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On the directions of Borel of meromorphic functions of finite order $> \frac{1}{2}$

by

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The object of this paper is to prove the following:

THEOREM IV. *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions* (E) [1]. *Suppose that, in an angle A of vertex 0 and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, we have*

$$\overline{\lim_{r \to \infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0,$$

for a value of a.

There exists, in an arbitrary angle A' containing A and of vertex 0, at least one semi-infinite line D such that for an arbitrary angle Ω of vertex 0 and of bisector D, we have

$$\overline{\lim_{r \to \infty}} \frac{n(r, \pi, \Omega)}{V(r)} > 0,$$

for all elements π, except at the most two, in the family $K(\eta, f)$, where $K(\eta, f)$ denotes the aggregate of all the distinct constants and the meromorphic functions $\pi(z)$ satisfying

$$T(r, \pi) < \eta(r)V(r), \quad r > r_0(\pi), \quad \lim_{r \to \infty} \eta(r)V(r) = \infty,$$

where $\eta(r)$ is an infinitesimal.

In reality, the foregoing theorem is a complement to the theorem due to Valiron as follows:

THEOREM of Valiron [2]). *$f(z)$ is a meromorphic function of*

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1) See p. [3] . . .

2) Acta Math. 47 (1926), 137—138.

finite order $\varrho > \frac{1}{2}$. *Let* $V(r)$ *be a continuous function satisfying the conditions* (E). *Suppose that, in an angle* A *of vertex* 0 *and of measure* $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, *we have*

$$\overline{\lim_{r\to\infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0$$

for a value of a, *and let* A' *be an arbitrary angle containing* A *and of vertex* 0.

There exist three positive finite numbers h, h_1, h_2 *and an infinite sequence of positive numbers* (R_m), *such that*

$$\lim_{m\to\infty} \frac{\log T(R_m, f)}{\log R_m} = \varrho, \quad R_{m+1} > hR_m$$

in relation with the following property: in the region Δ_m, *being the common region of* A' *and the circular ring* $R_m < |z| < hR_m$, *the function* $f(z)$ *takes* $kT(R_m, f)$ *times all values* c *except at the most two provided that* $m > m_0$ *where* $h_1 < k < h_2$.

In the whole paper, $n(r, \varphi, \Omega)$ denotes the number of zeros of the function $f(z) - \varphi(z)$ in the common part of the region Ω and the circle $|z| \leqq r$; and $N(r, \varphi, \Omega)$ denotes the corresponding density

$$\int_0^r \frac{n(r, \varphi, \Omega) - n(0, \varphi, \Omega)}{r} dr + n(0, \varphi, \Omega) \log r.$$

1. The present work is based principally upon the following THEOREM of Rauch [3]. *Let* $f(z)$, $P(z)$, $Q(z)$, $R(z)$ *be four distinct meromorphic functions in a region* (Δ) *and* (D) *a region contained in* (Δ). *Divide the region* (D) *into* p *partial regions* (D_i) *and let* (Γ_i) *be the circle concentric to the smallest circle containing* (D_i) *and radius* 20 *times larger. Suppose that the following conditions are satisfied:*

(C_1) *The number of the zeros of* $f(z) - c$ *in* (D) *is superior to* M *for all values of* c *in a certain circle of radius* $\frac{1}{2}$ *on the Riemann sphere.*

$$(C_2) \quad \frac{1}{\text{area of } (\Gamma_i)} \iint_{(\Gamma_i)} \overset{+}{\log} \left(|P(z) + Q(z)| + \frac{1}{|P(z) - Q(z)|} + \frac{1}{|Q(z) - R(z)|} + \frac{1}{|R(z) - P(z)|} \right) d\sigma < \frac{M}{p}.$$

[3] Journ. de Math. (9) 12 (1933), 133

$$(C_3)\quad \begin{cases} \textit{The number of the zeros and poles of the functions } P(z), \\ Q(z),\ R(z),\ P(z)-Q(z),\ Q(z)-R(z),\ R(z)-P(z) \textit{ in} \\ \textit{in } (\Delta) \textit{ is inferior to } \varepsilon \frac{M}{p}, \textit{ where } \varepsilon \textit{ is a numerical constant} \\ < \alpha < 1. \end{cases}$$

Then there exists at least one circle (Γ_i) *in which the number of the zeros of the function* $f(z) - \pi(z)$ *is superior to* $c_1 \frac{M}{p}$ *for at least one function* $\pi(z)$ *out of the three* $P(z)$, $Q(z)$, $R(z)$, *where* c_1 *is a numerical constant.*

This theorem is valid when $\frac{M}{p}$ *is superior to a numerical constant.*

It is to be remarked that, for the condition (C_3), the numbers of the zeros and the poles of the constans 0 and ∞ are counted as zero.

2. Let $f(z)$ be a meromorphic function of positive finite order ϱ. We are going to show that there exist continuous functions $V(r)$ adjoined to the characteristic function $T(r,f)$ of $f(z)$, satisfying the following conditions:

$$(E)\quad \begin{cases} \lim\limits_{r\to\infty} \dfrac{V(hr)}{V(r)} = h^{\varrho}, \quad \text{for every } h > 0, \\ \lim\limits_{r\to\infty} \dfrac{\log V(r)}{\log r} = \varrho, \\ T(r,f) \leqq V(r), \quad r > r_0, \\ \overline{\lim\limits_{r\to\infty}} \dfrac{T(r,f)}{V(r)} = 1. \end{cases}$$

In fact, after Valiron[4]), there exist continuous functions $\varrho(r)$ differentiable in adjacent closed intervals of which the end points are finite in number at finite distance, satisfying the following conditions:

$$(1)\quad \begin{aligned} &\lim_{r\to\infty} \varrho(r) = \varrho, \quad \lim_{r\to\infty} (r\varrho'(r)\log r) = 0, \\ &T(r,f) \leqq r^{\varrho(r)}, \quad \overline{\lim_{r\to\infty}} \frac{T(r,f)}{r^{\varrho(r)}} = 1. \end{aligned}$$

$\varrho(r)$ is known as a proximate order of $f(z)$. If we put $V(r) = r^{\varrho(r)}$, evidently, the last three conditions in (E) are satisfied. Let us show that it satisfies also the first condition.

[4]) C. R. **194** (1932), 1305—1306.

The case $h = 1$ is trivial, and the case $h < 1$ follows immediately from the case $h > 1$. Consider this case. Put

$$r = e^x, \quad V(r) = e^{x\omega(x)}, \quad \omega(x) = \varrho(e^x),$$

then

$$\lim_{x \to \infty} \omega(x) = \varrho,$$

and from (1) the first condition in (E) is equivalent to

$$\lim_{x \to \infty} [(H + x)\omega(H + x) - x\omega(x)] = \varrho H, \quad H = \log h. \tag{2}$$

But

$$\lim_{x \to \infty} [(H + x)\omega(H + x) - x\omega(x)] =$$

$$= \lim_{x \to \infty} \int_x^{x+H} (x\omega(x))' dx = \lim_{x \to \infty} \int_x^{x+H} x\omega'(x) dx + H\varrho,$$

and from (1)

$$\lim_{x \to \infty} \int_x^{x+H} x\omega'(x) dx = 0,$$

hence we have (2) and we see that the function $V(r) = r^{\varrho(r)}$ satisfies all the conditions (E).

3. In this section, we shall prove the theorem as follows which is analogous to the foregoing theorem of Valiron and also of fundamental importance:

THEOREM I. *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions* (E). *Suppose that for an angle A of vertex* 0 *and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, we have*

$$\overline{\lim_{r \to \infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0 \tag{3}$$

for a value of a.

There exists at least one sequence (R_n) of values of r, $\lim_{n \to \infty} R_n = \infty$, such that

$$n[S(n, B), f = c] > K(\varrho, k')V(R_n) \tag{4}$$

for all values of c in a certain circle (C_n) of radius $\frac{1}{2}$ on the Riemann sphere, where $n[S(n, B), f = c]$ denotes the number of the zeros of $f(z) - c$ in the common part $S(n, B)$ of the ring $\frac{R_n}{1 + S} \leq |z| \leq R_n$

and an arbitrary angle B containing A, of vertex 0 and of measure $\frac{\pi}{k'}$ $\left(\frac{1}{2} < k' < k\right)$, and s is a suitably chosen positive constant.

This theorem is established in modifying a method due to Valiron as follows.

From the second fundamental theorem of R. Nevanlinna, we see easily that for every value of a except at the most two, there exists at least one angle A of vertex 0 and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, such that (3) is satisfied. Hence, that hypothesis is possible.

It follows from (3),

$$n(r, a, A) < (1 + \varepsilon)\varrho e V(r) \tag{5}$$

for $r > r_0$, and

$$n(r, a, A) > \beta H(\varrho) V(r) \tag{6}$$

for a sequence of values of r tending to infinity.

Let B be an arbitrary angle containing A, of vertex 0 and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$. Without loss of generality, we may suppose that the bisectors of A and B coincide with the positive real axis. Make the transformation $Z = z^{-k'}$, where Z is real when z is so, and then the transformation $Z = 1 - z$, so that the function $f(z)$ in the angle B corresponds to a meromorphic function $F(z)$ in the unit circle $|z| < 1$. From (6) we have

$$n(r, F = a) > \beta H_1(\varrho) V(r) \tag{7}$$

for a sequence of values of r tending to 1. Hence

$$\begin{aligned} T\left(r, \frac{1}{F-a}\right) \geqq N(r, a) > (1-r)n(2r-1, F=a) > \\ > \beta H_2(\varrho)(1-r)V\left[\left(\frac{1}{1-r}\right)^{\frac{1}{k'}}\right] \end{aligned} \tag{8}$$

for a sequence of values of r tending to 1. But

$$T\left(r, \frac{1}{F-a}\right) = T(r, F) + h(r, a)$$

$h(r, a)$ being bounded when a is fixed, therefore

$$T(r, F) > \beta H_3(\varrho)(1-r)V\left[\left(\frac{1}{1-r}\right)^{\frac{1}{k'}}\right] \tag{9}$$

for a sequence of values of r tending to 1. On the other hand, it is known that

$$(10) \qquad T(r, F) < H_4(\varrho)\,(1-r)V\left[\left(\frac{1}{1-r}\right)^{\frac{1}{k'}}\right]$$ [5])

for $r > r_0$, which shows that the hypothesis is essential and that $F(z)$ is of finite order.

The foregoing calculations are due to Valiron.

Now, from another well known theorem of Valiron [6]), we deduce that there exists at least one circle $C(r)$ of radius $\frac{1}{2}$ on the Riemann sphere, such that for all values c in that circle, we have

$$(11) \qquad N(r, F = c) > \frac{\beta}{4} H_3(\varrho)(1-r)V\left[\left(\frac{1}{1-r}\right)^{\frac{1}{k'}}\right]$$

for all values of $r > r_0^1$ in the sequence for which (9) is satisfied, and that for all values of $R > R_0$,

$$(12) \qquad T\left(R, \frac{1}{F-c}\right) < T(R, F) + C(F),$$

$$(13) \qquad T\left(R, \frac{1}{f-c}\right) < T(R, f) + C_1(f)$$

$C(F)$, $C_1(f)$ being constants depending only upon $F(z)$ and $f(z)$ respectively.

From (9), (10), (12), we have for all values of c in $C(r)$

$$(14) \qquad n(r, F = c) > H(\varrho, k')V(r)$$

for at least one sequence of values of r tending toward ∞. In passing back to $f(z)$, it follows from (14), for all values of c in (C_n),

$$(15) \qquad n(R_n, c, B) > H_1(\varrho, k')V(R_n)$$

for at least one sequence (R_n) tending toward infinity with n.

Moreover, from (13) we have

$$(16) \qquad n\left(\frac{R_n}{1+S}, c, B\right) < \mu\left(\frac{2}{1+S}\right)^{\varrho} V(R_n) \qquad \left(\frac{R_n}{1+S} > R_0\right)$$

for all values of c in (C).

It follows from (15) and (16), by choosing s such that $H_1(\varrho, k') - \mu\left(\frac{2}{1+S}\right)^{\varrho} > K(\varrho, k')$,

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[5]) VALIRON, l. c. [2]), 136.

[6]) VALIRON, l. c. [2]), 123—124.

$$n(S(n, B), f = c) > K(\varrho, k')V(R_n),$$

where $n(S(n, B), f - c)$ denotes the number of zeros of the function $f(z) - c$ in the region $S(n, B)$ common to B and the ring $\frac{R_n}{1+S} < |z| \leqq R_n$, and for all values of c in (C_n).

Thus the foregoing theorem is proved.

4. For the sake of convenience, we shall employ the following notation. Let $T^*(r, \varphi)$ be $T(r, \varphi)$ if $\varphi(z) \not\equiv \infty$, and be 0 if $\varphi(z) \equiv \infty$. Let $C^*(\varphi)$ be $C(\varphi) = T\left(r, \frac{1}{\varphi}\right) - T(r, \varphi)$ if $\varphi(z) \not\equiv 0, \infty$, and be 0 if $\varphi(z) \equiv 0, \infty$. $T(r, \varphi)$ is the characteristic function of $\varphi(z)$.

THEOREM II. *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions* (E). *Suppose that for an angle A of vertex* 0 *and of measure $\frac{\pi}{k}\left(\frac{1}{2} < k < \varrho\right)$, we have*

$$\overline{\lim_{r \to \infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0$$

for a value of a, and let (R_n) be the sequence of values of r in Theorem I.

There exists, in an arbitrary angle A' containing A and of vertex 0, *at least one sequence of circles $\Gamma(n)$*

$$|z - x(n)| = \alpha\, |x(n)|, \qquad \frac{R_n}{1+S} < |x(n)| < R_n, \tag{17}$$

such that if $P(z)$, $Q(z)$, $R(z)$ are any three distinct meromorphic functions satisfying the conditions

$$\begin{cases} T^*[(1+\alpha)R_n, \varphi] < \alpha^4 V(R_n), \quad \varphi(z) \equiv P(z), Q(z), R(z); \\ C^*(\psi) > -\alpha^4 V(R_n), \\ \psi(z) \equiv P(z), Q(z), R(z), P(z)-Q(z), Q(z)-R(z), R(z)-P(z), \end{cases} \tag{18}$$

then we have

$$n(\Gamma(n), f - \pi) > a^3 V(R_n) \tag{19}$$

for at least one function out of the three $P(z)$, $Q(z)$, $R(z)$, where $n(\Gamma(n), f - \pi)$ denotes the number of the zeros of $f(z) - \pi(z)$ in $\Gamma(n)$.

This theorem is valid when $\frac{1}{a}$ *and* $\alpha^4 V(r)$ *are greater than a certain constant.*

Let B be an angle having the properties given in Theorem I, and contained in A'.

We are going to apply the theorem of Rauch. Let the region $S(n, B)$ defined in theorem I be the region (D) and let $M = K(\varrho, k')V(R_n)$. Then by Theorem I, the condition (C_1) in the Theorem of Rauch is satisfied. Divide the angle B into equal sectors of measure $\frac{\pi}{\alpha_1 k'}$ by semi-infinite lines issued from the origin and describe the circles

$$|z| = \frac{R_n}{1+S}(1+\alpha_1)^i \qquad \left(\frac{R_n}{1+S}(1+\alpha_1)^q = R_n;\ i = 1,2,\ldots,q\right).$$

The region (D) is thus divided into

$$p = c(k')\,\frac{\log(1+S)}{\alpha_1^2}$$

similar curvilinear rectangles $D_i(n)$, where α_1 is sufficiently small and $c(k')$ is a constant depending only upon k'. Let $D_i(n)$ be the partial regions (D_i) in the theorem of Rauch and $\Gamma_i(n)$ the corresponding circles Γ_i. Then the circles $\Gamma_i(n)$ are contained in the ring

$$(1 - 15\alpha_i)\frac{R_n}{1+S} < |z| < (1 + 15\alpha_1)R_n$$

which is taken for each n as the region (Δ).

Let $\varrho_i(n)$ be the modulus of the center of $\Gamma_i(n)$, then its radius is $k_i(n)\alpha_1\varrho_i(n)$, $k_i(n)$ lying between two numerical constants h_1 and h_2.

In modifying suitably a method due to Rauch [7]), we see that the conditions (C_2), (C_3) in his theorem given above are satisfied if the conditions (18) have been imposed, where α is taken to be the largest of $16\alpha_1$ and $h_2\alpha_1$, and when $\frac{1}{\alpha}$ and $\alpha^4 V(R_n)$ are greater than a certain constant.

Hence, by the theorem of Rauch, for each large integer n, there exists at least one circle defined by (17), such that (19) is satisfied.

It is evident that when α is less than a certain constant, all the circles $\Gamma(n)$ are contained in A'.

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[7]) RAUCH, l. c., 133—138.

5. Let $H(\alpha, f)$ be the family of meromorphic functions $\pi(z)$ satisfying the condition

$$T^*[(1+\alpha)r, \pi] < \alpha^4 V(r), \qquad r > r_0(\pi).$$

Consider a certain infinite sequence of circles $\Gamma(n)$ in theorem II. We are going to show that for all functions $\pi(z)$ of the family $H(\alpha, f)$, except at the most two, we have

$$n(\Gamma(n'), f - \pi) > \alpha^3 V(R_{n'}), \qquad n' > n_0(\pi).$$

Suppose that there are two functions $P(z)$, $Q(z)$ in the family $H(\alpha, f)$, such that

$$n(\Gamma(n'),\ f - P) \leqq \alpha^3 V(R_{n'}),$$
$$n(\Gamma(n'),\ f - Q) \leqq \alpha^3 V(R_{n'})$$

for an infinite sequence of values of n'. Let $R(z)$ be a function in the family $H(\alpha, f)$ distinct from $P(z)$ and $Q(z)$. Evidently when $n' > n_0(R)$ the conditions (18) are satisfied, hence by theorem II, we have

$$n(\Gamma(n'),\ f = R) > \alpha^3 V(R_{n'}), \qquad n' > n_0(R).$$

The statement is therefore proved and we have the following

THEOREM III. *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions* (E). *Suppose that in an angle A of vertex* 0 *and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, we have*

$$\overline{\lim_{r\to\infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0$$

for a value of a. Let $H(\alpha, f)$ be the family of meromorphic functions satisfying the condition

$$T^*[(1+\alpha)r, \pi] < \alpha^4 V(r), \qquad r > r_0(\pi).$$

There exists, in an arbitrary angle A' containing A and of vertex 0, *at least one sequence of circles $\Gamma(n)$ defined by* (17) *such that*

$$n(\Gamma(n),\ f - \pi) > \alpha^3 V(r), \qquad n > n_0(\pi), \tag{20}$$

for all functions $\pi(z)$ of the family $H(\alpha, f)$ except at the most two.

This theorem is valid when $\frac{1}{\alpha}$ and $\alpha^4 V(r)$ are greater than a certain constant.

6. Let A_α be the smallest of the angles contained in A' and of vertex 0 in which there are an infinite number of the circles $\Gamma(n)$ in theorem III. Then

$$\overline{\lim_{r\to\infty}} \frac{n(r,\pi,A_\alpha)}{V(r)} \geqq \frac{\alpha^3}{2(1+\alpha)^\varrho} \tag{21}$$

from (20).

Let (α_n) be a sequence of values of α, tending to 0 with $\frac{1}{n}$. Let (D_{α_n}) be the bisector of A_{α_n} and D a limit-line of the semi-lines D_{α_n}. An arbitrary angle Ω of vertex 0 and of bisector D contains then an infinity of the angles A_{α_n}.

Let $K(r,f)$ be the family of meromorphic functions $\pi(z)$ satisfying the condition

$$T^*(r,\pi) \leqq \eta(r)V(r), \qquad r > r_0(\pi), \quad \lim \eta(r)V(r) = \infty,$$

where $\eta(r)$ is an infinitesimal. It is evident that the family $K(\eta,f)$ is contained in the family $H(\alpha,f)$ for every fixed value of α. Hence from (21) we have

$$\overline{\lim_{r\to\infty}} \frac{n(r,\pi,\Omega)}{V(r)} > 0$$

for all elements $\pi(z)$ in the family $K(\eta,f)$ except at the most two.

It is also evident that the family $K(\eta,f)$ is the aggregate of all the distinct constants, and the meromorphic functions (non-degenerated to constants) $\pi(z)$ satisfying

$$T(r,\pi) \leqq \eta(r)V(r), \qquad r \geqq r_0(\pi).$$

We have therefore the following

THEOREM IV.[8]) *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions* (E). *Suppose that, in an angle A of vertex 0 and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2}<k<\varrho\right)$, we have*

$$\overline{\lim_{r\to\infty}} \frac{N(r,a,A)}{V(r)} = \beta > 0,$$

for a value of a.

There exists, in an arbitrary angle A' containing A and of vertex 0 at least one semi-line (D) issued from 0, such that for an arbitrary angle Ω of vertex 0 and of bisector D, we have

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[8]) This theorem has been stated in a Note in Comptes Rendus **206** (1938), 811—812.

$$\overline{\lim_{r\to\infty}} \frac{n(r, \pi, \Omega)}{V(r)} > 0$$

for all elements π *of the family* $K(r, f)$ *except at the most two.*

It is to be remarked that the foregoing theorem and the theorem IX [9]) in the Thèse of Rauch do not contain each other and that the family $K(\alpha, f)$ in the latter must be stated analogously to the family $K(\eta, f)$ in our theorem IV.

Finally, the writer wishes to thank Prof. Valiron for his useful criticisms.

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[9]) Rauch, l. c. [3]), 157.