) in \( G \). A 1-complex \( K^1 \) we call regular if any two 1-cells in \( K^1 \) have at most their endpoints in common. The set of all points contained in either of the 1-cells of \( K^1 \) we call the corresponding set of \( K^1 \) and denote it by \( |K^1| \). The corresponding set of a regular complex in \( G \) consists obviously of a finite system of (non-intersecting) arcs; the corresponding set of a
regular geometrical (flat) 1-complex consists of a finite number of connected polygon lines.
A 0-complex is by definition a finite set of 0-cells (points) in $G$.

6. Let $I_2$ be the group of integers reduced mod 2. Its elements (classes) we denote by 0 and 1, and we apply to these the usual (algebraic) operations mod 2. We define now in $G$ 0- and 1-dimensional chains mod 2 with respect to the systems $[x^0]$ and $[x^1]$ of all 0- and 1-cells as variables.

An $r$-chain $C^r$ mod 2 ($r = 0$ or $= 1$) is a linear form in which each $r$-cell of $[x^r]$ is associated with a coefficient belonging to the group $I_2$ and such that at most a finite number of $r$-cells in $C^r$ are taken with a coefficient $\neq 0$. Thus, omitting terms with the coefficient 0, we can write $C^r$ in the form

$$C^r = \sum_i x^r_i \quad (r = 0, 1)$$

where $i$ varies between 1 and some integer. An $r$-chain with all coefficients 0 we call an $r$-chain 0 or an empty chain. Since all variables of dimension $r$ occur in a chain, we can define the summation of chains. The sum of two chains mod 2 is again a chain mod 2. The $r$-chains mod 2 of the region $G$ form thus an additive group $L^r$ in which each element coincides with its inverse, the element 0 being the empty $r$-chain $^6$). It is obvious that $r$-chains mod 2 can also be considered as $r$-complexes for which addition mod 2 is defined. The notion of a corresponding set can thus be applied to a chain $C^r$ mod 2; the corresponding set of $C^r$ we denote by $|C^r|$.

7. The boundary (mod 2) of a 1-cell $x^1 = P_0P_1$ with the endpoints $P_0$, $P_1$ is the linear form $P_0 + P_1$, and the boundary $C^1$ of a chain $C^1$ is the sum of the boundaries of all 1-cells in $C^1$. The boundary $C^0$ of a 0-chain $C^0$ is by definition 0.

An $r$-chain $C^r$ ($r = 0, 1$) such that $C^r = 0$ is called an $r$-cycle. Thus all 0-chains are 0-cycles. If a 0-cycle $z^0$ is the boundary of a chain $C^1$, we write also $C^1 \rightarrow z^0$. The $r$-cycles mod 2 form obviously a subgroup $Z^r$ of the group $L^r$.

$^6$) The group of $r$-chains defined above forms obviously a special case of the usual group of singular chains mod 2 on a generalized (infinite) polyhedron. In the case of a region it is usual to assume (as above) that the corresponding sets of all complexes are contained in the region.
A cycle \( z^0 \) is called a boundary cycle and is said to be homologous \( 0 \mod 2 \),
\[
z^0 \sim 0 \pmod{2},
\]
if there exists a complex \( C^1 \) such that \( C^1 = z^0 \). This relation can also be written in the form \( C^1 \to z^0 \). Two cycles \( z^0_1 \) and \( z^0_2 \) are said to be homologous to each other \( (\mod 2) \), if there exists a chain \( C^1 \) such that
\[
C^1 
\sim
z^0_1 + z^0_2.
\]
A 0-cycle \( 0 \) is by definition \( \sim 0 \). Thus each cycle \( z^0 \) is homologous to itself,
\[
z^0 \sim z^0.
\]
Since any two 0-cells in \( G \) can be joined by a 1-cell in \( G \), it is easy to see that a 0-cycle \( z^0 \) is \( \sim 0 \) if, and only if, the number of its points (0-cells) is \( \equiv 0 \mod 2 \). The 0-cycles homologous \( 0 \) form a subgroup of the group \( Z^0 \), which we denote by \( H^0 \).

Throughout this paper all boundary and homology relations are understood \( \mod 2 \).

8. Let \( \mathcal{G} \) be a divisor of \( \Gamma \) defined by a sequence
\[
(G_m) = G_1 \supset G_2 \supset \ldots \supset G_m \supset \ldots
\]
of decreasing regions. A sequence
\[
C^r = (C^r_k) = C^r_1, \ldots, C^r_k, \ldots
\]
of \( r \)-dimensional chains \( (r = 0 \) or \( = 1 \) \) we call an \( r \)-dimensional limit chain in \( \mathcal{G} \) if for each \( m \) almost all corresponding sets \( |C^r_k|, k = 1, 2, \ldots \), are contained in \( G_m \). The sum
\[
\big| C^r \big| = \sum_{k=1}^{\infty} \big| C^r_k \big|
\]
we call the corresponding set of the limit chain \( C^r \). By \( t \) we denote the topological limit of the set \( C^r \). The set \( t \) which lies on \( \Gamma \) we call the boundary limit of the chain \( C^r \).

If \( C^r \) is a limit chain in \( \mathcal{G} \), then any subsequence of chains
\[
C^{r'} = (C^{r'}_{\mu_k}) = C^{r'}_{\mu_1}, C^{r'}_{\mu_2}, \ldots, C^{r'}_{\mu_k}, \ldots \quad (\mu_k > \mu_{k-1})
\]
of \( (C^r_{\mu_k}) \) forms a limit chain in \( \mathcal{G} \), which we call a subordinate chain of \( C^r \) in \( \mathcal{G} \). If \( C^{r'} \) is a subordinate chain of \( C^r \) we write
\[
C^{r'} < C^r.
\]
The system of all subordinate chains of \( C^r \) we denote by \( \{ C^r \} \). The limit chains \( C^r = (C^r_k) \) and \( C^{r'} = (C^{r'}_k) \) we consider as being identical,

\[ C^{r'} = C^r, \]

if almost all chains \( C^r_k \) and \( C^{r'}_k \) are identical, i.e. if there exists an integer \( k_0 \) such that

\[ C^{r'}_{k_0+k} = C^r_{k_0+k} \]

for each \( k = 1, 2, \ldots \)

A 0-dimensional limit chain

\[ z_f = z_{1}, z_{2}, \ldots, z_{k}, \ldots \]

in \( \mathcal{E}_G \) we call a limit 0-cycle \(^8\) in \( \mathcal{E}_G \), if for each \( m \) we can define an integer \( k_m \) such that

\[ z_{k_m+k} \sim z_{k_m+k+1} \text{ in } G^m \]

for each \( k = 1, 2, \ldots \). A limit 0-cycle \( z_f \) in \( \mathcal{E}_G \) is by definition 0 if almost all cycles \( z_k \) are 0. The sum of two limit cycles \( z_f = (z_k) \) and \( z'_f = (z'_k) \) is by definition the limit 0-cycle

\[ z_f + z'_f = (z_k + z'_k). \]

It is obvious that the sum of two limit cycles in \( \mathcal{E}_G \) is a limit cycle in \( \mathcal{E}_G \). Thus the limit 0-cycles in \( \mathcal{E}_G \) form a group which we denote by \( Z(\mathcal{E}_G) \).

A limit cycle \( z_f = (z_k) \) is homologous 0 in \( \mathcal{E}_G \),

\[ z_f \sim 0 \text{ in } \mathcal{E}_G, \]

if there exists a limit chain \( C^1 = (C^1_k) \) in \( \mathcal{E}_G \) such that

\[ C^1_k \rightarrow z_k \text{ for } k = 1, 2, \ldots \]

Two limit cycles \( z'_f \) and \( z''_f \) are homologous to each other,

\[ z'_f \sim z''_f \text{ in } \mathcal{E}_G, \]

if \( z'_f + z''_f \sim 0 \text{ in } \mathcal{E}_G \). The limit cycle 0 is by definition \( \sim 0 \) in \( \mathcal{E}_G \).

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\(^7\) More generally, if \( M(C^r) \) is an arbitrary aggregate of a limit chain, we denote by \( \{ M(C^r) \} \) the sum \( \sum_{C^r \in M(C^r)} \{ C^r \} \); of all systems for all \( C^r \)'s in \( M(C^r) \).

\(^8\) Each chain \( z_k \) of the limit chain \( z_f \) is obviously a (0-dimensional) cycle. We write \( z \) for \( z^0 \), omitting the dimensional index number 0.

\(^9\) It is sufficient to assume this homology to be valid in \( G \). In general, if \( G' \subset G \) and \( |z| \subset G' \), then from \( z \sim 0 \text{ in } G \) it follows that \( z \sim 0 \text{ in } G' \). If \( C^r \rightarrow z \text{ in } G \), we can easily construct a chain \( C^1 \) isomorphic with \( C^1 \) such that \( C^1 \rightarrow z \text{ in } G' \). (This construction is obviously possible, because \( C \) is 1-dimensional.)
We get thus

$$z_f \sim z_f \quad \text{in } G_f.$$  

The limit 0-cycles homologous 0 in $G_f$ form a subgroup $H(G_f)$ of $Z(G_f)$. It is easy to see that the factor group $Z(G_f) - H(G_f)$ is isomorphic with the group of integers reduced mod 2.

9. **The groups of $\alpha$- and $\beta$-cycles.** We define now subgroups of "convergent" and "divergent" limit cycles in a region. The following definition of $\alpha$- and $\beta$-cycles makes this distinction clear.

A topological transformation $J$ of the interval $0 \leq x < 1$ contained in $G$ we call a path in $G$, if the closure $\overline{J}$ of $J$ is a simple arc consisting of $J$ and (precisely) one single point $t$ lying on the boundary $\Gamma$.

A linear combination

$$x_f = x_f^1 + \ldots + x_f^i + \ldots + x_f^k$$

of a finite number of limit 0-cycles $x_f^i = (x_k^i)$ we call an $\alpha$-cycle if for each $i$ the corresponding set $|x_f^i|$ of the cycle $x_f^i$ lies on a path $J_i$ in $G$.

From this definition it follows easily that for each $i$ there exists a limit chain $C_1^i$ lying entirely on $J$ and such that

$$C_1^i \rightarrow z_1^i + z_1^i.$$  

A limit 0-cycle $y_f = (y_k)$ we call a $\beta$-cycle in $G$ if there exists no $\alpha$-cycle lying on the corresponding set $|y_f|$ of $y_f$.

If $z_f$ is an arbitrary limit cycle, then it can easily be proved that there exists always an $\alpha$- or a $\beta$-cycle lying on the corresponding set $|z_f|$ of $z_f$. A limit cycle $z_f$, which is either an $\alpha$- or a $\beta$-cycle, we call briefly a pure cycle.

A limit 0-cycle 0 is by definition both an $\alpha$- and a $\beta$-cycle. With addition defined in the usual sense, the totality of all $\alpha$-cycles in $G$ forms a group which we denote by $A(\Gamma_G)$; this group obviously cannot be empty. Similarly the system of all $\beta$-cycles forms a group which we denote by $B(\Gamma_G)$.

If $G_f$ is an arbitrary divisor of $\Gamma$, then the systems of all $\alpha$- and $\beta$-cycles in $G_f$ form also groups, which we denote by $A(G_f)$ and $B(G_f)$ correspondingly. It is clear that for an arbitrary $G_f \neq \Gamma_f$ either of these groups can be empty. The common part of the

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10) We could define $\alpha$- and $\beta$-cycles with respect to an arbitrary divisor $G_f$ of $\Gamma$. But this is not essential, as it can easily be shown that the definition of $\alpha$- and $\beta$-cycles is independent of the choice of divisors.
groups $A(\mathcal{G}_c)$ and $B(\mathcal{G}_c)$ is 0. The totality of cycles which are either in $A(\mathcal{G}_c)$ or in $B(\mathcal{G}_c)$ is a group which can be considered as the direct sum

$$\Sigma = A(\mathcal{G}_c) + B(\mathcal{G}_c)$$

of the groups $A(\mathcal{G}_c)$ and $B(\mathcal{G}_c)$.

$\alpha$- and $\beta$-homologies. We define now homology relations with respect to the groups $A(\Gamma_c)$ and $B(\Gamma_c)$. These relations are fundamental for our theory, and are very similar in both cases.

An $\alpha$-cycle $x_\ell$ is said to be $\alpha$-homologous 0,

$$x_\ell \sim_\alpha 0 \quad \text{in } G,$$

if $x_\ell$ bounds a limit chain $C^1$ in $G$ such that each pure cycle on the corresponding set $|C^1|$ is an $\alpha$-cycle.

Two $\alpha$-cycles $x_\ell'$ and $x_\ell''$ are said to be $\alpha$-homologous to each other

$$x_\ell' \sim_\alpha x_\ell'',$$

if the cycle $x_\ell' + x_\ell''$ is $\alpha$-homologous 0 in $G$.

A $\beta$-cycle $y_\ell$ is said to be $\beta$-homologous 0,

$$y_\ell \sim_\beta 0 \quad \text{in } G,$$

if $y_\ell$ bounds a limit chain $C^1$ such that each pure cycle on $|C^1|$ is a $\beta$-cycle.

Two $\beta$-cycles $y_\ell'$ and $y_\ell''$ are said to be $\beta$-homologous to each other in $G$,

$$y_\ell' \sim_\beta y_\ell'',$$

if the chain $y_\ell' + y_\ell''$ is $\beta$-homologous 0 in $G$.

The limit 0-cycle 0 is by definition $\alpha$- and $\beta$-homologous 0. If the limit cycles $z_\ell'$ and $z_\ell''$ are $\alpha$- or $\beta$-homologous 0 in $G$, then the sum $z_\ell' + z_\ell''$ is $\alpha$- or $\beta$-homologous 0 as well. Thus the $\alpha$-cycles $\alpha$-homologous 0 and the $\beta$-cycles $\beta$-homologous 0 form subgroups of the groups $A(\Gamma_c)$ and $B(\Gamma_c)$, which we denote by $H_\alpha(\Gamma_c)$ and $H_\beta(\Gamma_c)$.

It is obvious that the notions of $\alpha$- and $\beta$-homologies can be extended to an arbitrary divisor $\mathcal{G}_\ell$ of $\Gamma$, and we can define similarly the groups $H_\alpha(\mathcal{G}_c)$ and $H_\beta(\mathcal{G}_c)$.

We define further a simple but important subgroup of the group $H_\alpha(\Gamma_c)$.

Let $x_\ell$ be an $\alpha$-cycle, and let $t$ be the (finite) set of its limit points on $\Gamma$. We say, $x_\ell$ is locally homologous 0 if $x_\ell$ bounds a limit chain $C^1$ such that the limit set of $|C^1|$ on $\Gamma$ is contained
in $t$. It is easy to see that in this case all pure cycles on $|C^1|$ are $\alpha$-cycles.

Two $\alpha$-cycles $x'_j$ and $x''_j$ are said to be locally homologous to each other if $x'_j + x''_j$ is locally $\sim 0$. The $\alpha$-cycles $0$ are locally $\sim 0$ by definition. Thus the $\alpha$-cycles locally $\sim 0$ form a subgroup of the group $H_2(\Gamma_G)$, which we denote by $H^I_2(\Gamma_G)$.

Similarly we can define the subgroup $H^I_{\alpha}(E_G)$ of $H(E_G)$, where $E_G$ is an arbitrary divisor of $\Gamma$. We shall be mainly concerned with the groups $H^I_{\alpha}(\Gamma_G)$, and $H^I_{\alpha}(\Gamma_G)$, making no use of the group $H_2(\Gamma_G)$.

§ 3. Limit spaces of the region.

10. We shall now consider the group $\Sigma$ of all pure ($\alpha$- and $\beta$-) cycles as an abstract space 11). We shall define in $\Sigma$ a (generalized) topology, which is fundamental in the theory of prime ends.

Let $S$ be an arbitrary aggregate of limit cycles of $\Sigma$. The closure $\overline{S}$ is defined as a set of limit cycles of $\Sigma$:

$$\overline{S} = (S)^1 + (S)^2$$

consisting of two parts: the closure $(S)^1$ of the first kind, and the closure $(S)^2$ of the second kind. If $S$ is empty, we write $S = 0$ and set $\overline{S} = \overline{0} = 0$. If $S \neq 0$, we define $(S)^1$ and $(S)^2$ as follows:

Let $z_f = z_1, z_2, \ldots, z_k, \ldots$ be a limit cycle of the group $\Sigma$, and let $t$ be its boundary limit (§ 2, 8). Consider a spherical neighbourhood 12) $U(t, \epsilon)$ of $t$, i.e. the sum of all spherical regions (with a diameter $\leq \epsilon$) containing points of $t$. The open set

$$U(t, \epsilon)G = U(z_f, \epsilon)$$

which obviously contains almost all cycles $z_k$, we call an $\epsilon$-neighbourhood of $z_f$ in $G$. If we omit a finite number of cycles $z_k$ in $z_f$, the limit cycle $z_f$ will be replaced by an identical cycle, which we denote again by $z_f$. We can therefore assume that all cycles $z_k$ of $z_f$ lie in $U(z_f, \epsilon)$, $\epsilon$ being arbitrarily small.

Now let $\xi_f = \xi_1, \xi_2, \ldots, \xi_k, \ldots$ be an arbitrary limit cycle of the group $\Sigma$.

11) More generally we could consider the group $Z(\Gamma_G)$ of all limit cycles as an abstract space. The results of this paragraph can be extended to this group, which is of some interest from the abstract point of view. In this paper we shall however make no further use of this group.

12) This notion is purely auxiliary; the space defined by means of these neighbourhoods is of no importance, and must not be confused with the abstract space $\Sigma^*$ defined below.
The cycle $\zeta_f$ is called a closure cycle of the first kind with respect to $S$ if each (arbitrarily small) neighbourhood $U(\zeta_f, \varepsilon)$ of $\zeta_f$ contains a cycle $\zeta = z_1, z_2, \ldots, z_k, \ldots$ such that

$$\zeta_f \sim \zeta \quad \text{in} \ U(\zeta_f, \varepsilon).$$

The aggregate of all closure cycles of $S$ of the first kind we denote by $(S)^1_1$, and call it the closure of the first kind of $S$.

We make use now of the following notation. Let $U(z_f)$ be a neighbourhood of $z_f = z_1, z_2, \ldots, z_k, \ldots$, and let $z$ be a cycle (mod 2) of the group $Z$. If for each $k = 1, 2, \ldots$

$$z \sim z_k \quad \text{in} \ U(z_f),$$

we write briefly

$$z \sim z_f \quad \text{in} \ U(z_f).$$

The cycle $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_\lambda$ is called a closure cycle of the second kind with respect to $S$, if each $\varepsilon$-neighbourhood $U(\zeta_f, \varepsilon)$ contains a limit cycle $z^{\lambda} = z_1^{\lambda}, z_2^{\lambda}, \ldots, z_k^{\lambda}, \ldots$ of $S$ for each $\lambda$ such that

$$\zeta^{\lambda}_f \sim z^{\lambda}_f \quad \text{in} \ U(\zeta_f, \varepsilon),$$

or, more precisely, if

$$\zeta^{\lambda}_f \sim z^{\lambda}_k \quad \text{in} \ U(\zeta_f, \varepsilon) \quad \text{for each} \quad \lambda = 1, 2, \ldots.$$ 

Here, as always, we must bear in mind the convention by which the limit cycles $\zeta_f$ and $z_f$ can be replaced by identical cycles with a finite number of cycles $\zeta^{\lambda}_f$ and $z^{\lambda}_k$ ($\lambda = 1, 2, \ldots$) omitted. The aggregate of all closure cycles of the second kind we denote by $(S)^2$, and call it the closure of the second kind of $S$.

The closure $\Sigma = (S)^1 + (S)^2$ of $S$ is now defined. The group $\Sigma$ can therefore be considered as a general topological space $\Sigma^*$, which we denote by $\Sigma^*$.

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13) See § 2, 7. The limit cycles $z_f = z_1, z_2, \ldots, z_k, \ldots$ and $z'_f = z'_1, z'_2, \ldots, z'_k, \ldots$ are identical if $z_k = z'_k$ for almost all $k$. In particular, if we omit in $z_f$ a finite number of cycles $z_k$, we obtain a limit cycle identical with $z_f$, which we denote again by $z_f$. We make use of this definition throughout this paper without special references.

14) A set $E$ of elements ("points") is called a general topological space if to each $S \subset E$ there corresponds an $\bar{S} \subset E$ (the closure of $S$). $E$ is called a topological space if the above correspondence satisfies the four axioms of Kuratowski. (See Alexandroff-Hoff, Topologie I.) The equivalence of Kuratowski's axioms with the usual Hausdorff axioms of a topological neighbourhood space can easily be proved. Sufficient and necessary conditions for $E$ to be a neighbourhood space (in the sense of H. Weyl) were given by A. Markoff (Alexandroff-Hoff, p. 42).
We introduce now the following notations: by $L(S)$ we denote the part of $\bar{S}$ which is not in $S$; correspondingly we denote by $L_1(S)$ and $L_2(S)$ the parts of $(S)^1$ and $(S)^2$ not in $S$. Further we use the notations $L_\alpha(S)$, $L_\beta(S)$, $L_1^\alpha(S)$, $L_1^\beta(S)$ etc. to represent the part of $L$, $L^1(S)$, $L^2(S)$ etc. consisting of all $\alpha$- and all $\beta$-cycles of these aggregates respectively.

11. We shall establish now some elementary properties of the space $\Sigma^*$.

I. If $z_f$ is an arbitrary limit cycle, then $z_f + z_f = 0$, and thus $z_f \sim z_f$ in $I_0$; obviously we can consider a limit cycle $0$ to be $\sim 0$ in any arbitrarily small neighbourhood $U(z_f, \varepsilon)$. Thus

$$z_f \sim z_f \quad \text{in } U(z_f, \varepsilon),$$

and by definition of the closure of the first kind we get

$$S \subset (S)^1 \subset \bar{S}.$$

II. The group $\Sigma^*$ is a neighbourhood space, i.e. the general topological correspondence in $\Sigma^*$ can be induced by means of a full system of neighbourhoods of all elements in $\Sigma^*$.

For if $0$ is the empty set in $\Sigma^*$, and $S'$ and $S''$ are two arbitrary sets in $\Sigma^*$, then

1. $\overline{0} = 0$.

2. from $S' \subset S''$ it follows that $(S')^1 \subset (S')^2 \subset (S'')^2$, and therefore $\overline{S'} \subset \overline{S''}$.

Thus the necessary and sufficient (A. Markoff's) conditions for $\Sigma^*$ to be a neighbourhood space \(^{15}\) are fulfilled.

If $z_f$ is an arbitrary limit cycle, and $E$ an arbitrary set of limit cycles such that $E \cap z_f$ then the set

$$V(z_f) = \Sigma^* - E$$

is defined as a neighbourhood of $z_f$. The system of all sets such as $\Sigma^* - E$ forms the full system of neighbourhoods of $z_f$.

III. $\Sigma^*$ satisfies the following relations,

$$\overline{S} = (S)^1 + (S)^2 = S + \overline{(S)^1}^2 = S + \overline{(S)^2}^1.$$

\(^{15}\) It may be noted here that the neighbourhoods in $\Sigma^*$ are not necessarily open.
According to I the set $S$ can be omitted in the first of these two relations. It is sufficient to verify

\begin{align*}
(1) \quad & ((S)^1)^2 \subset \bar{S} \\
(2) \quad & ((S)^2)^1 \subset \bar{S}.
\end{align*}

The inverse relations follow at once from I and from Markoff's conditions in II.

Suppose (1) is not fulfilled. Then there exists a limit cycle $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ such that

$$\zeta_f \subset (S)^2 \text{ and } \zeta_f \subset ((S)^1)^2.$$ \hfill (1)

We can therefore define an arbitrarily small neighbourhood $U(\zeta_f, \varepsilon)$ such that each cycle $\zeta_k$ is $\sim$ to a limit cycle $\xi_f^k$ of $(S)^1$. For a given $k$ let $U(\xi_f^k, \delta)$ be an arbitrarily small neighbourhood of $\xi_f^k$. This neighbourhood must contain a cycle $z_f^k$ of $S$ such that

$$\zeta_f^k \sim z_f^k \quad \text{in } U(\xi_f^k, \varepsilon),$$

$k$ being arbitrary, and $\varepsilon$ arbitrarily small. Thus (1) is proved.

Suppose now that (2) is not fulfilled. Then there exists a limit cycle $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ such that

$$\zeta_f \subset (S)^1 \text{ and } \zeta_f \subset ((S)^2)^1.$$ \hfill (2)

There exists therefore a sufficiently small neighbourhood $U(\zeta, \varepsilon)$ in which no cycle of $S$ is $\sim \zeta_f$. Therefore we can choose a neighbourhood $U(\zeta_f, \frac{1}{2}\varepsilon)$ in which a cycle $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ satisfies the relations

$$\zeta_f \sim \xi_f, \quad \xi_k \sim z_f^k, \quad k = 1, 2, \ldots,$$

where $z_f^k$ denotes some cycle of $S$. Thus

$$\zeta_k \sim z_f^k \quad \text{in } U(\zeta_f, \varepsilon), \quad k = 1, 2, \ldots,$$

and since $\varepsilon$ is arbitrarily small, we get $\zeta_f \subset (S)^2 \subset \bar{S}$.

IV. The group $\Sigma^*$ satisfies the axiom (c) of a topological space.

$$\bar{S} = \bar{\bar{S}}.$$ \hfill (3)

We have $^{16}$

$$\bar{S} = [(S)^1 + (S)^2]^1 + [(S)^1 + (S)^2]^2 =$$
$$= [(S)^1]^1 + [(S)^2]^1 + [(S)^1 + (S)^2]^2.$$ \hfill (4)

$^{16}$ It is easy to see that the closures of the first kind satisfy the distributive law $(S' + S'')^1 = (S')^1 + (S'')^1$ and also the relation $[(S)^1]^1 = (S)^1$. 

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But

\[ [(S)^1]^1 = (S)^1 \subset \overline{S} \]

and (according to III)

\[ ((S)^2) \subset \overline{S}. \]

It is thus sufficient to consider the last term in the above expression of \( \overline{S} \). If \( \zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots \) is an arbitrary cycle of \([ (S)^1 + (S)^2 ] \), then in an (arbitrarily small) \( \frac{1}{2}\varepsilon \)-neighbourhood \( U(\zeta_f, \frac{1}{2}\varepsilon) \) of \( \zeta_f \) we have for each \( k \)

\[ \zeta_k \sim \zeta_f^k \quad \text{in} \quad U(\zeta_f, \frac{1}{2}\varepsilon) \]

where \( \zeta_f^k \) is a cycle of \( S \), or of \( (S)^1 \), or of \( (S)^2 \). It is easy to see that in each of these cases there exists a cycle \( z_f^k \) of \( S \) such that

\[ \zeta_k \sim z_f^k \quad \text{in} \quad U(\zeta_f, \varepsilon) \]

where \( U(\zeta_f, \varepsilon) \) is an \( \varepsilon \)-neighbourhood of \( \zeta_f \). It follows that

\[ (\overline{S})^2 \subset (S)^2 \subset \overline{S}. \]

V. We have seen above that the group \( \Sigma^* \) satisfies the axioms

(a) \( 0 = 0 \),
(b) \( S \subset \overline{S} \),
(c) \( \overline{S} = \overline{S} \).

d of a topological space as well as Markoff's conditions and the relations in III, the latter referring to the twofold character of closures. But the axiom

(d) \( \overline{S'} + \overline{S''} = \overline{S'} + \overline{S''} \)

expressing the distributive law of closures is not fulfilled, since the inverse of the relation

\[ \overline{S'} + \overline{S''} \subset \overline{S'} + \overline{S''}. \]

is not valid. But this disadvantage is sufficiently outweighed by the following property of the space \( \Sigma^* \):

If \( \zeta_f \) is a cycle of the closure \( \overline{S'} + \overline{S''} \) of the sum \( S' + S'' \), then there exists always a subordinate cycle \( \zeta'_f < \zeta_f \) of \( \zeta_f \) contained in the sum of closures \( \overline{S'} + \overline{S''} \).

VI. The elements ("points") in \( \Sigma^* \) are not closed, and the corresponding separation axiom is not fulfilled. But this again
is not a serious disadvantage; the closures of the limit cycles are very simple units, of which all aggregates to be considered henceforth are built.

If \( z_f = z_1, z_2, \ldots, z_k, \ldots \) is an arbitrary limit cycle, and \( t \) its boundary limit on \( L_t \), then the boundary limit of an arbitrary cycle \( \zeta_f \) of the closure \( \tilde{z}_f \) of \( z_f \) is contained in \( t \).

If \( x_f \) is an \( \alpha \)-cycle \( \sim 0 \), then the following relations are valid:

\[
(1) \quad L_\beta(x_f) = 0,
\]

and

\[
(2) \quad (x_f)^1 = \bar{x}_f.
\]

Further, if \( x_f \) is arbitrary, and \( x^* < x_f \) a subordinate cycle of \( x_f \), then

\[
(3) \quad \bar{x}^* = \bar{x}_f,
\]

and finally,

\[
(4) \quad \bar{x}_f \supset \{\bar{x}_f\}^{17}).
\]

The proof of these relations depends on the following property of \( \alpha \)-cycles, which can easily be verified:

We represent \( x_f \) in the form \( x_f = \sum_{i=1}^{\lambda} x^i_f \), where \( x^i_f = (x^i_k) \) is an \( \alpha \)-cycle lying on a path \( J_i \). There exists then a limit chain \( C^1(f) = (C^1_k(f)) \) on \( J_i \) for each \( i \) such that

\[
C^1_k(f) \to x^i_k + x^i_{k+1} \quad (k = 1, 2, \ldots).
\]

The proof of (1) is quite elementary; (2) follows easily from (1), and the proofs of (3) and (4) are similar. The proof of (4) is as follows:

Let \( U(x_f, \varepsilon) \) be an arbitrary \( \varepsilon \)-neighbourhood of \( x_f \), and let \( \mu_1, \mu_2, \ldots, \mu_k, \ldots \) be an arbitrary sequence of increasing integers \( (\mu_k > \mu_{k-1}) \). We can define a limit chain \( (C^1_{\mu_k})_k \),

\[
C^1_{\mu_k} \to x_k + x_{\mu_k} \quad (k = 1, 2, \ldots)
\]

such that almost all chains \( C^1_{\mu_k} \) are in \( U(x_f, \varepsilon) \). The limit cycle \( (x_{\mu_k})_k \) is obviously contained in \( (x_f)^1 \). If \( \zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots \) is an arbitrary cycle of \( \bar{x}_f \), we have

\[
\zeta_{\mu_k} \sim x_{\mu_k} \sim x_k \quad \text{in} \ U(x_f, \varepsilon).
\]

\[17) \text{If} \quad \zeta_f \text{is an arbitrary limit cycle of} \bar{x}_f, \text{then} \{\zeta_f\} \text{denotes the system of all subordinate cycles of} \zeta_f. \{\bar{x}_f\} \text{denotes then the sum} \sum_{\zeta_f \in \bar{x}_f} \{\zeta_f\}.
\]
Any subordinate cycle $\zeta'_f < \zeta_f$ is obviously among the chains $(\zeta_{\mu_k})$, and we get

$$\zeta'_f \subset (x_f)^1 \subset x_f.$$

VII. From the definition of closures in $\Sigma^*$ it follows at once that

$$(S)^2 \supset \{(S)^2\},$$

i.e. all subordinate cycles of a cycle of $(S)^2$ are contained in $(S)^2$.

With regard to the closure of the first kind, we have

$$(S)^1_\alpha \subset (S)^2$$

If $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ is in $(S)^1_\alpha$, and if $U(\zeta_f, \frac{1}{2}\epsilon)$ is an $\frac{1}{2}\epsilon$-neighbourhood of $\zeta_f$, then

$$\zeta_k \sim z_k \text{ in } U(\zeta_f, \frac{1}{2}\epsilon)$$

where $z_f = z_1, z_2, \ldots, z_k, \ldots$ is a cycle of $S$. But for any $k = k_0$ we have

$$\zeta_{k_0} \sim z_{k_0} + \epsilon \text{ in } U(\zeta_f, \epsilon), \quad v \neq 1, 2, \ldots$$

Thus we have

$$\zeta_{k_0} \sim z_{k_0} + \epsilon \quad (v = 1, 2, \ldots),$$

i.e.

$$\zeta_{k_0} \sim z_f$$

and therefore

$$\zeta_f \subset (S)^3.$$

12. A slightly different, but practically equivalent, topological relation in the group $\Sigma^*$ can be obtained by a more restricted definition of closures. For each aggregate $S$ of limit cycles we set

$$\overline{S} = S + L^*_\beta(S) + L^\alpha(S),$$

thus omitting in the closure $\beta$-cycles of the first kind and $\alpha$-cycles of the second kind not in $S$. The space obtained by means of this definition we denote by $'\Sigma^*$. In $'\Sigma^*$ each $\beta$-cycle $\sim 0$ locally (i.e. in an arbitrary neighbourhood of its boundary limit) is closed.

Let $z_f = z_1, z_2, \ldots, z_k, \ldots$ be a $\beta$-cycle $\sim 0$ in the neighbourhoods $U(z_f, \epsilon_v); \epsilon_v \to 0$ with $v \to \infty$. Any cycle $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ of $\tilde{z}_f$ must obviously be a $\beta$-cycle, and such that the boundary sets of $z_f$ and $\zeta_f$ coincide. According to the above definition of $\overline{S}$ it is sufficient to show that $\zeta_f$ is not in $L^2_\beta(S)$. Otherwise we have

$$\zeta_k \sim z_f \quad \text{for each } k \text{ and each } v.$$
Since \( z_f \sim 0 \) in \( U(z_f, \varepsilon_v) \), there exists for each \( v \) a component \( h_v \) of \( U(z_f, \varepsilon_v) \) containing points of a set \( |z_2| \) for arbitrarily large values of \( \lambda \). It is therefore possible to construct a decreasing sequence of components
\[
h_1 \supset h_2 \supset \ldots \supset h_v \supset \ldots
\]
containing points of the set \( |z_f| \). It follows that a path \( J \) in \( G \) meets points of \( |z_f| \) arbitrarily near \( \Gamma \), and this contradicts the definition of \( \beta \)-cycles.

\section*{§ 4. The prime ends.}

13. \( \alpha \)- and \( \beta \)-homology groups. We consider now the factor groups
\[
\mathfrak{A} = A(\Gamma_\alpha) - H^1_\alpha(\Gamma_\alpha)
\]
and
\[
\mathfrak{B} = B(\Gamma_\beta) - H_\beta(\Gamma_\alpha).
\]

If \( \zeta_f \) is an \( \alpha \)- or \( \beta \)-cycle \( \neq 0 \) of the group \( A(\Gamma_\alpha) \) or \( B(\Gamma_\beta) \), we denote by \( C(\zeta_f) \) the class of the group \( \mathfrak{A} \) or \( \mathfrak{B} \) containing \( \zeta_f \). By \( \overline{C}(\zeta_f) \) we denote the closure of \( C(\zeta_f) \) in \( \Sigma^* \). The cycle \( \zeta \) is called soluble if any two subordinate cycles \( \zeta' < \zeta_f \) and \( \zeta'' < \zeta_f \) satisfy the relation
\[
\overline{C}(\zeta'_f)\overline{C}(\zeta''_f) \neq 0.
\]

If \( \zeta_f \) is soluble, then all cycles of \( C(\zeta_f) \) are soluble. A class of soluble cycles is called a soluble class.

A class \( C(\zeta_f) \) of \( \mathfrak{A} \) or \( \mathfrak{B} \) such that for any two cycles \( \zeta'_f < \zeta_f \) and \( \zeta''_f < \zeta_f \)
\[
C(\zeta'_f) = C(\zeta''_f)
\]
we call an (\( \alpha \)- or \( \beta \)-) oscillator.

Let \( x_f \) be an arbitrary \( \alpha \)-cycle. From the definition of local homologies it follows that
\[
C(x_f) = (x_f)^1.
\]

According to VI, § 3, we have
\[
(x_f)^1 = (x_f)^2 = \overline{\alpha}_f.
\]

Thus we get the identities
\[
(x) \quad (x_f)^1 = (x_f)^2 = \overline{\alpha}_f = C(x_f) = \overline{C}(x_f).
\]
Again according to VI, § 3, we have

\[ \bar{x}' = \bar{x}'' \]

where \( x' \) and \( x'' \) are subordinate cycles of \( x_i \). Thus

\[ C(x'_i) = C(x''_i) \]

which implies that \( x_i \) is soluble, and that all elements of \( \mathfrak{A} \) are oscillators.

**14. The space of convergent limit sequences.** Let

\[ P_f = (P_k) = P_1, P_2, \ldots, P_k, \ldots \]

be a convergent limit sequence (see § 1) in \( \Gamma_0 \). It is clear that \( P_f \) can be considered as a limit cycle \( z_f = (z_k) \mod 2 \), in which each cycle \( z_k = P_k \) consists of a single point \( P_k \) taken with the coefficient 1. Sequences \( P_f \) which are in this sense pure (\( \alpha \)- or \( \beta \)-) cycles we call pure (\( \alpha \)- or \( \beta \)-) sequences. All subordinate sequences \( P'_f \) satisfy the relation \( P'_f \sim 0 \) in \( \Gamma_G \).

Let \( F_\alpha \) and \( F_\beta \) be the aggregates of all \( \alpha \)- and \( \beta \)-sequences in \( \Gamma_0 \). The aggregate

\[ F = F_\alpha + F_\beta \]

is contained in the group space \( \Sigma^* \). Therefore we can consider \( F \) as a subspace of \( \Sigma^* \) in which a general topological correspondence is induced by \( \Sigma^* \); if \( S \subset F \) and \( \overline{S} \) is the closure of \( S \) in \( \Sigma^* \), then, the closure of \( S \) in \( F \) is \( \overline{SF} \). Obviously all topological relations of § 3 apply also to the space \( F \).

Let \( C(P_f) \) be the class of the factor group \( \mathfrak{A} \) or \( \mathfrak{B} \) containing \( P_f \). The aggregate

\[ \Delta(P_f) = C(P_f)F \]

we call a conjugate (\( \alpha \)- or \( \beta \)-) class in \( F \). Since the conjugate classes are contained in the classes of the groups \( \mathfrak{A} \) or \( \mathfrak{B} \), it is clear that the classes \( \Delta \) are distinct, i.e. if \( \Delta' \neq \Delta'' \) then \( \Delta' \Delta'' = 0 \).

A limit sequence \( P_f \) is called soluble if for any two sub-ordinate sequences \( P'_f \subset P_f \) and \( P''_f \subset P_f \)

\[ \overline{\Delta(P'_f)} \overline{\Delta(P''_f)} \neq 0. \]

In the case of conjugate \( \alpha \)-classes all identities (\( \alpha \)) in section 13 are valid in the space \( F \) as well. In particular all \( \alpha \)-sequences are soluble. In addition we can give a simple interpretation of all
aggregates in ($\alpha$). For we can always construct $^{18)}$ a divisor $G_{\alpha} = P_{\alpha}$ consisting of a single (accessible) point $P$ such that

$$(\alpha') \quad f(P_{\alpha}) = \Delta = \overline{\Delta} = \overline{P}_{\alpha}.$$ 

15. Complexes of conjugate classes in $F$. Let $P_{\alpha}$ be a soluble sequence, and let $\Delta$ be the conjugate class containing $P_{\alpha}$. As usual we denote by $\{P_{\alpha}\}$ the system of all subordinate sequences of $P_{\alpha}$. Let $\{\Delta\}$ be the system of all conjugate classes which contain at least one sequence in $\{P_{\alpha}\}$. Let $\{\overline{\Delta}\}$ be the system of all closures of the classes $\Delta$ in $\{\Delta\}$. The sum

$$\Delta_1 = \sum_{\Delta \in \{\overline{\Delta}\}} (\overline{\Delta})$$

of all classes $\overline{\Delta}$ in $\{\overline{\Delta}\}$ we call a complex of the order 1 $^{19)}$.

We state now (without proof) the first fundamental theorem in the theory of prime ends.

**Theorem I.** Each limit sequence $P_{\alpha}$ in $G$ contains a soluble subordinate sequence $P'_{\alpha} < P_{\alpha}$ (Primendentheorie, § 10.) In particular there exists a subordinate sequence $P'_{\alpha} < P_{\alpha}$ such that the conjugate class $\Delta \supset P'_{\alpha}$ satisfies the relation $L_{\alpha}^1(\Delta) \neq 0$ $^{20)}$.

We define now complexes $\Delta_{\varphi}$ in $F$ of the order $\varphi$, where $\varphi$ is an arbitrary finite or transfinite ordinal number.

Let

$$1, 2, \ldots, \ldots, \eta, \ldots$$

be the ordered system of all ordinals $< \varphi$. Assume all complexes $\Delta_{\eta}$ of all orders $\eta < \varphi$ to have been defined. Let $M(\Delta_{\eta})$ be a system of complexes ($\eta < \varphi$ but otherwise arbitrary) and let $M(\overline{\Delta}_{\eta})$ be the system of closures of all complexes in $M(\overline{\Delta}_{\eta})$. The sum

$$\Delta_{\varphi} = \sum_{\overline{\Delta}_{\eta} \in M(\overline{\Delta}_{\eta})} (\overline{\Delta}_{\eta})$$

of all closures $\overline{\Delta}_{\eta}$ in $M(\overline{\Delta}_{\eta})$ we call a complex of the order $\varphi$ if

---

$^{18)}$ We consider a sequence $U(P, e_{\nu})$ of decreasing spherical neighbourhoods of $P$. It is easy to show that (for each $\nu$) a component $h_{\nu}$ of $U(P, e)$ contains $P_{\alpha}$. The decreasing sequence $h_1 \supset h_2 \supset \ldots \supset h_{\nu} \supset \ldots$ defines the divisor $P_{\alpha}$, which coincides with $P$ and is, in general, not regular.

$^{19)}$ If $\Delta$ is a conjugate $\alpha$-class, the complex $\Delta_1$ coincides with $\Delta$ and thus with all aggregates in ($\alpha$) or ($\alpha'$).

$^{20)}$ The second part of Theorem I is proved in my paper „Über die Struktur der Komplexe beliebiger Ordnung in der Theorie der Primenden“ [Math. Annalen 106 (1932)].
1. for each decomposition of the system \( M(\bar{A}_\eta) \) into two proper parts

\[
M(\Delta_\eta) = M'(\Delta_\eta) + M''(\Delta_\eta)
\]

there exists a complex \( \Delta'_\eta \in M'(\Delta_\eta) \) and a complex \( \Delta''_\eta \in M''(\Delta) \) such that

\[
\bar{\Delta}_\eta_1 \bar{\Delta}_\eta_2 \neq 0.
\]

2. there exists no system \( N(\Delta_\eta) \supset M(\Delta_\eta) \) of which \( M(\Delta_\eta) \) forms a proper part, i.e. if \( N(\Delta_\eta) \) satisfies condition 1, then \( N(\Delta_\eta) = M(\Delta_\eta) \).

A complex \( \Delta_\eta \) of the order \( \eta \) is said to be saturated if there exists no complex of an order \( > \eta \) containing \( \Delta_\eta \).

A regular divisor \( \mathcal{G}_G \) is called a prime end of the order \( \eta \) if there exists a complex \( \Delta_\eta \) of the order \( \eta \) such that

1. \( \Delta_\eta \subset f(\mathcal{G}_G) \)
2. \( \Delta_\eta \) contains a subordinate sequence \( P'_f < P_f \) of each sequence \( P_f \) of \( f(\mathcal{G}_G) \).

We can now state the second and third fundamental theorems in the theory of prime ends.

**Theorem II.** There exists a full system of distinct regular divisors \( M(\mathcal{E}_G) \) in \( G \) such that each divisor \( \mathcal{E}_\eta \) is a prime end (of finite or transfinite order \( \eta \)). (Primendentheorie, § 11.)

**Theorem III.** The order \( \eta \) of each prime end \( \mathcal{E}_\eta \) is enumerable, i.e. \( \eta \) is an ordinal of the second Cantor class. (Primendentheorie, § 12.)

16. **Problems.** We have considered above the groups of 0-dimensional cycles. There exists however no simple generalization of these groups to higher dimensions, and it is clear that our theory is essentially dependent on their 0-dimensionality. Further, if we examine the definition of the ends we can make an interesting observation:

**This definition depends on the 0-dimensional Betti groups of the region \( G \).** For a divisor \( \mathcal{G}_G \) of \( G \) is defined by a decreasing sequence \( G_1 \supset G_2 \supset \ldots \supset G_k \supset \ldots \) of part regions (§ 1, 2). But if \( G_k \) is a part region, its 0-dimensional Betti group \( B(\mathcal{G}_k) \) is isomorphic with the Betti group \( B(G) \). Now we can consider decreasing sequences of part regions

\[
G_1^{(r)} \supset G_2^{(r)} \supset \ldots \supset G_k^{(r)} \supset \ldots
\]
such that

\[ B^r(G_k) \approx B^r(G), \quad k = 1, 2, \ldots, \]

where \( B^r \) denotes the \( r \)-th Betti group. The sequence \( (G_k^{(r)}) \) defines a divisor \( G_c^{(r)} \) depending on the \( r \)-dimensional Betti group of \( G \). More generally we can define divisors of the type \( G_c^{(0,1,\ldots,r)} \).

If we consider now full systems of distinct ends (§ 1, 2) of the type \( G_c^{(r)} \) or \( G_c^{(0,1,\ldots,r)} \), then certain of these systems, namely, the full systems of indivisible ends \(^{21}\) seem to have a special significance \(^{22}\).

Another problem is that of decomposition of the group \( Z(G_c) \) of limit cycles. If \( M(E_{\ell}^r) \) is the system of prime ends, then \( Z(E_{\ell}^r) \) is a subgroup of \( Z(I_c^r) \) for each \( E_{\ell}^r \). In what sense can the system of all \( Z(E_{\ell}^r) \) be considered as a decomposition of \( Z(I_c^r) \)?

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\(^{21}\) A full system of distinct ends \( M(G_c) \) we call indivisible if there exists no full system \( N(G_c') \neq M(G_c) \) such that each \( G_c' \subset G_c \).

\(^{22}\) Even in the case where the boundary \( I_c \) of \( G \) is a complex; the reader may consider the example of a region \( G \) bounded by the sum of a sphere and a segment in \( G \) with one endpoint on the boundary. From the point of view of connectivity groups the region and its boundary are indistinguishable from a spherical region and a sphere. But the segment is obviously an ideal element of the type \( G_c^{(r)} \).