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On certain useful vectors in differential geometry

by

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1. Definitions. We propose to consider briefly a set of vectors, by means of which some of the formulae and proofs of elementary Differential Geometry of a surface in Euclidean 3-space are considerably simplified. The position vector $\mathbf{r}$ of the current point on the surface is a function of two parameters, u and v. Let suffixes 1, 2 denote differentiations with respect to u, v respectively. Thus

$$\mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial u}, \qquad \mathbf{r}_2 = \frac{\partial \mathbf{r}}{\partial v}$$

and the unit normal vector $\mathbf{n}$ to the surface at the point considered is such that $\mathbf{n} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H}$, where $H^2 = EG - F^2$, E, F, G being fundamental magnitudes of the first order for the surface. Then the scalar triple product

$$[\mathbf{r}_1, \mathbf{r}_2, \mathbf{n}] = H\mathbf{n} \cdot \mathbf{n} = H$$

and the reciprocal system of vectors [1]) to $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{n}$ is the system

$$\frac{\mathbf{r}_2 \times \mathbf{n}}{H}, \qquad \frac{\mathbf{n} \times \mathbf{r}_1}{H}, \qquad \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H}.$$

The third of these is clearly $\mathbf{n}$. The first and second we shall denote by $\mathbf{p}$ and $\mathbf{q}$. Thus

$$(1) \qquad \begin{cases} H^2\mathbf{p} = \mathbf{r}_2 \times (\mathbf{r}_1 \times \mathbf{r}_2) = G\mathbf{r}_1 - F\mathbf{r}_2 \\ H^2\mathbf{q} = (\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{r}_1 = E\mathbf{r}_2 - F\mathbf{r}_1 \end{cases}$$

and these vectors are, of course, tangential to the surface. It is easily verified that

$$(2) \qquad \mathbf{p}^2 = \frac{G}{H^2}, \qquad \mathbf{q}^2 = \frac{E}{H^2}, \qquad \mathbf{p} \cdot \mathbf{q} = -\frac{F}{H^2},$$

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[1]) Cf. the writer's Elementary Vector Analysis [Bell and Sons 1921], 65.

while from their definitions it follows that

$$(3) \qquad \begin{cases} \quad \mathbf{p}\cdot\mathbf{r}_1 = 1, \qquad \mathbf{q}\cdot\mathbf{r}_2 = 1 \\ 0 = \mathbf{p}\cdot\mathbf{r}_2 = \mathbf{q}\cdot\mathbf{r}_1 = \mathbf{p}\cdot\mathbf{n} = \mathbf{q}\cdot\mathbf{n}. \end{cases}$$

2. Derivatives of Vectors. In terms of $\mathbf{p}$ and $\mathbf{q}$ the derivatives[2]) of $\mathbf{n}$ with respect to u, v are given very compactly by the equations

$$(4) \qquad \begin{cases} \mathbf{n}_1 = -L\mathbf{p} - M\mathbf{q}, \\ \mathbf{n}_2 = -M\mathbf{p} - N\mathbf{q}, \end{cases}$$

L, M, N being the fundamental magnitudes of the second order. The second derivatives of $\mathbf{r}$ with respect to u and v, when expressed in terms of $\mathbf{n}$, $\mathbf{r}_1$, $\mathbf{r}_2$, involve as coefficients certain functions usually denoted by the Christoffel symbols. In terms of the reciprocal system of vectors, however, these derivatives may be simply expressed

$$(5) \qquad \begin{cases} \mathbf{r}_{11} = L\mathbf{n} + \frac{1}{2}E_1\mathbf{p} + \left(F_1 - \frac{1}{2}E_2\right)\mathbf{q}, \\ \mathbf{r}_{12} = M\mathbf{n} + \frac{1}{2}E_2\mathbf{p} + \frac{1}{2}G_1\mathbf{q}, \\ \mathbf{r}_{22} = N\mathbf{n} + \left(F_2 - \frac{1}{2}G_1\right)\mathbf{p} + \frac{1}{2}G_2\mathbf{q}. \end{cases}$$

To prove the first of these we observe that the coefficient of $\mathbf{p}$ in the expression for $\mathbf{r}_{11}$ must, in virtue of (3), have the value

$$\mathbf{r}_{11}\cdot\mathbf{r}_1 = \frac{1}{2}\frac{\partial}{\partial u}(\mathbf{r}_1^2) = \frac{1}{2}E_1.$$

Similarly the coefficient of $\mathbf{q}$ must have the value

$$\mathbf{r}_{11}\cdot\mathbf{r}_2 = \frac{\partial}{\partial u}(\mathbf{r}_1\cdot\mathbf{r}_2) - \frac{1}{2}\frac{\partial}{\partial v}(\mathbf{r}_1^2) = F_1 - \frac{1}{2}E_2$$

and likewise for the others. The above formulae (5) should be compared with the usual[3])

$$(6) \qquad \begin{cases} \mathbf{r}_{11} = L\mathbf{n} + l\mathbf{r}_1 + \lambda\mathbf{r}_2, \\ \mathbf{r}_{12} = M\mathbf{n} + m\mathbf{r}_1 + \mu\mathbf{r}_2, \\ \mathbf{r}_{22} = N\mathbf{n} + n\mathbf{r}_1 + \nu\mathbf{r}_2, \end{cases}$$

whose coefficients l, m, n, λ, μ, ν, when expressed in terms of E, F, G, are much less simple than those in (5).

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[2]) Cf. the writer's Differential Geometry, Vol. 1, 61 [Cambridge University Press 1927]. This book will be indicated briefly by D. G.

[3]) D. G., 90.

The derivatives of $\mathbf{p}$ and $\mathbf{q}$ with respect to u, v are given by

$$(7)\qquad \begin{cases} H^2\mathbf{p}_1 = (GL - FM)\mathbf{n} - H^2(l\mathbf{p}+m\mathbf{q}), \\ H^2\mathbf{p}_2 = (GM - FN)\mathbf{n} - H^2(m\mathbf{p}+n\mathbf{q}), \\ H^2\mathbf{q}_1 = (EM - FL)\mathbf{n} - H^2(\lambda\mathbf{p}+\mu\mathbf{q}), \\ H^2\mathbf{q}_2 = (EN - FM)\mathbf{n} - H^2(\mu\mathbf{p}+\nu\mathbf{q}). \end{cases}$$

To verify these take, for instance, the coefficient of $\mathbf{p}$ in the first. In virtue of (3) this must have the value

$$\mathbf{p}_1 \cdot \mathbf{r}_1 = \frac{\partial}{\partial u}(\mathbf{p} \cdot \mathbf{r}_1) - \mathbf{p} \cdot \mathbf{r}_{11} = -l$$

by (6) and similarly for the others.

3. Differential Invariants. In terms of $\mathbf{p}$ and $\mathbf{q}$ the surface gradient[4]) $\nabla\varphi$, of a scalar point-functions φ, is simply

$$(8)\qquad \nabla\varphi = \mathbf{p}\frac{\partial\varphi}{\partial u} + \mathbf{q}\frac{\partial\varphi}{\partial v}$$

and the surface divergence and rotation[5]) of a vector $\mathbf{h}$ are

$$(9)\qquad \begin{cases} \operatorname{div}\mathbf{h} = \mathbf{p} \cdot \mathbf{h}_1 + \mathbf{q} \cdot \mathbf{h}_2, \\ \operatorname{rot}\mathbf{h} = \mathbf{p}\times\mathbf{h}_1 + \mathbf{q}\times\mathbf{h}_2. \end{cases}$$

As an illustration of the use of these we observe that they give immediately, in virtue of (4),

$$\operatorname{div}\mathbf{n} = \frac{(2FM - EN - GL)}{H^2} = -J, \qquad \operatorname{rot}\mathbf{n} = 0,$$

where J is the first curvature of the surface. Also the surface Laplacian of $\mathbf{r}$, (or the differential parameter of $\mathbf{r}$ of the second order), is[6])

$$\nabla^2\mathbf{r} = \frac{1}{H}\frac{\partial}{\partial u}(H\mathbf{p}) + \frac{1}{H}\frac{\partial}{\partial v}(H\mathbf{q}) = \mathbf{p}_1 + \mathbf{q}_2 + \frac{1}{H}(H_1\mathbf{p}+H_2\mathbf{q}) = J\mathbf{n}$$

in virtue of (7).

Again, if a vector $\mathbf{F}$ is expressed as a sum of components in the directions of $\mathbf{p}$, $\mathbf{q}$, $\mathbf{n}$ in the form $\mathbf{F} = P\mathbf{p} + Q\mathbf{q} + R\mathbf{n}$, it is easily verified that

$$(10)\qquad \mathbf{n}\cdot\operatorname{rot}\mathbf{F} = \frac{1}{H}(Q_1 - P_2).$$

[4]) D. G., 223.

[5]) D. G., 225, 228.

[6]) D. G., 231.

This leads to a short proof of the Circulation Theorem [7]) for a closed curve on the surface, viz.

$$\iint \mathbf{n} \cdot \operatorname{rot} \mathbf{F} \, dS = \oint \mathbf{F} \cdot d\mathbf{r}, \tag{11}$$

where the surface integral is taken over the enclosed region of the surface, and the line integral round the boundary. For the first member of (11) has the value

$$\iint (Q_1 - P_2) \, du \, dv = \oint Q \, dv + \oint P \, du$$

as in the writer's Differential Geometry, Vol 1, 243; and this may be written

$$\oint (P\mathbf{p} + Q\mathbf{q} + R\mathbf{n}) \cdot (\mathbf{r}_1 \, du + \mathbf{r}_2 \, dv) = \oint \mathbf{F} \cdot d\mathbf{r}$$

as required.

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[7]) D. G. 243.